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GALERKIN METHOD APPLIED TO $p(\cdot)$-BI-LAPLACE EQUATION WITH VARIABLE EXPONENT

Abstract. In this article, a Galerkin mixed finite element method is proposed to find the numerical solutions of high order $p(\cdot)$-bi-Laplace equations. The well-posedness of the problem in suitable Lebesgue–Sobolev spaces with variable exponent owing to nonlinear monotone operator theory is investigated. Some a priori error estimates are shown by using the Galerkin orthogonality properties and variable exponent Lebesgue–Sobolev continues embedding.

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Key words and phrases. $p(\cdot)$-bi-Laplace equation, Galerkin method, weak solution, error estimate.

რეზიუმე. სტატიაში შემოთავაზებულია გალიორკინის შერეული სასრული ელემენტების მეთოდი მაღალი რიგის $p(\cdot)$-ბი-ლაპლასის განტოლებების რიცხვითი ამონახსნების საპოვნელად. არაწრფივ მონოტონურ ოპერატორთა თეორიის გამოყენებით გამოკვლეულია დასმული ამოცანის კორექტულობა შესაფერის ცვლად მაჩვენებლიან ლებეგ-სობოლევის სივრცეებში. ცდომილების ზოგიერთი აპრიორული შეფასება ნაჩვენებია გალიორკინის ორთოგონალურობის თვისებებისა და ცვლადმაჩვენებლიანი ლებეგ-სობოლევის უწყვეტი ჩადგმის გამოყენებით.

1 Introduction

We consider a bounded open domain Ω of $\mathbb{R}^n$ with a Lipschitz-continuous boundary $\partial\Omega$. Our aim is to prove the existence and uniqueness of a weak solution u and some a priori error estimates to the differential $p(x)$-Bilaplace equation

$$\begin{cases} \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big) = f & \text{in } \Omega, \\ u = \Psi, \ \ \nabla u = \nabla\Psi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f and Ψ are the given functions in $L^{q(\cdot)}(\Omega)$ and $W^{2,\infty}(\Omega)$, respectively. Here, $p(\cdot) : \Omega \to \mathbb{R}$ denotes the variable exponent which is assumed to be in $L^\infty_+(\Omega)$ such that $1 < p^- \leq p(x) \leq p^+ < \infty$, where $p^- = \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ = \sup_{x\in\overline{\Omega}} p(x)$ a.e. in Ω. During the last decades, the high-order PDEs with variable exponent has undergone rapid development. From a mathematical point of view, equation (1.1) can be considered as a natural generalization of $p(\cdot)$-bi-Laplace equation

$$\triangle\big(|\triangle u|^{p-2}\triangle u\big) = f,$$

which falls within the framework of nonlinear PDEs, where the exponent p is constant. One of our motivation for studying (1.1) comes from applications in the area of elasticity, more precisely, it can be used in modelling of travelling waves in suspension bridges (see [6, 8]). Other interesting applications are related to improve the visual quality of damaged and noisy images if $1 < p^- \leq p^+ < 2$ (see, e.g., [14] and the references therein). Note that in the case $p(x) = 2$, problem (1.1) becomes $\triangle^2 u = f$ which models the deformations of a thin homogeneous plate embedded along its beam and subjected to a distribution f of a load normal to the plate (cf. [1]). Among the most recent works concerning the p-Laplace equation, we can review Lazer et al. [8], where the authors tried to demonstrate the existence of periodic solutions for models of nonlinear supported bending beams and periodic flexing in floating beam. In [5], the authors used discontinuous Galekin method to approximate a biharmonic problem. They also gave an a priori analysis of the error in norm L^2. In [11], the author has studied a p-biharmonic problem using discontinuous Galerkin finite element Hessian. An imagery problem caused by a $p(\cdot)$-Laplace operator with $1 \leq p(\cdot) \leq 2$ has been considered in [14]. To solve the problem, the authors regularized the proposed PDE to be able to use a fixed point iterative method.

