Memoirs on Differential Equations and Mathematical Physics

Volume 92, 2024, 69–90

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MIXED TYPE TRANSMISSION DYNAMICAL PROBLEMS WITH INTERIOR CRACKS OF THE THERMO-PIEZOELECTRICITY THEORY WITHOUT ENERGY DISSIPATION

Abstract. In the paper, we study a mixed type interaction dynamical problem with interior cracks between thermo-elastic and thermo-piezoelastic bodies. The model under consideration is based on the Green–Naghdi theory of thermo-piezoelectricity without energy dissipation. This theory allows the thermal waves to propagate only with a finite speed. Using the Laplace transform, potential theory and the method of boundary pseudodifferential equations, we prove the existence and uniqueness of solutions and analyze their smoothness.

2020 Mathematics Subject Classification. 35B65, 35S15, 47G30, 74A15, 74F05, 74F15, 74G40.

Key words and phrases. Thermo-piezoelectricity without energy dissipation, bodies with microstructure, transmission problem, potential method, pseudodifferential equations.

რეზიუმე. ნაშრომში შესწავლილია შერეული ტიპის თერმო-დრეკადი და თერმო-ელექტროდრეკადი სხეულების ურთიერთქმედების დინამიკური ტრანსმისიის ამოცანა შიგა ბზარებით. განხილული თერმო-ელექტროდრეკადი მოდელი ეფუძნება გრინ-ნახდის თეორიას ენერგიის დისიპაციის გარეშე. ამ თეორიაში დასაშვებია თერმული ტალღების გავრცელება სასრული სიჩქარით. ლაპლასის გარდაქმნის, პოტენციალთა და სასაზღვრო ფსევდოდიფერენციალურ განტოლებათა მეთოდის გამოყენებით მტკიცდება ამოცანის ამონახსნთა არსებობისა და ერთადერთობის თეორემები და შესწავლილია მათი სიგლუვე.

1 Introduction

In this paper, we investigate the transmission dynamical problem, i.e., the mixed type interaction dynamical problem with interior cracks between thermo-elastic and thermo-piezoelastic bodies. The model under consideration is based on the Green–Naghdi theory of thermo-piezoelectricity without energy dissipation. This theory allows the thermal waves to propagate only with a finite speed.

Other models of thermo-piezoelectricity, in particular, Foigt and Mindlin's model are well known. Our model is refined, it takes into account microrotation and microstretch of a particle.

Almost complete historical and bibliographical notes in this direction can be found in [23], where the dynamical equations of the thermo-piezoelectricity without energy dissipation are derived on the basis of the Green–Naghdi theory established in [21, 22] and Eringen's results obtained in [19, 20]. In the present paper, we consider the pseudo-oscillation equations obtained by the Laplace transform from the dynamical equations derived by Ieşan in [23] for homogeneous isotropic solids possessing thermo-piezoelectricity properties without energy dissipation. In [5], it is studied the mixed and crack type pseudo-oscillation problem of thermo-piezoelectricity without energy dissipation.

The basic dynamical problems of the classical elasticity and thermo-elasticity with either the Dirichlet or Neumann type boundary conditions on the whole boundary were developed in [24]. The mixed type dynamical problems of the classical elasticity for anisotropic bodies were studied in [25]. The mixed and crack type dynamical problems of the electro-magneto-elasticity were studied in [6] and the mixed boundary-transmission dynamical problems of generalized thermo-electro-magneto-elasticity theory for piecewise homogeneous composed structures were studied in [8].

In [16], a three-dimensional fluid-solid dynamical interaction problem is considered, when an anisotropic elastic body occupying a bounded region is immersed into an inviscid fluid occupying an unbounded domain. In the solid region, it is considered the generalized Green–Lindsay's model of the thermo-electro-magneto-elasticity theory. In this direction, see [9–15].

Using the Laplace transform, potential theory and the method of boundary pseudodifferential equations, we prove the existence and uniqueness theorems of solutions in the appropriate function spaces. Further, we the analyze regularity of solutions of a mixed type dynamical transmission problem with interior cracks near the exceptional curve, where different type boundary conditions collide near the crack edges. The regularity of solutions near the crack edges is $C^m([0,\infty), C^{\frac{1}{2}})$ and for the temperature of elasticity body is $C^m([0,\infty), C^{\frac{3}{2}})$, $m \geq 2$ (for the definition of these classes, see Section 3 of this paper). The regularity of solutions near the curve, where different type boundary conditions meet, depends on the material constants and does not depend on the geometry of the exceptional curve. If these constants meet certain conditions, then the smoothness of solutions is $C^m([0,\infty), C^{\frac{1}{2}})$, $m \geq 2$ (cf. [1–5, 7, 17]).

The Dirichlet, Neumann and mixed type transmission pseudo-oscillation problems of thermo-piezoelectricity without energy dissipation are studied in [17], and the mixed type transmission pseudo-oscillation problem with interior cracks of thermo-piezoelectricity without energy dissipation is studied in [18].

2 Thermo-elastic field equations and thermo-piezoelastic field equations without energy dissipation

The model under consideration is based on the Green–Naghdi theory of thermo-piezoelectricity without energy dissipation.

Consider disjoint bounded domains Ω_1 and Ω_2 in the Euclidean space $\mathbb{R}^3$ with $C\infty$-smooth boundaries $\partial\Omega_1 = S_1$ and $\partial\Omega_2 = S_1 \cup S_2$ ($S_1 \cap S_2 = \varnothing$). $S_2 = \overline{S}_2^{(D)} \cup \overline{S}_2^{(N)}$, $S_2 = \overline{S}_2^{(D)} \cap \overline{S}_2^{(N)} = \varnothing$, $\ell = \partial S_2^{(D)} = \partial S_2^{(N)} \in C^\infty$. We assume that the solids under consideration contain interior cracks. We identify the crack surfaces as two-dimensional, two-sided manifolds Σ_k, $k = 1, 2$, with the crack edges $\ell_c^{(k)} := \partial\Sigma_k$, $k = 1, 2$. We assume that Σ_k, $k = 1, 2$, are the proper parts of closed surfaces $S_0^{(k)} \subset \Omega_k$, $k = 1, 2$, surrounding domains $\overline{\Omega}_0^{(k)} \subset \Omega_k$ and that Σ_k and $\ell_c^{(k)}$, $k = 1, 2$, are C^∞-smooth. Denote $\Omega_{\Sigma_k} := \Omega_k \setminus \overline{\Sigma}_k$, $k = 1, 2$.

Throughout the paper, $n=(n_1,n_2,n_3)$ stands for the exterior unit normal vector to $\partial\Omega_1=S_1$ and $S_0^{(1)}=\partial\Omega_0^{(1)}$. A vector $\nu=(\nu_1,\nu_2,\nu_3)$ is exterior unit normal vector to $\partial\Omega_2=S_1\cup S_2$ and $S_0^{(2)}=\partial\Omega_0^{(2)}$.

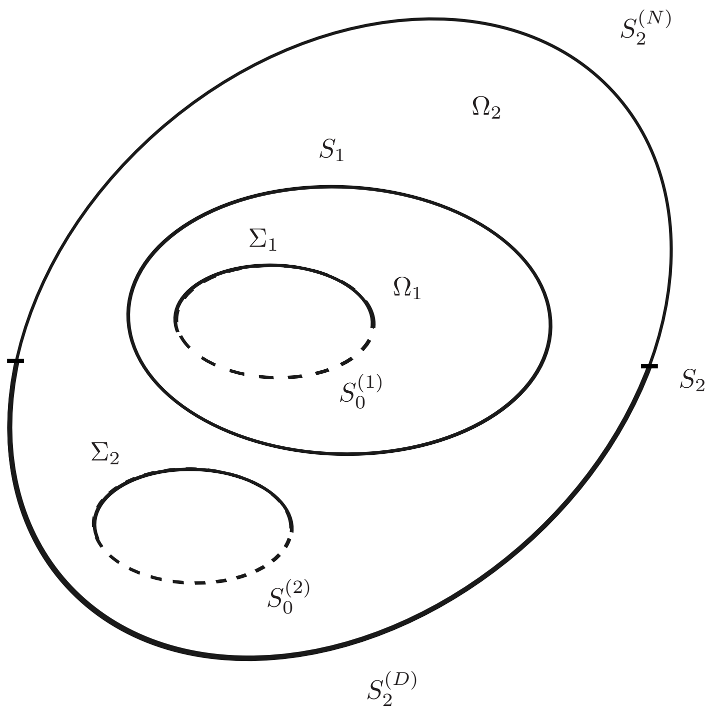

Figure 1

Suppose the domain Ω_1 is filled with a homogeneous thermo-elastic material, then the system of governing differential equations of dynamics with respect to the sought vector function $U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top$, where $u^{(1)}=(u_1^{(1)},u_2^{(1)},u_3^{(1)})^\top$ is the displacement vector and $\vartheta^{(1)}$ is the temperature, has the following form (see [24]):

$$(\mu^{(1)}+\varkappa^{(1)})\Delta u^{(1)}+(\lambda^{(1)}+\mu^{(1)})\operatorname{grad}\operatorname{div}u^{(1)}-\rho_1\partial_t^2u^{(1)}-\beta_0^{(1)}\operatorname{grad}\partial_t\vartheta^{(1)}=(F_1^{(1)},F_2^{(1)},F_3^{(1)})^\top, \tag{2.1}$$

$$k^{(1)}\Delta\vartheta^{(1)}-a^{(1)}\partial_t^2\vartheta^{(1)}-\beta_0^{(1)}\partial_t\operatorname{div}u^{(1)}=F_4^{(1)}, \tag{2.2}$$

where $(F_1^{(1)},F_2^{(1)},F_3^{(1)})^\top$ is a mass force density, $F_4^{(1)}$ is a heat source density, ρ_1 is the mass density, $\mu^{(1)}$, $\varkappa^{(1)}$, $\lambda^{(1)}$, $\beta_0^{(1)}$, $k^{(1)}$ and $a^{(1)}$ are the thermo-elastic constants satisfying the conditions

$$\varkappa^{(1)}>0,\ \ \varkappa^{(1)}+2\mu^{(1)}>0,\ \ \varkappa^{(1)}+2\mu^{(1)}+3\lambda^{(1)}>0,\ \ k^{(1)}>0,\ \ \rho_1>0,\ \ a^{(1)}>0,$$
$$\beta_0^{(1)}>0,\ \ \tau=\sigma+i\omega,\ \ \sigma>\sigma_0>0,\ \ \omega\in\mathbb{R}.$$

The stress operator for a homogeneous isotropic system of equations is defined as follows:

$$\begin{aligned}T^{(1)}=T^{(1)}(\partial_x,n,\partial_t)&=\big[T_{ij}^{(1)}(\partial_x,n,\partial_t)\big]_{4\times4}\\ &:=\begin{bmatrix}\big[\lambda^{(1)}n_i\partial_j+\mu^{(1)}n_j\partial_i+\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})n_k\partial_k\big]_{3\times3}, & \big[-\beta_0^{(1)}n\,\partial_t\big]_{3\times1}\\ [0]_{1\times3} & k^{(1)}n_l\partial_l\end{bmatrix}_{4\times4}.\end{aligned}$$

We can write the above system of equations (2.1), (2.2) for pseudo-oscillations of the theory of homogeneous isotropic thermo-elasticity in the following matrix form:

$$A^{(1)}(\partial_x,\tau)U^{(1)}=F^{(1)},$$

where $U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top$, $F^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)}, F_4^{(1)})^\top$, and $A^{(1)}(\partial x, \partial_t)$ is the 4-dimensional matrix differential operator of the generalized thermo-elasticity:

$$A^{(1)}(\partial_x, \partial_t) = \big[A_{ij}^{(1)}(\partial_x, \partial_t)\big]_{4\times 4}$$
$$:= \begin{bmatrix} \big[\delta_{ij}(\mu^{(1)} + \varkappa^{(1)})\Delta + (\lambda^{(1)} + \mu^{(1)})\partial_i\partial_j - \rho_1\delta_{ij}\,\partial_t^2\big]_{3\times 3}, & -\beta_0^{(1)}\partial_t[\partial_i]_{3\times 1} \\ -\beta_0^{(1)}\partial_t[\partial_j]_{1\times 3} & -a^{(1)}\partial_t^2 + k^{(1)}\Delta \end{bmatrix}_{4\times 4},$$

where δ_{ij} is the Kronecker delta.

The domain Ω_2 is filled with a thermo-electro-elastic material. The corresponding system of differential equations of pseudo-oscillations with respect to the sought vector function $U^{(2)}$ has the following form (see [23]):

$$(\mu^{(2)} + \varkappa^{(2)})\partial_j\partial_j u_i^{(2)} + (\lambda^{(2)} + \mu^{(2)})\partial_i\partial_j u_j^{(2)} - \rho_2\partial_t^2 u_i^{(2)} + \varkappa^{(2)}\varepsilon_{ijk}\partial_j\phi_k^{(2)}$$
$$+\lambda_0^{(2)}\partial_i\varphi^{(2)} - \beta_0^{(2)}\partial_t\partial_i\vartheta^{(2)} = -\rho_2 g_i^{(2)}, \quad i = 1, 2, 3, \tag{2.3}$$
$$k^{(2)}\partial_j\partial_j\vartheta^{(2)} - a^{(2)}\partial_t^2\vartheta^{(2)} - \beta_0^{(2)}\partial_t\partial_j u_j^{(2)} - c_0^{(2)}\partial_t\varphi^{(2)}$$
$$+\nu_1^{(2)}\partial_j\partial_j\varphi^{(2)} - \nu_3^{(2)}\partial_j\partial_j\psi^{(2)} = -\frac{1}{T_0}\,\rho_2 Q^{(2)}, \tag{2.4}$$
$$\gamma^{(2)}\partial_j\partial_j\phi_i^{(2)} + (\alpha^{(2)} + \beta^{(2)})\partial_j\partial_i\phi_j^{(2)} - I_0^{(2)}\partial_t^2\phi_i^{(2)}$$
$$+\varkappa^{(2)}\varepsilon_{ijk}\partial_j u_k^{(2)} - 2\varkappa^{(2)}\phi_i^{(2)} = -\rho_2 X_i^{(2)}, \quad i = 1, 2, 3, \tag{2.5}$$
$$(a_0^{(2)}\partial_j\partial_j - \xi_0^{(2)})\varphi^{(2)} - j_0^{(2)}\partial_t^2\varphi^{(2)} - \lambda_2^{(2)}\partial_j\partial_j\psi^{(2)} + \nu_1^{(2)}\partial_j\partial_j\vartheta^{(2)} + c_0^{(2)}\partial_t\vartheta^{(2)} - \lambda_0^{(2)}\partial_j u_j^{(2)} = -\rho_2 F^{(2)}, \tag{2.6}$$
$$\lambda_0^{(2)}\partial_j\partial_j\varphi^{(2)} + \chi^{(2)}\partial_j\partial_j\psi^{(2)} + \nu_3^{(2)}\partial_j\partial_j\vartheta^{(2)} = -g^{(2)}, \tag{2.7}$$

where $U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \vartheta^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top$, $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})^\top$ is the displacement vector, $\vartheta^{(2)}$ is the temperature, $\phi^{(2)} = (\phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)})^\top$ is the vector of microrotation, $\varphi^{(2)}$ is the microstretch, $\psi^{(2)}$ is the electric field potential, and $(g_1^{(2)}, g_2^{(2)}, g_3^{(2)})$ is the external body force per unit mass, $Q^{(2)}$ is the external rate of supply of heat per unit mass, $X_i^{(2)}$ is the external body couple per unit mass, $F^{(2)}$ is the microstretch body force, $g^{(2)}$ is the density of free charge, T_0 is the initial reference temperature, ε_{ijk} is the Levi–Civita symbol and ρ_2 is the mass density.

Due to the positiveness of internal energy, the coefficients of system (2.3)–(2.7) have to satisfy the following conditions:

$$\varkappa^{(2)} > 0, \;\; \varkappa^{(2)} + 2\mu^{(2)} > 0, \;\; \varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)} > 0,$$
$$\xi_0^{(2)}(\varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)}) > 3(\lambda_0^{(2)})^2, \;\; \gamma^{(2)} > |\beta^{(2)}|, \;\; a_0^{(2)}k^{(2)} - (\nu_1^{(2)})^2 > 0,$$
$$\beta^{(2)} + \gamma^{(2)} + 3\alpha^{(2)} > 0, \;\; \chi^{(2)} > 0, \;\; a^{(2)} > 0, \;\; k^{(2)} > 0, \;\; a_0^{(2)} > 0, \;\; a_0^{(2)}(\gamma^{(2)} - \beta^{(2)}) > 2(b_0^{(2)})^2,$$
$$(\gamma^{(2)} - \beta^{(2)})\big[a_0^{(2)}k^{(2)} - (\nu_1^{(2)})^2\big] + 4b_0^{(2)}\nu_1^{(2)}\nu_2^{(2)} - 2a_0^{(2)}(\nu_2^{(2)})^2 - 2k^{(2)}(b_0^{(2)})^2 > 0,$$
$$\rho_2 > 0, \;\; I_0^{(2)} > 0, \;\; j_0^{(2)} > 0, \;\; \beta_0^{(2)} > 0.$$

Denote by

$$A^{(2)}(\partial_x, \partial_t) = \big[A_{ij}^{(2)}(\partial_x, \partial_t)\big]_{9\times 9}$$

the matrix differential operator generated by the left-hand side expressions in (2.3)–(2.7),

$$A_{ij}^{(2)}(\partial_x, \partial_t) = \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\partial_l\partial_l + (\lambda^{(2)} + \mu^{(2)})\partial_i\partial_j - \rho_2\delta_{ij}\partial_t^2,$$
$$A_{i4}^{(2)}(\partial_x, \partial_t)) = -\beta_0^{(2)}\partial_t\partial_i, \quad A_{i,j+4}^{(2)}(\partial_x, \partial_t) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l, \quad A_{i8}^{(2)}(\partial_x, \partial_t) = \lambda_0^{(2)}\partial_i,$$
$$A_{i9}^{(2)}(\partial_x, \partial_t) = 0, \quad A_{4j}^{(2)}(\partial_x, \partial_t) = -\beta_0^{(2)}\partial_t\partial_j, \quad A_{44}^{(2)}(\partial_x, \partial_t) = k^{(2)}\partial_l\partial_l - a^{(2)}\partial_t^2,$$
$$A_{4,j+4}^{(2)}(\partial_x, \partial_t) = 0, \quad A_{48}^{(2)}(\partial_x, \partial_t) = \nu_1^{(2)}\partial_l\partial_j - c_0^{(2)}\partial_t, \quad A_{49}^{(2)}(\partial_x, \partial_t) = -\nu_3^{(2)}\partial_l\partial_l,$$

$$A^{(2)}_{i+4,j}(\partial_x,\partial_t)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l,\quad A^{(2)}_{i+4,4}(\partial_x,\partial_t)=0,$$
$$A^{(2)}_{i+4,j+4}(\partial_x,\partial_t)=\delta_{ij}\gamma^{(2)}\partial_l\partial_l+(\alpha^{(2)}+\beta^{(2)})\partial_i\partial_j-(2\varkappa^{(2)}+I^{(2)}_0\partial_t^2)\delta_{ij},$$
$$A^{(2)}_{i+4,8}(\partial_x,\partial_t)=0,\quad A^{(2)}_{i+4,9}(\partial_x,\partial_t)=0,\quad A^{(2)}_{8j}(\partial_x,\partial_t)=-\lambda^{(2)}_0\partial_j,$$
$$A^{(2)}_{84}(\partial_x,\partial_t)=\nu^{(2)}_1\partial_l\partial_l+c^{(2)}_0\partial_t,\quad A^{(2)}_{8,j+4}(\partial_x,\partial_t)=0,\quad A^{(2)}_{88}(\partial_x,\partial_t)=a^{(2)}_0\partial_l\partial_l-(\xi^{(2)}_0+j^{(2)}_0\partial_t^2),$$
$$A^{(2)}_{89}(\partial_x,\partial_t)=-\lambda^{(2)}_2\partial_l\partial_l,\quad A^{(2)}_{9j}(\partial_x,\partial_t)=0,\quad A^{(2)}_{94}(\partial_x,\partial_t)=\nu^{(2)}_3\partial_l\partial_l,$$
$$A^{(2)}_{9,j+4}(\partial_x,\partial_t)=0,\quad A^{(2)}_{98}(\partial_x,\partial_t)=\lambda^{(2)}_2\partial_l\partial_l,\quad A^{(2)}_{99}(\partial_x,\partial_t)=\chi^{(2)}\partial_l\partial_l,\ \ i,j=1,2,3.$$