The paper is structured as follows. We present in Section 2 some basic notations and material needed for our work. Section 3 is devoted to the existence and uniqueness of a weak solution to the problem under investigation in suitable Lebesgue–Sobolev spaces with variable exponent using the nonlinear monotone operators theory. In Section 4, the Galerkin mixed finite element method and $\inf - \sup$ condition are given. Finally, we show some a priori error estimates with the help of Ritz projection operator and Galerkin orthogonality properties, which are presented in Section 5.

2 Preliminaries

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ as follows:

$$L^{p(\cdot)}(\Omega) = \Big\{ u : \ \ \Omega \to \mathbb{R}, \ \ u \ \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty \Big\}.$$

Note that $L^{p(\cdot)}(\Omega)$ equipped with the Luxembourg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{ \gamma > 0, \ \int_\Omega \Big|\frac{u(x)}{\gamma}\Big|^{p(x)}\,dx \leq 1 \Big\}$$

is a Banach space. Note that all definitions and properties of Lebesgue and Sobelev spaces with variable exponent given below are taken from references [2–4, 7, 12].

Definition 2.1. Let $u : \Omega \to \mathbb{R}$ be a measurable function, then the expression

$$\rho_{p(\cdot)}(u) = \int\limits_{\Omega} |u(x)|^{p(x)}\, dx$$

is called modular of u.

Definition 2.2. For some $p \in L^{\infty}_{+}(\Omega)$ and $m \in \mathbb{N} - \{0\}$, we introduce the exponent variable Sobolev space

$$W^{m,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega);\ \ D^{\alpha} u \in L^{p(\cdot)}(\Omega),\ \ \forall\, \alpha \in \mathbb{N}^n \text{ and } |\alpha| \le m \Big\}$$

equipped with the norm

$$\|u\|_{m,p(\cdot)} = \sum_{|\alpha| \le m} \|D^{\alpha} u\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 2.1.

(1) Let p, q and $r \in L^{\infty}_{+}(\Omega)$, $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{q(\cdot)}(\Omega)$ such that

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)}\,.$$

Then

$$\|uv\|_{L^{r(\cdot)}(\Omega)} \le \Big(\frac{1}{(\frac{p}{r})^{-}} + \frac{1}{(\frac{q}{r})^{-}} \Big) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)}.$$

(2) Suppose that $p(x) \le q(x)$ a.e. in Ω. Then

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega).$$

(3) $\|u\|_{L^{p(\cdot)}(\Omega)} = k \Longleftrightarrow \rho_{p(\cdot)}\big(\frac{u}{k}\big) = 1.$

(4) $\big(\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \underset{n\to\infty}{\longrightarrow} 0\big) \Longleftrightarrow \big(\rho_{p(\cdot)}(u_n - u) \underset{n\to\infty}{\longrightarrow} 0\big).$

(5) Let $p, q \in L^{\infty}_{+}(\Omega)$ and $m \in \mathbb{N}^{*}$ with $p(x) \le q(x)$ a.e. in Ω. Then

$$W^{m,q(\cdot)}(\Omega) \hookrightarrow W^{m,p(\cdot)}(\Omega).$$

Definition 2.3 (see [2, Definition 4.1.1, p. 98])**.** A function $\beta : \Omega \to \mathbb{R}$ is locally log-Hölder continuous on Ω if $\exists C > 0$ such that

$$|\beta(x) - \beta(y)| \le \frac{C}{\log(e + \frac{1}{|x-y|})}\,, \quad \forall\, x, y \in \Omega.$$

If

$$|\beta(x) - \beta_{\infty}| \le \frac{C}{\log(e + |x|)}$$

for some $\beta_{\infty} \ge 1$, $c > 0$ and all $x \in \Omega$, then we say that β satisfies the log-Hölder decay condition (at infinity). We denote by $P^{\log}(\Omega)$ the class of variable exponents which are log-Hölder continuous, i.e., which satisfy the local log-Hölder continuity condition and the log-Hölder decay condition.

Definition 2.4 (see [2, Definition 11.2.1])**.** Let $p \in P^{\log}(\Omega)$. We also define

$$W_0^{2,p(\cdot)}(\Omega) := \overline{C_0^{\infty}(\Omega)}^{W^{2,p(\cdot)}(\Omega)}$$

Similarly, we define

$$W_{\Psi}^{2,p(\cdot)}(\Omega) = \Psi + W_0^{2,p(\cdot)}(\Omega) = \Big\{ \varphi \in W^{2,p(\cdot)}(\Omega); \varphi_{\setminus \partial\Omega} = \Psi \text{ and } \nabla \varphi_{\setminus \partial\Omega} = \nabla \Psi \Big\}.$$

Remark 2.2.

(i) Note that if $p^{-} > 1$, then the spaces $W^{2,p(\cdot)}(\Omega)$ and $W_0^{2,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

(ii) (Poincaré inequality) Let $p \in L^{\infty}(\Omega)$ with $p^{-} \ge 1$, there exists $C(\Omega, p(\cdot))$ such that

$$\|u\|_{p(\cdot)} \le C \|\nabla u\|_{p(\cdot)}, \quad \forall\, u \in W_0^{1,p(\cdot)}(\Omega)$$

3 Existence and uniqueness of the weak solution to $p(\,\cdot\,)$-Bi-Laplacien with variable exponent

Definition 3.1. A function u is a weak solution of problem (1.1) if it satisfies

$$\int_\Omega \big(|\triangle u|^{p(x)-2}\triangle u\big)\triangle v\,dx = \int_\Omega f v\,dx,\quad \forall\, v\in W_0^{2,p(\,\cdot\,)}(\Omega).$$

Theorem 3.1. *For $f\in L^{q(\,\cdot\,)}(\Omega)$, problem (1.1) admits a unique weak solution u in $W_\Psi^{2,p(\,\cdot\,)}(\Omega)$.*

Proof. We prove the theorem in $W_0^{2,p(\,\cdot\,)}(\Omega)$ because if $u\in W_\Psi^{2,p(\,\cdot\,)}(\Omega)$, then $u-\Psi\in W_0^{2,p(\,\cdot\,)}(\Omega)$ and we can take $u-\Psi$ instead of u. We apply the monotone operators theory and prove that

$$\triangle^2_{p(x)} := \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big) : W_0^{2,p(\,\cdot\,)}(\Omega)\to (W_0^{2,p(\,\cdot\,)}(\Omega))^* \tag{3.1}$$

is a hemicontinuous, coercive and monotone operator.

Let us define the functional A on $W_0^{2,p(\,\cdot\,)}(\Omega)$ by

$$A(u)=\int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}\,dx.$$

We have

$$\begin{aligned}(A'(u),v) &= \frac{d}{dt}\,\{A(u+tv)\}_{t=0} = \frac{d}{dt}\bigg\{\int_\Omega \frac{1}{p(x)}\,|\triangle(u+tv)|^{p(x)}\,dx\bigg\}_{t=0}\\
&= \bigg\{\int_\Omega \frac{1}{p(x)}\,\triangle v\cdot p(x)|\triangle(u+tv)|^{p(x)-1}\,dx\bigg\}_{t=0} = \int_\Omega \big(|\triangle u|^{p(x)-2}\triangle u\big)\triangle v\,dx\\
&= \int_\Omega \triangle(|\triangle u|^{p(x)-2}\triangle u)v\,dx = \big(\triangle^2_{p(x)}u,v\big),\quad \forall\, v\in W_0^{2,p(\,\cdot\,)}(\Omega)\end{aligned} \tag{3.2}$$

which implies that $A(\,\cdot\,)$ is differentiable in Gateau sense and $A'=\triangle^2_{p(x)}$. Therefore, $\triangle^2_{p(x)}$ is a hemicontinuous operator.