The stress differential operator of thermo-electro-elasticity is defined as follows:

$$T^{(2)}=T^{(2)}(\partial_x,\nu,\partial_t):=\big[T^{(2)}_{ij}(\partial_x,\nu,\partial_t)\big]_{9\times 9},$$

where

$$T^{(2)}_{ij}(\partial_x,\nu,\partial_t)=\lambda^{(2)}\nu_i\partial_j+\mu^{(2)}\nu_j\partial_i+\delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\nu_k\partial_k,\quad T^{(2)}_{i4}(\partial_x,\nu,\partial_t)=-\beta^{(2)}_0\nu_i\partial_t,$$
$$T^{(2)}_{i,j+4}(\partial_x,\nu,\partial_t)=-\varkappa^{(2)}\varepsilon_{ijk}\nu_k,\quad T^{(2)}_{i8}(\partial_x,\nu,\partial_t)=\lambda^{(2)}_0\nu_i,\quad T^{(2)}_{i9}(\partial_x,\nu,\partial_t)=0,$$
$$T^{(2)}_{4,j}(\partial_x,\nu,\partial_t)=0,\quad T^{(2)}_{44}(\partial_x,\nu,\partial_t)=k^{(2)}\nu_l\partial_l,\quad T^{(2)}_{4,j+4}(\partial_x,\nu,\partial_t)=-\nu^{(2)}_2\varepsilon_{ljk}\nu_l\partial_k,$$
$$T^{(2)}_{48}(\partial_x,\nu,\partial_t)=\nu^{(2)}_1\nu_k\partial_k,\quad T^{(2)}_{49}(\partial_x,\nu,\partial_t)=-\nu^{(2)}_3\nu_k\partial_k,\quad T^{(2)}_{i+4,j}(\partial_x,\nu,\partial_t)=0,$$
$$T^{(2)}_{i+4,4}(\partial_x,\nu,\partial_t)=\nu^{(2)}_2\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{i+4,j+4}(\partial_x,\nu,\partial_t)=\alpha^{(2)}\nu_i\partial_j+\beta^{(2)}\nu_j\partial_i+\delta_{ij}\gamma^{(2)}\nu_k\partial_k,$$
$$T^{(2)}_{i+4,8}(\partial_x,\nu,\partial_t)=b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{i+4,9}(\partial_x,\nu,\partial_t)=\lambda^{(2)}_1\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{8j}(\partial x,\nu,\partial_t)=0,$$
$$T^{(2)}_{84}(\partial_x,\nu,\partial_t)=\nu^{(2)}_1\nu_k\partial_k,\quad T^{(2)}_{8,j+4}(\partial_x,\nu,\partial_t)=-b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{88}(\partial_x,\nu,\partial_t)=a^{(2)}_0\nu_k\partial_k,$$
$$T^{(2)}_{89}(\partial_x,\nu,\partial_t)=-\lambda^{(2)}_2\nu_k\partial_k,\quad T^{(2)}_{9j}(\partial x,\nu,\partial_t)=0,\quad T^{(2)}_{94}(\partial_x,\nu,\partial_t)=\nu^{(2)}_3\nu_k\partial_k,$$
$$T^{(2)}_{9,j+4}(\partial_x,\nu,\partial_t)=\lambda^{(2)}_1\varepsilon_{ljk}\nu_l\partial_k,\quad T^{(2)}_{98}(\partial_x,\nu,\partial_t)=\lambda^{(2)}_2\nu_k\partial_k,$$
$$T^{(2)}_{99}(\partial_x,\nu,\partial_t)=\chi^{(2)}\nu_k\partial_k,\ \ i,j=1,2,3.$$

The system of equations (2.3)–(2.7) can be written in a matrix form

$$A^{(2)}(\partial_x,\partial_t)U^{(2)}=\Phi,$$

where

$$U^{(2)}=\Big(u^{(2)}_1,u^{(2)}_2,u^{(2)}_3,\vartheta^{(2)},\phi^{(2)}_1,\phi^{(2)}_2,\phi^{(2)}_3,\varphi^{(2)},\psi^{(2)}\Big)^\top,$$
$$\Phi=-\Big(\rho_2g^{(2)}_1,\rho_2g^{(2)}_2,\rho_2g^{(2)}_3,\frac{1}{T_0}\,\rho_2Q^{(2)},\rho_2X^{(2)}_1,\rho_2X^{(2)}_2,\rho_2X^{(2)}_3,\rho_2F^{(2)},g^{(2)}\Big)^\top$$

and $A^{(2)}(\partial_x,\tau)$ is the 9-dimensional matrix differential operator corresponding to system (2.3)–(2.7).

3 Formulation of the mixed type dynamical transmission problem

3.1 Formulation of the mixed type transmission dynamical problem with interior cracks $(TM)_{c,t}$

We are looking for a solution

$$U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top=(u^{(1)},u^{(1)}_4)^\top,$$
$$U^{(2)}=\big(u^{(2)},\vartheta^{(2)},\phi^{(2)},\varphi^{(2)},\psi^{(2)}\big)^\top=\big(u^{(2)},u^{(2)}_4,u^{(2)}_5,\dots,u^{(2)}_9\big)^\top$$

of the dynamical equations

$$A^{(1)}(\partial_x,\partial_t)U^{(1)}=\Phi_1 \ \text{ in } \Omega_{\Sigma_1}\times[0,\infty),$$
$$A^{(2)}(\partial_x,\partial_t)U^{(2)}=\Phi_2 \ \text{ in } \Omega_{\Sigma_2}\times[0,\infty),$$

which satisfy on the surface S_1 the following transmission conditions:

$$\{u_j^{(1)}\}^+-\{u_j^{(2)}\}^+=f_j^{(1)} \ \text{ on } S_1\times[0,\infty),\ \ j=\overline{1,4},$$
$$\big\{T^{(1)}(\partial_x,n,\partial_t)U^{(1)}\big\}_j^+ + \{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\}_j^+=f_j^{(2)} \ \text{ on } S_1\times[0,\infty),\ \ j=\overline{1,4},\ \ \nu=-n,$$

and the boundary conditions

$$\{u_j^{(2)}\}^+=Q_j^{(2)} \ \text{ on } S_1\times[0,\infty),\ \ j=\overline{5,9},$$

while on the surface S_2, the mixed boundary conditions

$$\{U^{(2)}\}^+=p_2^{(D)} \ \text{ on } S_2^{(D)}\times[0,\infty),$$
$$\big\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\big\}^+=q_2^{(N)} \ \text{ on } S_2^{(N)}\times[0,\infty),$$

the crack boundary conditions on Σ_1

$$\big\{T^{(1)}(\partial_x,n,\partial_t)U^{(1)}\big\}_j^{\pm}=F_j^{(1),\pm} \ \text{ on } \Sigma_1\times[0,\infty),\ \ j=1,2,3,$$
$$\{u_4^{(1)}\}^+-\{u_4^{(1)}\}^+=G_4^{(1)} \ \text{ on } \Sigma_1\times[0,\infty),$$
$$\big\{T^{(1)}(\partial_x,n,\partial_t)U^{(1)}\big\}_4^+ - \big\{T^{(1)}(\partial_x,n,\partial_t)U^{(1)}\big\}_4^- = F_4^{(1)} \ \text{ on } \Sigma_1\times[0,\infty),$$

the crack boundary conditions on Σ_2

$$\big\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\big\}_j^{\pm}=F_j^{(2),\pm} \ \text{ on } \Sigma_2\times[0,\infty),\ \ j=1,2,3,5,6,7,$$
$$\{u_j^{(2)}\}^+-\{u_j^{(2)}\}^+=G_j^{(2)} \ \text{ on } \Sigma_2\times[0,\infty),\ \ j=4,8,9,$$
$$\big\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\big\}_j^+ - \big\{T^{2)}(\partial_x,\nu,\partial_t)U^{(2)}\big\}_4^- = F_j^{(2)} \ \text{ on } \Sigma_1\times[0,\infty),\ \ j=4,8,9,$$

and the initial conditions

$$u_j^{(1)}(x,0)=0,\quad \partial_t\, u_j^{(1)}(x,0)=0,\ \ x\in\Omega_1,\ \ j=\overline{1,4},$$
$$u_j^{(2)}(x,0)=0,\quad \partial_t\, u_j^{(2)}(x,0)=0,\ \ x\in\Omega_2,\ \ j=\overline{1,8}.$$

where

$$S_2=\overline{S}_2^{(D)}\cup\overline{S}_2^{(N)},\quad S_2^{(D)}\cap S_2^{(N)}=\varnothing,\ \ \ell_m=\partial S_2^{(D)}=\partial S_2^{(N)}\in C^\infty.$$

Remark 3.1. Taking into account the homogeneous initial conditions of the mixed type transmission dynamical problem $(MT)_t$, from 9-th equation of the basic dynamical system of equations, when $t=0$ and the boundary conditions, we can find the function $\psi^{(2)}(x,0)$ for $x\in\Omega_2$. Note that when formulating the mixed type transmission dynamical problem $(MT)_t$, we can consider the homogeneous initial conditions without loss of generality (see [3, 16]).

By H^s with $s\in\mathbb{R}$ we denote the Sobolev–Slobodetsky space. Let $\mathcal{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathcal{M}\subset\mathcal{M}_0$, we denote by $\widetilde{H}^s(\mathcal{M})$ the subspace of $H^s(\mathcal{M}_0)$,

$$\widetilde{H}^s(\mathcal{M})=\big\{g:\ \ g\in H^s(\mathcal{M}_0),\ \ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$

while $H^s(\mathcal{M})$ stand for the space of restriction on $\mathcal{M}$ of the functions from $H^s(\mathcal{M}_0)$.

Let $\mathbb{B}$ be some Banach space and let $a>0$ and $m\in\mathbb{N}\cup 0$.

Definition 3.1. By $C_a^m([0,\infty),\mathbb{B})$ we denote the set of all $\mathbb{B}$ valued functions which are m-times continuously differentiable on $[0,\infty)$ and satisfy the conditions

$$\frac{\partial^l u(0)}{\partial t^l}=0,\ \ l=0,\dots,m,\quad \Big\|\frac{\partial^l u(t)}{\partial t^l}\Big\|_{\mathbb{B}}=O(e^{\alpha t}),\ \ \forall\alpha>a>0,\ \ l=0,\dots,m.$$

Definition 3.2. By $C_{0,a}^m([0,\infty),\mathbb{B})$ we denote the set of all $\mathbb{B}$ valued functions which are m-times continuously differentiable on $[0,\infty)$ and the satisfy the conditions

$$\frac{\partial^l u(0)}{\partial t^l}=0,\ \ l=0,\dots,m-2,\quad \Big\|\frac{\partial^l u(t)}{\partial t^l}\Big\|_{\mathbb{B}}=O(e^{at}),\ \ l=0,\dots,m.$$

We study the solvability of the above formulated transmission dynamical problem in the spaces

$$C_a^m([0,\infty),[H^1(\Omega_1)]^4)\times C_a^m([0,\infty),[H^1(\Omega_2)]^9)\ \text{ with }\ m\geq 2\ \text{ and }\ a>0.$$

assuming that

$$\begin{gathered}
\Phi_1\in C_{0,a}^M([0,\infty),[L_2(\Omega_{\Sigma_1})]^4),\quad \Phi_1\in C_{0,a}^M([0,\infty),[L_2(\Omega_{\Sigma_2})]^9),\\
f_j^{(1)}\in C_{0,a}^{M+2}([0,\infty),H^{\frac12}(S_1)),\quad f_j^{(2)}\in C_{0,a}^{M+2}([0,\infty),H^{-\frac12}(S_1)),\ \ j=\overline{1,4},\\
Q_j^{(2)}\in C_{0,a}^{M+2}([0,\infty),H^{\frac12}(S_1)),\ \ j=\overline{5,9},\\
p^{(2)}\in C_{0,a}^{M+2}([0,\infty),[H^{\frac12}(S_1)]^9),\quad q^{(2)}\in_{0,a}^{M+2}([0,\infty),[H^{-\frac12}(S_1)]^9),\\
p_2^{(D)}\in C_{0,a}^{M+2}([0,\infty),[H^{\frac12}(S_2^{(D)})]^9),\quad q_2^{(N)}\in C_{0,a}^{M+2}([0,\infty),[H^{-\frac12}(S_2^{(N)}]^9),\\
F_j^{(1),\pm}\in C_{0,a}^{M+2}([0,\infty),H^{-\frac12}(\Sigma_1)),\ \ j=1,2,3,\\
F_j^{(2),\pm}\in C_{0,a}^{M+2}([0,\infty),H^{-\frac12}(\Sigma_1)),\ \ j=1,2,3,5,6,7,\\
G_4^{(1)}\in C_{0,a}^{M+2}([0,\infty),\widetilde H^{\frac12}(\Sigma_1)),\quad F_4^{(1)}\in C_{0,a}^{M+2}([0,\infty),\widetilde H^{-\frac12}(\Sigma_1)),\\
G_j^{(2)}\in C_{0,a}^{M+2}([0,\infty),\widetilde H^{\frac12}(\Sigma_2)),\quad F_j^{(2)}\in C_{0,a}^{M+2}([0,\infty),\widetilde H^{\frac12}(\Sigma_2)),\ \ j=4,8,9,
\end{gathered}$$

and the compatibility conditions

$$\begin{aligned}
&F_j^{(1),+}-F_j^{(1),-}\in C_{0,a}^{M+2}([0,\infty),\widetilde H^{-\frac12}(\Sigma_1)),\ \ j=1,2,3,\\
&F_j^{(2),+}-F_j^{(2),-}\in C_{0,a}^{M+2}([0,\infty),\widetilde H^{-\frac12}(\Sigma_2)),\ \ j=1,2,3,5,6,7,
\end{aligned}$$

are satisfied, where M is an appropriately chosen natural number. Further, note that the initial conditions are satisfied automatically.

4 Boundary-transmission problem of pseudo-oscillations

Using the Laplace transform

$$\widetilde f(\tau)=\int\limits_0^\infty e^{-\tau t}f(t)\,dt,\ \ \tau=\sigma+i\omega,\ \ \sigma=\operatorname{Re}\tau>a>0,\ \ \omega\in\mathbb{R},$$

the mixed type transmission dynamical problem with interior cracks can be reduced to the following boundary-transmission pseudo-oscillation $(TM)_\tau$ problem depending on the complex parameter τ.

We are looking for a solution

$$\begin{aligned}
&\widetilde U^{(1)}=(\widetilde u^{(1)},\widetilde\vartheta^{(1)})^\top=(\widetilde u^{(1)},\widetilde u_4^{(1)})^\top\in[H^1(\Omega_{\Sigma_1})]^4,\\
&\widetilde U^{(2)}=\big(\widetilde u^{(2)},\widetilde\vartheta^{(2)},\widetilde\phi^{(2)},\widetilde\varphi^{(2)},\widetilde\psi^{(2)}\big)^\top=\big(\widetilde u^{(2)},\widetilde u_4^{(2)},\widetilde u_5^{(2)},\dots,\widetilde u_9^{(2)}\big)^\top\in[H^1(\Omega_{\Sigma_2})]^9
\end{aligned}$$

of the pseudo-oscillation equations

$$A^{(1)}(\partial_x,\tau)\widetilde{U}^{(1)}=\widetilde{\Phi}_1 \text{ in } \Omega_1,$$
$$A^{(2)}(\partial_x,\tau)\widetilde{U}^{(2)}=\widetilde{\Phi}_2 \text{ in } \Omega_2,$$

which satisfy on the surface S_1 the following transmission conditions:

$$\{\widetilde{u}_j^{(1)}\}^+-\{\widetilde{u}_j^{(2)}\}^+=\widetilde{f}_j^{(1)} \text{ on } S_1,\ \ j=\overline{1,4},$$
$$\big\{T^{(1)}(\partial_x,n,\tau)\widetilde{U}^{(1)}\big\}_j^+ +\big\{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\big\}_j^+ = f_j^{(2)} \text{ on } S_1,\ \ j=\overline{1,4},\ \ \nu=-n,$$

and the boundary conditions

$$\{\widetilde{u}_j^{(2)}\}^+=\widetilde{Q}_j^{(2)} \text{ on } S_1,\ \ j=\overline{5,9},$$

while on the surface S_2, the mixed boundary conditions

$$\{\widetilde{U}^{(2)}\}^+=\widetilde{p}_2^{(D)} \text{ on } S_2^{(D)},\quad \big\{T^{(2)}\widetilde{U}^{(2)}\big\}^+=\widetilde{q}_2^{(N)} \text{ on } S_2^{(N)},$$

the crack boundary conditions on Σ_1

$$\big\{T^{(1)}(\partial_x,n,\tau)\widetilde{U}^{(1)}\big\}_j^{\pm}=\widetilde{F}_j^{(1),\pm} \text{ on } \Sigma_1,\ \ j=1,2,3,$$
$$\{\widetilde{u}_4^{(1)}\}^+-\{\widetilde{u}_4^{(1)}\}^+=\widetilde{G}_4^{(1)} \text{ on } \Sigma_1,$$
$$\big\{T^{(1)}(\partial_x,n,\tau)\widetilde{U}^{(1)}\big\}_4^+-\big\{T^{(1)}(\partial_x,n,\tau)\widetilde{U}^{(1)}\big\}_4^-=\widetilde{F}_4^{(1)} \text{ on } \Sigma_1,$$

the crack boundary conditions on Σ_2

$$\big\{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\big\}_j^{\pm}=\widetilde{F}_j^{(2),\pm} \text{ on } \Sigma_2,\ \ j=1,2,3,5,6,7,$$
$$\{\widetilde{u}_j^{(2)}\}^+-\{\widetilde{u}_j^{(2)}\}^+=\widetilde{G}_j^{(2)} \text{ on } \Sigma_2,\ \ j=4,8,9,$$
$$\big\{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\big\}_j^+-\big\{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\big\}_4^-=\widetilde{F}_j^{(2)} \text{ on } \Sigma_1,\ \ j=4,8,9,$$

and the compatibility conditions

$$F_j^{(1),+}-F_j^{(1),-}\in\widetilde{H}^{-\frac12}(\Sigma_1),\ \ j=1,2,3,$$
$$F_j^{(2),+}-F_j^{(2),-}\in\widetilde{H}^{-\frac12}(\Sigma_2),\ \ j=1,2,3,5,6,7,$$