On the other hand, using Hölder's inequality, we get

$$\begin{aligned}\sup_{\|v\|_{W_0^{2,p(\,\cdot\,)}(\Omega)}\le 1}\big|(\triangle^2_{p(x)}u,v)\big| &= \sup_{\|v\|_{W_0^{2,p(\,\cdot\,)}(\Omega)}\le 1}\bigg|\int_\Omega \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big)v\,dx\bigg|\\
&\le \sup_{\|v\|_{W_0^{2,p(\,\cdot\,)}(\Omega)}\le 1}\int_\Omega |\triangle u|^{p(x)-1}|\triangle v|\,dx \le C^{\frac{p(x)}{q(x)}} \le C^{\frac{p^+}{q^-}}.\end{aligned} \tag{3.3}$$

This proves that $\triangle^2_{p(\,\cdot\,)}$ is bounded on $W_0^{2,p(\,\cdot\,)}(\Omega)$. Next, from the inequality (see [10])

$$|b|^{p(\,\cdot\,)} \ge |a|^{p(\,\cdot\,)} + p|a|^{p(\,\cdot\,)-2}a(b-a)+\frac{|b-a|^{p(\,\cdot\,)}}{2^{p(\,\cdot\,)-1}-1}\quad\text{for } p\ge 2 \text{ and } a,b\in\mathbb{R}^n$$

it follows that

$$\begin{aligned}\big(\triangle^2_{p(x)}(u)-\triangle^2_{p(x)}(v),u-v\big) &= \int_\Omega \big(|\triangle u|^{p(x)-2}\triangle u-|\triangle v|^{p(x)-2}\triangle v\big)\triangle(u-v)\,dx\\
&= \int_\Omega |\triangle u|^{p(x)-2}\triangle u(\triangle u-\triangle v)\,dx - \int_\Omega |\triangle v|^{p(x)-2}\triangle v(\triangle u-\triangle v)\,dx\end{aligned}$$

$$\geq \frac{2}{p(x)(2^{p(x)-1}-1)} \int\limits_{\Omega} |\triangle u - \triangle v|^{p(x)}\, dx \geq \frac{2}{p^+(2^{p^+-1}-1)} \int\limits_{\Omega} |\triangle u - \triangle v|^{p(x)}\, dx. \quad (3.4)$$

Now, using Calderon–Zygmund and Poincaré inequalities, we find that the norm $\|\cdot\|_{W_0^{2,p(\cdot)}(\Omega)}$ is equivalent to the semi-norm $\|\triangle(\cdot)\|_{L^{p(\cdot)}(\Omega)}$ over the space $W_0^{2,p(\cdot)}(\Omega)$.

This allows us to write

$$\big(\triangle^2_{p(x)}(u) - \triangle^2_{p(x)}(v), u - v\big) \geq C(p^+)\|u - v\|^{p(x)}_{W_0^{2,p(\cdot)}(\Omega)},$$

from which we conclude the monotonicity of $\triangle^2_{p(x)}$. Similarly,

$$(\triangle^2_{p(x)}(u), u) \geq C(p^+)\|u\|^{p(x)}_{W_0^{2,p(\cdot)}(\Omega)}.$$

This proves the coercivity of $\triangle^2_{p(x)}$. Finally, by Hölder's inequality, we have

$$|(f, v)| = \left|\int\limits_{\Omega} f v\, dx\right| \leq C\|f\|_{q(x)}\|v\|_{p(x)}.$$

Taking into account that $L^{q^+}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W_0^{2,p(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, we arrive at

$$|(f, v)| \leq C\|f\|_{L^{q^+}(\Omega)}\|v\|_{W_0^{2,p(\cdot)}(\Omega)}.$$

Hence $f \in (W_0^{2,p(\cdot)}(\Omega))^*$. This achieves the proof. □

4 Galerkin mixed formulation

Set $X := W_\Psi^{2,p(\cdot)}(\Omega)$ and $M := W_0^{2,p(\cdot)}(\Omega)$. Let us introduce a new variable

$$\varphi = |\triangle u|^{p(x)-2}\triangle u.$$

This allows us to write problem (1.1) as follows:

$$\begin{cases} -\triangle u = |\varphi|^{q(x)-2}\varphi, \\ -\triangle \varphi = f. \end{cases} \quad (4.1)$$