are satisfied, where $\operatorname{Re}\tau>a>0$,

$$\widetilde{f}_j^{(1)}\in H^{\frac12}(S_1),\quad \widetilde{f}_j^{(2)}\in H^{-\frac12}(S_1),\ \ j=\overline{1,4},\quad \widetilde{Q}_j^{(2)}\in H^{\frac12}(S_1),\ \ j=\overline{5,9},$$
$$\widetilde{p}^{(2)}\in[H^{\frac12}(S_2)]^9,\quad \widetilde{q}^{(2)}\in[H^{-\frac12}(S_2)]^9,\quad \widetilde{p}_2^{(D)}\in[H^{\frac12}(S_2^{(D)})]^9,\quad \widetilde{q}_2^{(N)}\in[H^{-\frac12}(S_2^{(N)})]^9,$$
$$\widetilde{F}_j^{(1),\pm}\in H^{-\frac12}(\Sigma_1),\ \ j=1,2,3,\quad \widetilde{F}_j^{(2),\pm}\in H^{-\frac12}(\Sigma_1),\ \ j=1,2,3,5,6,7,$$
$$\widetilde{G}_4^{(1)}\in\widetilde{H}^{\frac12}(\Sigma_1),\quad \widetilde{F}_4^{(1)}\in\widetilde{H}^{-\frac12}(\Sigma_1),$$
$$\widetilde{G}_j^{(2)}\in\widetilde{H}^{\frac12}(\Sigma_2),\quad \widetilde{F}_j^{(2)}\in\widetilde{H}^{\frac12}(\Sigma_2),\ \ j=4,8,9,$$

and

$$\begin{aligned}A^{(1)}(\partial_x,\tau)&=\big[A_{ij}^{(1)}(\partial_x,\tau)\big]_{4\times4}\\&:=\begin{bmatrix}\big[\delta_{ij}(\lambda^{(1)}+\mu^{(1)})\Delta+(\lambda^{(1)}+\varkappa^{(1)})\partial_i\partial_j-\tau^2\rho_1\delta_{ij}\big]_{3\times3} & -\tau\beta_0^{(1)}[\partial_i]_{3\times1}\\ -\tau\beta_0^{(1)}[\partial_j]_{1\times3} & -\tau^2a^{(1)}+k^{(1)}\Delta\end{bmatrix}_{4\times4},\\
T^{(1)}=T^{(1)}(\partial_x,n,\tau)&=\big[T_{ij}^{(1)}(\partial_x,n,\tau)\big]_{4\times4}\\&:=\begin{bmatrix}\big[\lambda^{(1)}n_i\partial_j+\mu^{(1)}n_j\partial_i+\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})n_k\partial_k\big]_{3\times3} & [-\tau\beta_0^{(1)}n]_{3\times1}\\ [0]_{1\times3} & k^{(1)}n_l\partial_l\end{bmatrix}_{4\times4},\end{aligned}$$

the matrix differential pseudo-oscillation operator of thermo-electro-elasticity is defined as follows:

$$A^{(2)}(\partial_x,\tau)=\big[A^{(2)}_{ij}(\partial_x,\tau)\big]_{9\times 9},$$
$$A^{(2)}_{ij}(\partial_x,\tau)=\delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\partial_l\partial_l+(\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j-\tau^2\rho_2\delta_{ij},$$
$$A^{(2)}_{i4}(\partial_x,\tau))=-\tau\beta^{(2)}_0\partial_i,\quad A^{(2)}_{i,j+4}(\partial_x,\tau)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l,\quad A^{(2)}_{i8}(\partial_x,\tau)=\lambda^{(2)}_0\partial_i,\quad A^{(2)}_{i9}(\partial_x,\tau)=0,$$
$$A^{(2)}_{4j}(\partial_x,\tau)=-\tau\beta^{(2)}_0\partial_j,\quad A^{(2)}_{44}(\partial_x,\tau)=k^{(2)}\partial_l\partial_l-\tau^2a^{(2)},\quad A^{(2)}_{4,j+4}(\partial_x,\tau)=0,$$
$$A^{(2)}_{48}(\partial_x,\tau)=\nu^{(2)}_1\partial_l\partial_j-\tau c^{(2)}_0,\quad A^{(2)}_{49}(\partial_x,\tau)=-\nu^{(2)}_3\partial_l\partial_l,$$
$$A^{(2)}_{i+4,j}(\partial_x,\tau)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l,\quad A^{(2)}_{i+4,4}(\partial_x,\tau)=0,$$
$$A^{(2)}_{i+4,j+4}(\partial_x,\tau)=\delta_{ij}\gamma^{(2)}\partial_l\partial_l+(\alpha^{(2)}+\beta^{(2)})\partial_i\partial_j-(2\varkappa^{(2)}+\tau^2I^{(2)}_0)\delta_{ij},$$
$$A^{(2)}_{i+4,8}(\partial_x,\tau)=0,\quad A^{(2)}_{i+4,9}(\partial_x,\tau)=0,\quad A^{(2)}_{8j}(\partial_x,\tau)=-\lambda^{(2)}_0\partial_j,\quad A^{(2)}_{84}(\partial_x,\tau)=\nu^{(2)}_1\partial_l\partial_l+\tau c^{(2)}_0,$$
$$A^{(2)}_{8,j+4}(\partial_x,\tau)=0,\quad A^{(2)}_{88}(\partial_x,\tau)=a^{(2)}_0\partial_l\partial_l-(\xi^{(2)}_0+\tau^2j^{(2)}_0),$$
$$A^{(2)}_{89}(\partial_x,\tau)=-\lambda^{(2)}_2\partial_l\partial_l,\quad A^{(2)}_{9j}(\partial_x,\tau)=0,\quad A^{(2)}_{94}(\partial_x,\tau)=\nu^{(2)}_3\partial_l\partial_l,$$
$$A^{(2)}_{9,j+4}(\partial_x,\tau)=0\quad A^{(2)}_{98}(\partial_x,\tau)=\lambda^{(2)}_2\partial_l\partial_l,\quad A^{(2)}_{99}(\partial_x,\tau)=\chi^{(2)}\partial_l\partial_l,\ \ i,j=1,2,3,$$

the corresponding stress differential operator of thermo-electro-elasticity is defined as follows:

$$T^{(2)}=T^{(2)}(\partial_x,\nu,\tau):=\big[T^{(2)}_{ij}(\partial_x,\nu,\tau)\big]_{9\times 9},$$

where

$$T^{(2)}_{ij}(\partial_x,\nu,\tau)=\lambda^{(2)}\nu_i\partial_j+\mu^{(2)}\nu_j\partial_i+\delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\nu_k\partial_k,\quad T^{(2)}_{i4}(\partial_x,\nu,\tau)=-\tau\beta^{(2)}_0\nu_i,$$
$$T^{(2)}_{i,j+4}(\partial_x,\nu,\tau)=-\varkappa^{(2)}\varepsilon_{ijk}\nu_k,\quad T^{(2)}_{i8}(\partial_x,\nu,\tau)=\lambda^{(2)}_0\nu_i,\quad T^{(2)}_{i9}(\partial_x,\nu,\tau)=0,$$
$$T^{(2)}_{4,j}(\partial_x,\nu,\tau=0,\quad T^{(2)}_{44}(\partial_x,\nu,\tau)=k^{(2)}\nu_l\partial_l,\quad T^{(2)}_{4,j+4}(\partial_x,\nu,\tau)=-\nu^{(2)}_2\varepsilon_{ljk}\nu_l\partial_k,$$
$$T^{(2)}_{48}(\partial_x,\nu,\tau)=\nu^{(2)}_1\nu_k\partial_k,\quad T^{(2)}_{49}(\partial_x,\nu,\tau)=-\nu^{(2)}_3\nu_k\partial_k,\quad T^{(2)}_{i+4,j}(\partial_x,\nu,\tau)=0,$$
$$T^{(2)}_{i+4,4}(\partial_x,\nu,\tau)=\nu^{(2)}_2\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{i+4,j+4}(\partial_x,\nu,\tau)=\alpha^{(2)}\nu_i\partial_j+\beta^{(2)}\nu_j\partial_i+\delta_{ij}\gamma^{(2)}\nu_k\partial_k,$$
$$T^{(2)}_{i+4,8}(\partial_x,\nu,\tau)=b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{i+4,9}(\partial_x,\nu,\tau)=\lambda^{(2)}_1\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{8j}(\partial x,\nu,\tau)=0,$$
$$T^{(2)}_{84}(\partial_x,\nu,\tau)=\nu^{(2)}_1\nu_k\partial_k,\quad T^{(2)}_{8,j+4}(\partial_x,\nu,\tau)=-b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k,\quad T^{(2)}_{88}(\partial_x,\nu,\tau)=a^{(2)}_0\nu_k\partial_k,$$
$$T^{(2)}_{89}(\partial_x,\nu,\tau)=-\lambda^{(2)}_2\nu_k\partial_k,\quad T^{(2)}_{9j}(\partial x,\nu,\tau)=0,\quad T^{(2)}_{94}(\partial_x,\nu,\tau)=\nu^{(2)}_3\nu_k\partial_k,$$
$$T^{(2)}_{9,j+4}(\partial_x,\nu,\tau)=-\lambda^{(2)}_1\varepsilon_{ljk}\nu_l\partial_k,\quad T^{(2)}_{98}(\partial_x,\nu,\tau)=\lambda^{(2)}_2\nu_k\partial_k,$$
$$T^{(2)}_{99}(\partial_x,\nu,\tau)=\chi^{(2)}\nu_k\partial_k,\ \ i,j=1,2,3.$$

Now, let us formulate the existence and uniqueness and regularity theorems of the mixed type boundary-transmission pseudo-oscillation problem $(TM)_{c,\tau}$, which were proved in [18].

Theorem 4.1. *Let $S_1,S_2\in C^\infty$, $\tau=\sigma+i\omega$, $\sigma>\sigma_0>0$, $\omega\in\mathbb{R}$, and*

$$\widetilde{\Phi}_1\in[L_2(\Omega_1)]^4,\quad \widetilde{\Phi}_2\in[L_2(\Omega_2)]^9,$$
$$\widetilde{f}^{(1)}_j\in H^{\frac12}(S_1),\quad \widetilde{f}^{(2)}_j\in H^{-\frac12}(S_2),\ \ j=\overline{1,4},$$
$$\widetilde{Q}^{(2)}_j\in H^{\frac12}(S_1),\ \ j=\overline{5,9},\quad \widetilde{p}^{\,(D)}_2\in[H^{\frac12}(S^{(D)}_2)]^9,\quad \widetilde{q}^{\,(N)}_2\in[H^{-\frac12}(S^{(N)}_2)]^9,$$
$$\widetilde{F}^{(1),\pm}_j\in H^{-\frac12}(\Sigma_1),\ \ j=1,2,3,\quad \widetilde{F}^{(2),\pm}_j\in H^{-\frac12}(\Sigma_1),\ \ j=1,2,3,5,6,7,$$
$$\widetilde{G}^{(1)}_4\in\widetilde{H}^{\frac12}(\Sigma_1),\quad \widetilde{F}^{(1)}_4\in\widetilde{H}^{-\frac12}(\Sigma_1),\quad \widetilde{G}^{(2)}_j\in\widetilde{H}^{\frac12}(\Sigma_2),\quad \widetilde{F}^{(2)}_j\in\widetilde{H}^{\frac12}(\Sigma_2),\ \ j=4,8,9,$$

and let the compatibility conditions

$$\widetilde{F}^{(1),+}_j-\widetilde{F}^{(1),-}_j\in\widetilde{H}^{-\frac12}(\Sigma_1),\ \ j=1,2,3,$$

$$\widetilde{F}_j^{(2),+} - \widetilde{F}_j^{(2),-} \in \widetilde{H}^{-\frac{1}{2}}(\Sigma_2), \;\; j = 1,2,3,5,6,7,$$

be satisfied. Then the mixed boundary-transmission problem $(TM)_{c,\tau}$ *has a unique solution*

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [H^1(\Omega_{\Sigma_1})]^4 \times [H^1(\Omega_{\Sigma_2})]^9.$$

Let us introduce the notation

$$d := \frac{cb_0^{(2)} + p\lambda_1^{(2)} + q\nu_2^{(2)}}{2\gamma^{(2)}},$$

where

$$c := \frac{1}{2}\big(b_0^{(2)} b_{11} + \lambda_1^{(2)} b_{21} + \nu_2^{(2)} b_{31}\big), \quad p := \frac{1}{2}\big(b_0^{(2)} b_{12} + \lambda_1^{(2)} b_{22} + \nu_2^{(2)} b_{32}\big),$$

$$q := \frac{1}{2}\big(b_0^{(2)} b_{13} + \lambda_1^{(2)} b_{23} + \nu_2^{(2)} b_{33}\big), \quad [b_{jk}]_{3\times 3} := \begin{bmatrix} a_0^{(2)} & -\lambda_2^{(2)} & \nu_1^{(2)} \\ \lambda_2^{(2)} & \chi^{(2)} & \nu_3^{(2)} \\ \nu_1^{(2)} & -\nu_3^{(2)} & k^{(2)} \end{bmatrix}^{-1}.$$

Denote by $C_0^\infty(\overline{\Sigma}_k)$, $k = 1,2$, the space of functions vanishing along with all tangential (to Σ_k) derivatives at $\ell_c^{(k)} = \partial\Sigma_k$, $k = 1,2$.

The following regularity theorem holds.

Theorem 4.2. *Suppose* $S_1, S_2 \in C^\infty$, *and*

$$\widetilde{\Phi}_1 \in [C^\infty(\overline{\Omega}_1)]^4, \quad \widetilde{\Phi}_2 \in [C^\infty(\overline{\Omega}_2)]^9,$$
$$\widetilde{f}_j^{(1)} \in C^\infty(S_1), \quad \widetilde{f}_j^{(2)} \in C^\infty(S_1), \;\; j = \overline{1,4},$$
$$\widetilde{Q}_j^{(2)} \in C^\infty(S_1), \;\; j = \overline{5,9}, \quad \widetilde{p}_2^{(D)} \in [C^\infty(\overline{S}_2^{(D)})]^9, \quad \widetilde{q}_2^{(N)} \in [C^\infty(\overline{S}_2^{(N)})]^9,$$
$$\widetilde{F}_j^{(1),\pm} \in C^\infty(\overline{\Sigma}_1), \;\; j = 1,2,3, \quad \widetilde{F}_j^{(2),\pm} \in C^\infty(\overline{\Sigma}_2), \;\; j = 1,2,3,5,6,7,$$
$$\widetilde{G}_4^{(1)} \in C^\infty(\overline{\Sigma}_1), \quad \widetilde{F}_4^{(1)} \in C^\infty(\overline{\Sigma}_1), \quad \widetilde{G}_j^{(2)} \in C^\infty(\overline{\Sigma}_2), \quad \widetilde{F}_j^{(2)} \in C^\infty(\overline{\Sigma}_2), \;\; j = 4,8,9,$$
$$\widetilde{F}_j^{(1),+} - \widetilde{F}_j^{(1),-} \in C_0^\infty(\overline{\Sigma}_1), \;\; j = 1,2,3, \quad \widetilde{F}_j^{(2),+} - \widetilde{F}_j^{(2),-} \in C_0^\infty(\overline{\Sigma}_2), \;\; j = 1,2,3,5,6,7.$$

Let $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ *be a unique solution to the boundary-transmission problem* $(TM)_{c,\tau}$.

Then $\widetilde{u}^{(1)}$ *and* $\widetilde{U}^{(2)}$ *have the* $C^{\frac{1}{2}}$*-Hölder smoothness in one-sided interior and exterior neighborhoods of the surfaces* $S_0^{(1)}$ *and* $S_0^{(2)}$*, respectively, and* $\vartheta^{(1)}$ *has the* $C^{\frac{3}{2}}$*-smoothness in one-sided interior and exterior neighborhoods of the surface* $S_0^{(1)}$*. While*

1. *If* $d < 0$*, then the vector* $\widetilde{U}^{(2)}$ *belongs to the* C^{γ_1}*-Hölder class in a neighborhood of the line* $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$*, where* $\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg} 2\sqrt{-d}$*,* γ_1 *depends on the material constants, does not depend on the geometry of the exceptional line* $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$ *and may take any values from the interval* $(0, \frac{1}{2})$*;*

2. *If* $d \geq 0$*, then the vector* $\widetilde{U}^{(2)}$ *belongs to the* $C^{\frac{1}{2}}$*-Hölder class in a neighborhood of the line* ℓ*.*

In order to perform the inverse Laplace transform of solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ of boundary-transmission problem $(TM)_{c,\tau}$, i.e.,

$$U^{(q)}(\,\cdot\,, t) = \frac{1}{2\pi i} \int\limits_{\alpha - i\infty}^{\alpha+\infty} e^{\tau t} \widetilde{U}^{(q)}(\,\cdot\,, \tau)\, d\tau, \;\; q = 1,2, \;\; \alpha > a,$$

we need the estimates in τ of $\|\widetilde{U}^{(1)}(\,\cdot\,,\tau)\|_{[H^1(\Omega_1)]^4}$, $\|\widetilde{U}^{(2)}(\,\cdot\,,\tau)\|_{[H^1(\Omega_2)]^9}$, when $|\tau| \to \infty$ ($\operatorname{Re}\tau > a$).

Using the integration by parts formula for the data of the boundary-transmission pseudo-oscillation problem $(TM)_{c,\tau}$, we deduce for $\operatorname{Re}\tau > a > 0$ the following inequalities:

$$\begin{gathered}
\|\widetilde{\Phi}^{(1)}(\,\cdot\,,\tau)\|_{[L_2(\Omega_{\Sigma_1})]^4} \leq C|\tau|^{-M}, \quad \|\widetilde{\Phi}^{(2)}(\,\cdot\,,\tau)\|_{[L_2(\Omega_{\Sigma_2})]^9} \leq C|\tau|^{-M}, \\
\|\widetilde{f}_j^{(1)}(\,\cdot\,,\tau)\|_{H^{\frac12}(S_1)} \leq C|\tau|^{-M-2}, \quad \|\widetilde{f}_j^{(2)}(\,\cdot\,,\tau)\|_{H^{-\frac12}(S_1)} \leq C|\tau|^{-M-2}, \;\; j=\overline{1,4}, \\
\|\widetilde{Q}_j^{(2)}(\,\cdot\,,\tau)\|_{H^{\frac12}(S_1)} \leq C|\tau|^{-M-2}, \;\; j=\overline{5,9}, \\
\|\widetilde{p}^{(2)}(\,\cdot\,,\tau)\|_{[H^{\frac12}(S_2)]^9} \leq C|\tau|^{-M-2}, \quad \|\widetilde{q}^{(2)}(\,\cdot\,,\tau)\|_{[H^{-\frac12}(S_2)]^9} \leq C|\tau|^{-M-2}, \\
\|\widetilde{p}_2^{(D)}(\,\cdot\,,\tau)\|_{[H^{\frac12}(S_2^{(D)})]^9} \leq C|\tau|^{-M-2}, \quad \|\widetilde{q}_2^{(N)}(\,\cdot\,,\tau)\|_{[H^{-\frac12}(S_2^{(N)})]^9} \leq C|\tau|^{-M-2}, \\
\|\widetilde{F}_j^{(1),\pm}\|_{H^{-\frac12}(\Sigma_1)} \leq C|\tau|^{-M-2}, \;\; j=1,2,3, \\
\|\widetilde{F}_j^{(2),\pm}\|_{H^{-\frac12}(\Sigma_2)} \leq C|\tau|^{-M-2}, \;\; j=1,2,3,5,6,7, \\
\|\widetilde{G}_4^{(1)}\|_{H^{\frac12}(\Sigma_1)} \leq C|\tau|^{-M-2}, \quad \|\widetilde{F}_4^{(1)}\|_{H^{-\frac12}(\Sigma_1} \leq C|\tau|^{-M-2}, \\
\|\widetilde{G}_j^{(2)}\|_{H^{\frac12}(\Sigma_2)} \leq C|\tau|^{-M-2}, \quad \|\widetilde{F}_j^{(2)}\|_{H^{\frac12}(\Sigma_2)} \leq C|\tau|^{-M-2}, \;\; j=4,8,9,
\end{gathered} \tag{4.1}$$

where C is a constant independent on τ.