The weak formulation associated to (4.1) is: Find $(u, \varphi) \in X \times L^{q(\cdot)}$ satisfying

$$\begin{cases} a(\varphi, v) + c(u, v) = 0 & \forall\, v \in X, \\ c(\varphi, \mu) = l_M(\mu) & \forall\, \mu \in M, \end{cases} \quad (4.2)$$

where

$$a(\varphi, v) := \int\limits_{\Omega} |\varphi|^{q(x)-2}\varphi v\, dx, \quad c(\varphi, \mu) := \int\limits_{\Omega} -\triangle\varphi\mu\, dx, \quad l_M(\mu) := \int\limits_{\Omega} f\mu\, dx.$$

Proposition 4.1 (inf-sup condition)**.** *There exists $\gamma > 0$ such that*

$$\inf_{\mu\in M} \sup_{u\in W_0^{2,p(\cdot)}(\Omega)} \frac{c(u, \mu)}{\|u\|_X\|\mu\|_M} \geq \gamma.$$

Proof. We apply Proposition 2.4 from [11, p. 60]. Our aim is to show that $\forall \mu \in M$, there exists $u_\mu \in X$ such that

$$c(u_\mu, \mu) = \|\mu\|_M^{p(\cdot)} \ \text{ and } \ \|u_\mu\|_X \leq \frac{1}{\gamma}\, \|\mu\|_M.$$

It suffices to find a mapping $\mu \longrightarrow u_\mu$ from $W_0^{2,p(\cdot)}(\Omega)$ to $W_\Psi^{2,p(\cdot)}(\Omega)$ such that

$$(\nabla u_\mu, \nabla \mu) = \|\triangle u_\mu\|^{p(\cdot)}_{p(\cdot)} \ \text{ and } \ \|u_\mu\|_X \leq \frac{1}{\gamma}\, \|\mu\|_M.$$

We can see that with a choice of $u_\mu = |\triangle\mu|^{p(x)-2}\triangle\mu$ we arrive at the desired result. □

5 Discretization

We consider a triangulation Υ_h made of triangles T whose edges are denoted by e. We assume that the intersection of two different elements is either empty, or a vertex, or a whole edge e, and we also assume that this triangulation is regular in Ciarlet sense, i.e.,

$$\exists\, \sigma > 0; \ \ \frac{h_T}{\rho_T} \le \sigma, \ \ \forall\, T \in \Upsilon_h,$$

where h_T is the diameter of T and ρ_T is the diameter of its largest inscribed bull. We define $h = \max_{T\in\Upsilon_h} h_T$. The jump operator for function v across an edge/face at the point x is given by

$$[v(x)]_e = \begin{cases} \lim\limits_{\alpha\to 0^+} v(x+\alpha\eta_e) - v(x+\alpha\eta_e) & \text{if } \ e \in \zeta_h^{int}, \\ v(x) & \text{if } \ e \in \zeta_h - \zeta_h^{int}, \end{cases}$$

where ζ_h^{int} is the set of interior edges/faces. Let us define the broken Laplace operator

$$(\triangle_h v)_{\backslash T} := \triangle(v_{\backslash T}), \ \ \forall\, T \in \Upsilon_h.$$

For $h > 0$, we introduce the following spaces:

$$X^h = \big\{\phi \in C^0(\overline{\Omega}); \ \ \phi_{\backslash T} \in P^k(T), \ \ \forall\, T \in \Upsilon_h\big\},$$
$$X^h_\Psi = \big\{\phi \in X^h; \ \ \phi_{\backslash\partial\Omega} = \Pi\Psi\big\}$$

and the Ritz projection operator Π defined as follows:

$$\int_\Omega \nabla(\Pi v)\nabla\phi\, dx = \int_\Omega \nabla v\nabla\phi\, dx, \ \ \forall\, \phi \in X^h \cap H^1_0(\Omega).$$

Lemma 5.1. *Let $u \in W^{m+1,q(\cdot)}(\Omega)$, then for $m \ge 2$, we have*

$$\|u - \Pi u\|_{L^{q(\cdot)}(\Omega)} + \|h(\nabla u - \nabla(\Pi u))\|_{L^{q(\cdot)}(\Omega)} + \Big(\sum_{T\in\Upsilon} \|h^2(\triangle u - \triangle(\Pi u))\|^{q(x)}_{L^{q(\cdot)}(T)}\Big)^{\frac{1}{q(x)}} \le Ch^{m+1}|u|_{m+1,q(\cdot)}.$$