To obtain the similar estimates (see (4.1)) for the corresponding solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ of the boundary-transmission problem $(TM)_{c,\tau}$, we use the representation

$$\begin{gathered}
\widetilde{U}^{(1)} = V_1^{(1)} + V_2^{(1)}, \quad V_q^{(1)} = (v_{q,1}^{(1)}, \dots, v_{q,4}^{(1)})^\top = (v_q^{(1)}, v_{q,4}^{(1)})^\top, \;\; q=1,2, \\
\widetilde{U}^{(2)} = W_1^{(2)} + W_2^{(2)}, \quad W_q^{(2)} = (w_{q,1}^{(2)}, \dots, w_{q,9}^{(2)})^\top = (w_q^{(2)}, w_{q,4}^{(2)}, \dots, w_{q,9}^{(2)})^\top, \;\; q=1,2,
\end{gathered}$$

where $(V_1^{(1)}, W_1^{(2)})$ and $(V_2^{(1)}, W_2^{(2)})$ are solutions of the following boundary-transmission Problem 4.1 and Problem 4.2, respectively.

Problem 4.1. Find a vector function $(V_1^{(1)}, W_1^{(2)}) \in [H^1(\Omega_[\Sigma_1])]^4 \times [H^1(\Omega_{\Sigma_2})]^9$ satisfying the pseudo-oscillation differential equations

$$\begin{gathered}
A^{(1)}(\partial_x, \tau_0) V_1^{(1)} = \widetilde{\Phi}_1 \;\text{ in } \Omega_{\Sigma_1}, \\
A^{(2)}(\partial_x, \tau_0) W_1^{(2)} = \widetilde{\Phi}_2 \;\text{ in } \Omega_{\Sigma_2},
\end{gathered}$$

the boundary-transmission conditions on the surface S_1

$$\begin{gathered}
\{v_{1,j}^{(1)}\}^+ - \{w_{1,j}^{(1)}\}^+ = \widetilde{f}_j^{(1)} \;\text{ on } S_1, \;\; j=\overline{1,4}, \\
\big\{T^{(1)}(\partial_x, n, \tau_0) V_1^{(1)}\big\}_j^+ + \big\{T^{(2)}(\partial_x, \nu, \tau_0) W_1^{(2)}\big\}_j^+ = \widetilde{f}_j^{(2)} \;\text{ on } S_1, \;\; j=\overline{1,4}, \;\; \nu=-n,
\end{gathered}$$

and the boundary conditions

$$\{w_{1,j}^{(2)}\}^+ = \widetilde{Q}_j^{(2)} \;\text{ on } S_1, \;\; j=\overline{5,9},$$

while on the surface S_2, the mixed boundary conditions

$$\{W_1^{(2)}\}^+ = \widetilde{p}_2^{(D)} \;\text{ on } S_2^{(D)}, \quad \big\{T^{(2)}(\partial_x, \nu, \tau_0) W_1^{(2)}\big\}^+ = \widetilde{q}_2^{(N)} \;\text{ on } S_2^{(N)},$$

the crack boundary conditions on Σ_1

$$\begin{gathered}
\big\{T^{(1)}(\partial_x, n, \tau_0) V_1^{(1)}\big\}_j^\pm = \widetilde{F}_j^{(1),\pm} \;\text{ on } \Sigma_1, \;\; j=1,2,3, \\
\{v_{1,4}^{(1)}\}^+ - \{v_{1,4}^{(1)}\}^- = \widetilde{G}_4^{(1)} \;\text{ on } \Sigma_1, \\
\big\{T^{(1)}(\partial_x, n, \tau_0) V_1^{(1)}\big\}_4^+ - \big\{T^{(1)}(\partial_x, n, \tau_0) V_1^{(1)}\big\}_4^- = \widetilde{F}_4^{(1)} \;\text{ on } \Sigma_1,
\end{gathered}$$

the crack boundary conditions on Σ_2

$$\big\{T^{(2)}(\partial_x,\nu,\tau_0)W_1^{(2)}\big\}_j^{\pm} = \widetilde{F}_j^{(2),\pm} \ \text{ on } \Sigma_2, \ \ j=1,2,3,5,6,7,$$
$$\{w_{1,j}^{(2)}\}^+ - \{w_{1,j}^{(2)}\}^- = \widetilde{G}_j^{(2)} \ \text{ on } \Sigma_2, \ \ j=4,8,9,$$
$$\big\{T^{(2)}(\partial_x,\nu,\tau_0)W_1^{(2)}\big\}_j^{+} - \big\{T^{2)}(\partial_x,\nu,\tau_0)W_1^{(2)}\big\}_j^{-} = \widetilde{F}_j^{(2)} \ \text{ on } \Sigma_2, \ \ j=4,8,9,$$

where τ_0 is a fixed complex number such that $\operatorname{Re}\tau_0 > 0$.

Problem 4.2. Find a vector function $(V_2^{(1)}, W_2^{(2)}) \in [H^1(\Omega_{\Sigma_1})]^4 \times [H^1(\Omega_{\Sigma_2})]^9$ satisfying the pseudo-oscillation differential equations

$$A^{(1)}(\partial_x,\tau)V_2^{(1)} = \Psi^{(1)} \ \text{ in } \Omega_{\Sigma_1},$$
$$A^{(2)}(\partial_x,\tau)W_2^{(2)} = \Psi^{(2)} \ \text{ in } \Omega_{\Sigma_2},$$

the boundary-transmission conditions on the surface S_1

$$\{v_{2,j}^{(1)}\}^+ - \{w_{2,j}^{(1)}\}^- = 0 \ \text{ on } S_1, \ \ j=\overline{1,4},$$
$$\big\{T^{(1)}(\partial_x,n,\tau)V_2^{(1)}\big\}_j^{+} + \big\{T^{(2)}(\partial_x,\nu,\tau)W_2^{(2)}\big\}_j^{+} = G_j \ \text{ on } S_1, \ \ j=\overline{1,4}, \ \ \nu=-n,$$

and the boundary conditions

$$\{w_{2,j}^{(2)}\}^+ = 0 \ \text{ on } S_1, \ \ j=\overline{5,9},$$

while on the surface S_2, the mixed boundary conditions

$$\{W_2^{(2)}\}^+ = 0 \ \text{ on } S_2^{(D)}, \quad \big\{T^{(2)}(\partial_x,\nu,\tau)W_2^{(2)}\big\}^+ = q_2^{*(N)} \ \text{ on } S_2^{(N)},$$

the crack boundary conditions on Σ_1

$$\big\{T^{(1)}(\partial_x,n,\tau)V_2^{(1)}\big\}_j^{\pm} = F_j^{*(1),\pm} \ \text{ on } \Sigma_1, \ \ j=1,2,3,$$
$$\{v_{2,4}^{(1)}\}^+ - \{v_{2,4}^{(1)}\}^- = 0 \ \text{ on } \Sigma_1,$$
$$\big\{T^{(1)}(\partial_x,n,\tau)V_2^{(1)}\big\}_4^{+} - \big\{T^{(1)}(\partial_x,n,\tau)V_2^{(1)}\big\}_4^{-} = F_4^{*(1)} \ \text{ on } \Sigma_1,$$

the crack boundary conditions on Σ_2

$$\big\{T^{(2)}(\partial_x,\nu,\tau)W_2^{(2)}\big\}_j^{\pm} = F_j^{*(2),\pm} \ \text{ on } \Sigma_2, \ \ j=1,2,3,5,6,7,$$
$$\{w_{2,j}^{(2)}\}^+ - \{w_{2,j}^{(2)}\}^- = 0 \ \text{ on } \Sigma_2, \ \ j=4,8,9,$$
$$\big\{T^{(2)}(\partial_x,\nu,\tau)W_2^{(2)}\big\}_j^{+} - \big\{T^{(2)}(\partial_x,\nu,\tau)W_2^{(2)}\big\}_j^{-} = F_j^{*(2)} \ \text{ on } \Sigma_1, \ \ j=4,8,9,$$

where

$$\Psi^{(1)} := \big[A^{(1)}(\partial_x,\tau_0) - A^{(1)}(\partial_x,\tau)\big]V_1^{(2)}, \quad \Psi^{(2)} := \big[A^{(2)}(\partial_x,\tau_0) - A^{(2)}(\partial_x,\tau)\big]W_1^{(2)},$$
$$G_j := \Big\{\big[T^{(1)}(\partial_x,n,\tau_0) - T^{(1)}(\partial_x,n,\tau)\big]V_1^{(1)}\Big\}_j^{+} + \Big\{\big[T^{(2)}(\partial_x,\nu,\tau_0) - T^{(2)}(\partial_x,\nu,\tau)\big]W_1^{(2)}\Big\}_j^{+}, \ \ j=\overline{1,4},$$