Proof. See [10]. □

The discrete formulation of (4.2) is to seek a solution $(u_h, \varphi_h) \in X^h_\Psi \times X^h$ such that

$$\begin{cases} a(\varphi_h, v) + c_h(u_h, v) = 0, \\ c_h(\varphi_h, \mu) = \int_\Omega f\mu, \ \ \forall\, (v,\mu) \in X^h \times X^h_0, \end{cases} \tag{5.1}$$

where c_h is given by

$$c_h(\varphi_h, \mu) = \sum_{T\in\Upsilon_h} \int_T \nabla\varphi_h\nabla\mu\, dx - \int_{\partial\Omega} \nabla\Psi\cdot\eta\mu\, dx = \int_\Omega \nabla\varphi_h\nabla\mu\, dx - \int_{\partial\Omega} \nabla\Psi\cdot\eta\mu\, dx. \tag{5.2}$$

Substituting (5.2) into (5.1), the discrete problem consists in finding $(u_h, \varphi_h) \in X^h_\Psi \times X^h$ satisfying for $(v,\mu) \in X^h \times X^h_0$

$$\int_\Omega |\varphi_h|^{q(x)-2}\varphi_h v\, dx + \int_\Omega \nabla u_h\nabla v\, dx = \int_{\partial\Omega} \nabla\Psi\cdot\eta v\, dx \int_\Omega \nabla\varphi_h\nabla\mu\, dx = \int_\Omega f\mu\, dx.$$

Denote $e_\varphi = \varphi - \varphi_h$ and $e_u = u - u_h$. Now, we are able to announce the following error estimate theorem.

Theorem 5.1. *There exists a constant C such that for $m \geq 2$, we have*

$$\|e_\varphi\|_{L^{q^-}(\Omega)} + \|e_u\|^{p^- -1}_{W_h^{2,p^-}(\Omega)} \leq C\Big(h^{\frac{q(x)}{2}(m+1)}|\varphi|^{\frac{q(\cdot)}{2}}_{m+1,q(\cdot)} + h^{m+1}|\varphi|_{m+1,q(\cdot)} + h^{m-1}|u|_{m+1,p(\cdot)} + h^{m-1}|u|_{m+1,p(\cdot)}\Big),$$

where $(u,\varphi) \in W_\Psi^{m+1,p(\cdot)}(\Omega) \times W^{m+1,q(\cdot)}(\Omega)$ is the exact solution of (4.2) and $(u_h, \varphi_h) \in X_\Psi^h \times X^h$ is the approximate solution of (5.1).

Proof. It is clear that

$$\|u - u_h\|_{W_h^{2,p(\cdot)}(\Omega)} \leq \|u_h - \Pi u\|_{W_h^{2,p(\cdot)}(\Omega)} + \|\Pi u - u\|_{W_h^{2,p(\cdot)}(\Omega)}. \tag{5.3}$$

Using the discrete form of inf-sup condition and Galerkin orthogonality properties, we get

$$\|u_h - \Pi u\|_{W_h^{2,p(\cdot)}(\Omega)} \leq \sup_{\mu \in X_0^h(\Omega), \mu \neq 0} \frac{c_h(u - \Pi u_h, \mu)}{\|\mu\|_{L_h^{q(\cdot)}(\Omega)}} = \sup_{\mu \in X_0^h(\Omega), \mu \neq 0} \frac{a(\varphi_h, \mu) - a(\varphi_h, \mu)}{\|\mu\|_{L_h^{q(\cdot)}(\Omega)}}.$$

In view of the properties of $a(\cdot\,,\cdot)$ (see [1, Proposition 3.1]) we can write

$$\sup_{\mu \in X_0^h(\Omega), \mu \neq 0} \frac{a(\varphi_h, \mu) - a(\varphi_h, \mu)}{\|\mu\|_{L_h^{q(\cdot)}(\Omega)}} \leq C\,\frac{\big(\int_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\,|\varphi - \varphi_h|\,dx\big)^{\frac{1}{p(x)}}\|\mu\|_{L^{q(\cdot)}(\Omega)}}{\|\mu\|_{L_h^{q(\cdot)}(\Omega)}}$$
$$\leq C\bigg(\int_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\,|\varphi - \varphi_h|\,dx\bigg)^{\frac{1}{p(x)}}$$

and

$$\int_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\,|\varphi - \varphi_h|\,dx \leq C\bigg(\int_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\,|\varphi - \varphi_h|\,dx\bigg)^{\frac{1}{p(x)}}\|\varphi - \varphi_h\|_{L^{q(\cdot)}(\Omega)}. \tag{5.4}$$

By the ϵ-Young inequality, we obtain that the right-hand side of (5.4) is estimated by

$$\frac{C^{q(x)}}{q(x)\epsilon^{q(x)}}\,\|\varphi - \varphi_h\|^{q(x)}_{L^{q(\cdot)}(\Omega)} + \frac{\epsilon^{p(x)}}{p(x)}\int_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\,dx. \tag{5.5}$$

Choosing ϵ such that $\frac{\epsilon^{p(x)}}{p(x)} \prec 1$ (for example, we can choose $\epsilon = (\frac{p(x)}{3})^{\frac{1}{p(x)}}$), we find that

$$\int_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\,|\varphi - \varphi_h|\,dx \leq C\|\varphi - \varphi_h\|^{q(x)}_{L^{q(\cdot)}(\Omega)}.$$

So, we get

$$\|u_h - \Pi u\|_{W_h^{2,p(\cdot)}(\Omega)} \leq C\|\varphi - \varphi_h\|^{\frac{q(x)}{p(x)}}_{L^{q(\cdot)}(\Omega)} \leq C\|\varphi - \varphi_h\|^{2\,\frac{q(x)-1}{2}}_{L^{q(\cdot)}(\Omega)}. \tag{5.6}$$

On the other hand, a simple calculation gives

$$a(\varphi, \varphi - \varphi_h) - a(\varphi_h, \varphi - \varphi_h) = a(\varphi, \varphi - v) - a(\varphi_h, \varphi - v) + a(\varphi, v - \varphi_h) - a(\varphi_h, v - \varphi_h). \tag{5.7}$$

Subtracting (5.1) from (4.2), we get

$$\begin{cases} a(\varphi, v) - a(\varphi_h, v) + c_h(u - u_h, v) = 0, & \forall\, v \in X^h, \\ c_h(\varphi - \varphi_h, \mu) = 0, & \forall\, \mu \in X_0^h. \end{cases}$$

This allows us to rewrite (5.7) as follows:

$$a(\varphi,\varphi-\varphi_h)-a(\varphi_h,\varphi-\varphi_h)=a(\varphi,\varphi-v)-a(\varphi_h,\varphi-v)+c_h(u-u_h,\varphi_h-v)=J_1+J_2,$$

where

$$J_1=a(\varphi,\varphi-v)-a(\varphi_h,\varphi-v)\ \text{ and }\ J_2=c_h(u-u_h,\varphi_h-v).$$

Now, using the properties of $a(\,\cdot\,,\,\cdot\,)$ (see [1, Proposition 3.1]) once more, the ϵ-Young inequality shows that

$$\frac{C_1}{2}\,\frac{\|\varphi-\varphi_h\|^2_{L^{q(\cdot)}(\Omega)}}{\|\varphi\|^{2-q(x)}_{L^{q(\cdot)}(\Omega)}+\|\varphi_h\|^{2-q(x)}_{L^{q(\cdot)}(\Omega)}}+\frac{C_2}{2}\int\limits_\Omega\big||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\big|\,|\varphi-\varphi_h|\,dx$$
$$\le a(\varphi,\varphi-\varphi_h)-a(\varphi_h,\varphi-\varphi_h)=J_1+J_2 \tag{5.8}$$

and

$$\begin{aligned}J_1&\le C_3\Big(\int\limits_\Omega\big||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\big|\,|\varphi-\varphi_h|\,dx\Big)^{\frac{1}{p(x)}}\|\varphi-v\|_{L^{q(\cdot)}(\Omega)}\\&\le\frac{C_3^{q(x)}}{\epsilon^{q(x)}q(x)}\,\|\varphi-v\|^{q(x)}_{L^{q(\cdot)}(\Omega)}+\frac{\epsilon^{p(x)}}{p(x)}\int\limits_\Omega\big||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\big|\,|\varphi-\varphi_h|\,dx,\end{aligned}$$

where C_1, C_2 and C_3 are the same constants as in Proposition 3.1 of [13]. If we choose ϵ such that $\frac{\epsilon^{p(x)}}{p(x)}=\frac{C_2}{2}$, we arrive at

$$J_1\le C(q^-,q^+)\|\varphi-v\|^{q(x)}_{L^{q(\cdot)}(\Omega)}+\frac{C}{2}\int\limits_\Omega\big||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\big|\,|\varphi-\varphi_h|\,dx. \tag{5.9}$$

Moreover,

$$J_2=c_h(u-u_h,\varphi_h-v)=c_h(u-\Pi u,\varphi_h-v), \tag{5.10}$$

in view of

$$c_h(\mu,\varphi_h-v)=0,\ \ \forall\,\mu\in X_0^h.$$

The continuity of c_h implies that

$$\begin{aligned}J_2&=c_h(u-\Pi u,\varphi_h-v)\le C\|u-\Pi u\|_{W_h^{2,p(x)}(\Omega)}\|\varphi_h-v\|_{L^{q(\cdot)}(\Omega)}\\&\le\frac{C\varepsilon^2}{2}\,\|\varphi_h-v\|^2_{L^{q(\cdot)}(\Omega)}+\frac{C}{2\epsilon^2}\,\|u-\Pi u\|^2_{W_h^{2,p(x)}(\Omega)}\\&\le\frac{C}{2\epsilon^2}\,\|u-\Pi u\|^2_{W_h^{2,p(x)}(\Omega)}+\frac{C\varepsilon^2}{2}\,\big(\|\varphi_h-\varphi\|_{L^{q(\cdot)}(\Omega)}+\|\varphi-v\|_{L^{q(\cdot)}(\Omega)}\big)^2\\&\le\frac{C}{2\epsilon^2}\,\|u-\Pi u\|^2_{W_h^{2,p(x)}(\Omega)}\\&\qquad+\frac{C\varepsilon^2}{2}\,\big(\|\varphi_h-\varphi\|^2_{L^{q(\cdot)}(\Omega)}+\|\varphi-v\|^2_{L^{q(\cdot)}(\Omega)}+2\|\varphi_h-\varphi\|_{L^{q(\cdot)}(\Omega)}\|\varphi-v\|_{L^{q(\cdot)}(\Omega)}\big)\\&\le\frac{C}{2\epsilon^2}\,\|u-\Pi u\|^2_{W_h^{2,p(x)}(\Omega)}+C\varepsilon^2\big(\|\varphi_h-\varphi\|^2_{L^{q(\cdot)}(\Omega)}+\|\varphi-v\|^2_{L^{q(\cdot)}(\Omega)}\big).\end{aligned} \tag{5.11}$$

Gathering estimates (5.9)–(5.11), substituting in (5.8) and taking ϵ sufficiently small, we obtain

$$\|\varphi_h-\varphi\|^2_{L^{q(\cdot)}(\Omega)}\le C\|\varphi-v\|^{q(x)}_{L^{q(\cdot)}(\Omega)}+C\|u-\Pi u\|^2_{W_h^{2,p(x)}(\Omega)}+C\|\varphi-v\|^2_{L^{q(\cdot)}(\Omega)}. \tag{5.12}$$

Using the properties of Π, we obtain the estimate of e_φ. Now, substituting (5.12) into (5.6), taking into account (5.3), Lemma 5.1 and the continuous embedding of $L^{q(\cdot)}(\Omega)$ into $L^{q^-}(\Omega)$, we arrive at the desired estimate for e_u. Thus the proof is completed. □

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