i.e.,

$$\Psi^{(1)} = \begin{bmatrix} \rho_1(\tau^2-\tau_0^2)v_1^{(1)} + (\tau-\tau_0)\beta_0^{(1)}[\partial_i]_{3\times1}v_{1,4}^{(1)} \\ (\tau-\tau_0)\beta_0^{(1)}\partial_i v_{1,i}^{(1)} + (\tau^2-\tau_0^2)a^{(1)}v_{1,4}^{(1)} \end{bmatrix},$$
$$\Psi^{(2)} = \begin{bmatrix} \rho_2(\tau^2-\tau_0^2)w_1^{(2)} + (\tau-\tau_0)\beta_0^{(2)}[\partial_i]_{3\times1}w_{1,4}^{(2)} \\ (\tau-\tau_0)\beta_0^{(2)}\partial_i w_{1,i}^{(2)} + (\tau^2-\tau_0^2)a^{(2)}w_{1,4}^{(2)} + (\tau-\tau_0)c_0^{(2)}w_{1,8}^{(2)} \\ I_0^{(2)}(\tau^2-\tau_0^2)\big[w_{1,j+4}^{(2)}\big]_{3\times1} \\ (\tau-\tau_0)c_0^{(2)}w_{1,4}^{(2)} + (\tau^2-\tau_0^2)j_0^{(2)}w_{1,8}^{(2)} \\ 0 \end{bmatrix},$$

$$
\begin{aligned}
G_j &= (\tau-\tau_0)\beta_0^{(1)} n_j v_{1,4}^{(1)} + (\tau-\tau_0)\beta_0^{(2)}\nu_j w_{1,4}^{(2)}, \ \ j=1,2,3, \quad G_4 = 0,\\
q_2^{*(N)} &:= \Big\{ \big[T^{(2)}(\partial_x,\nu,\tau_0) - T^{(2)}(\partial_x,\nu,\tau)\big] W_1^{(2)} \Big\}^+,\\
F_j^{*(1),\pm} &:= \Big\{ \big[T^{(1)}(\partial_x,n,\tau_0) - T^{(1)}(\partial_x,n,\tau)\big] V_1^{(1)} \Big\}_j^\pm, \ \ j=1,2,3,\\
F_4^{*(1)} &:= \Big\{ \big[T^{(1)}(\partial_x,n,\tau_0) - T^{(1)}(\partial_x,n,\tau)\big] V_1^{(1)} \Big\}_4^+\\
&\qquad - \Big\{ \big[T^{(1)}(\partial_x,n,\tau_0) - T^{(1)}(\partial_x,n,\tau)\big] V_1^{(1)} \Big\}_4^-,\\
F_j^{*(2),\pm} &:= \Big\{ \big[T^{(2)}(\partial_x,\nu,\tau_0) - T^{(2)}(\partial_x,\nu,\tau)\big] W_1^{(2)} \Big\}_j^\pm, \ \ j=1,2,3,5,6,7,\\
F_j^{*(2)} &:= \Big\{ \big[T^{(2)}(\partial_x,\nu,\tau_0) - T^{(2)}(\partial_x,\nu,\tau)\big] W_1^{(2)} \Big\}_j^+\\
&\qquad - \Big\{ \big[T^{(2)}(\partial_x,\nu,\tau_0) - T^{(2)}(\partial_x,\nu,\tau)\big] W_1^{(2)} \Big\}_j^-, \ \ j=4,8,9,
\end{aligned}
$$

where

$$
\begin{aligned}
T^{(1)}(\partial_x,n,\tau) - T^{(1)}(\partial_x,n,\tau_0) &= \begin{bmatrix} [0]_{3\times3} & \big[-\beta_0^{(1)} n(\tau-\tau_0)\big]_{3\times1} \\ [0]_{1\times3} & 0 \end{bmatrix}_{4\times4},\\
T^{(2)}(\partial_x,n,\tau) - T^{(2)}(\partial_x,\nu,\tau_0) &= \begin{bmatrix} [0]_{3\times3} & \big[-\beta_0^{(2)} \nu(\tau-\tau_0)\big]_{3\times1} & [0]_{3\times5} \\ [0]_{6\times3} & [0]_{6\times1}, & [0]_{6\times5} \end{bmatrix}_{9\times9}.
\end{aligned}
$$

By Theorem 4.1 Problem 4.1 is uniquely solvable in $[H^1(\Omega_{\Sigma_1})]^4\times[H^1(\Omega_{\Sigma_2})]^9$, then the following estimates hold:

$$
\begin{aligned}
&\|V_1^{(1)}\|_{[H^1(\Omega_{\Sigma_1})]^4} + \|W_1^{(2)}\|_{[H^1(\Omega_{\Sigma_2})]^9}\\
&\quad\le C'\bigg(\|\widetilde{\Phi}_1\|_{[L_2(\Omega_{\Sigma_1})]^4} + \|\widetilde{\Phi}_2\|_{[L_2(\Omega_{\Sigma_2})]^9} + \sum_{j=1}^4 \Big(\|\widetilde{f}_j^{(1)}\|_{H^{\frac12}(S_1)} + \|\widetilde{f}_j^{(2)}\|_{H^{-\frac12}(S_1)}\Big)\\
&\qquad + \sum_{j=5}^9 \|\widetilde{Q}_j^{(2)}\|_{H^{\frac12}(S_1)} + \|\widetilde{p}_2^{(D)}\|_{[H^{\frac12}(S_2^{(D)})]^9} + \|\widetilde{q}_2^{(N)}\|_{[H^{-\frac12}(S_2^{(N)})]^9}\\
&\qquad + \sum_{j=1}^3 \|\widetilde{F}_j^{(1),\pm}\|_{H^{-\frac12}(\Sigma_1)} + \|\widetilde{G}_4^{(1)}\|_{H^{\frac12}(\Sigma_1)} + \|\widetilde{F}_4^{(1)}\|_{H^{-\frac12}(\Sigma_1)}\\
&\qquad + \sum_{j=1,\ j\neq4}^7 \|\widetilde{F}_j^{(2),\pm}\|_{H^{-\frac12}(\Sigma_2)} + \sum_{j=4,8,9} \Big(\|\widetilde{G}_j^{(2)}\|_{H^{-\frac12}(S_1)} + \|\widetilde{F}_j^{(2)}\|_{H^{-\frac12}(\Sigma_2)}\Big)\bigg),
\end{aligned}
$$

where the constant C' does not depend on τ.

Taking into account estimates (4.1), we obtain

$$
\|V_1^{(1)}\|_{[H^1(\Omega_{\Sigma_1})]^4} + \|W_1^{(2)}\|_{[H^1(\Omega_{\Sigma_2})]^9} \le C_1|\tau|^{-M}, \ \ \operatorname{Re}\tau > a > 0. \tag{4.2}
$$

where the constant C_1 does not depend on τ.

It is clear that estimate (4.2) implies the estimates for the data of Problem 4.2 with respect to τ:

$$
\begin{gathered}
\|\Psi^{(1)}\|_{[L_2(\Omega_{\Sigma_1})]^4} \le C|\tau|^{-M+2}, \quad \|\Psi^{(2)}\|_{[L_2(\Omega_{\Sigma_2})]^9} \le C|\tau|^{-M+2},\\
\|G_j\|_{H^{-\frac12}(S_1)} \le C|\tau|^{-M+1}, \ \ j=\overline{1,4},\\
\|q_2^{*(N)}\|_{[H^{-\frac12}(S_2^{(N)})]^9} \le C|\tau|^{-M+1}, \quad \|F_j^{*(1),\pm}\|_{H^{-\frac12}(\Sigma_1)} \le C|\tau|^{-M+1}, \ \ j=1,2,3,\\
\|F_4^{*(1)}\|_{H^{-\frac12}(\Sigma_1)} \le C|\tau|^{-M+1}, \quad \|F_j^{*(2),\pm}\|_{H^{-\frac12}(\Sigma_2)} \le C|\tau|^{-M+1}, \ \ j=1,2,3,5,6,7,\\
\|F_j^{*(2),\pm}\|_{H^{-\frac12}(\Sigma_2)} \le C|\tau|^{-M+1}, \ \ j=4,8,9.
\end{gathered}
\tag{4.3}
$$

Now let us consider Problem 4.2. Suppose $(V_2^{(1)}, W_2^{(2)})$ is a solution to the Problem 4.2. Let us write Green's formulas for the vector functions $V_2^{(1)}$ and $W_2^{(2)}$ in the domains Ω_{Σ_1} and Ω_{Σ_2}, respectively:

$$\int\limits_{\Omega_{\Sigma_1}} A^{(1)}(\partial_x,\tau)V_2^{(1)}\cdot V_2^{(1)}\,dx + \int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(V_2^{(1)},\overline{V}_2^{(1)})\,dx = \Big\langle \{T^{(1)}V_2^{(1)}\}^+, \{V_2{}^{(1)}\}^+ \Big\rangle_{S_1\cup S_0^{(1)}}, \tag{4.4}$$

$$\int\limits_{\Omega_{\Sigma_2}} A^{(2)}(\partial_x,\tau)W_2^{(2)}\cdot W_2{}^{(2)}\,dx + \int\limits_{\Omega_{\Sigma_2}} E_\tau^{(2)}(W_2^{(2)},\overline{W}_2^{(2)})\,dx = \Big\langle \{T^{(2)}W_2^{(2)}\}^+, \{W_2{}^{(2)}\}^+ \Big\rangle_{S_1\cup S_2\cup S_0^{(2)}}, \tag{4.5}$$

where

$$V_2^{(1)} = (v_{2,1}^{(1)},\dots,v_{2,4}^{(1)})^\top = (v_2^{(1)}, v_{2,4}^{(1)})^\top, \quad v_2^{(1)} = (v_{2,1}^{(1)}, v_{2,2}^{(1)}, v_{2,3}^{(1)})^\top,$$
$$W_2^{(2)} = (w_{2,1}^{(2)},\dots,w_{2,9}^{(2)})^\top = \big(w_2^{(2)}, w_{2,4}^{(2)}, \phi_2^{(2)}, w_{2,8}^{2}, w_{2,9}^{(2)}\big)^\top,$$
$$w_2^{(2)} = (w_{2,1}^{(2)}, w_{2,2}^{(2)}, w_{2,3}^{(2)})^\top, \quad \phi_2^{(2)} = (w_{2,5}^{(2)}, w_{2,6}^{(2)}, w_{2,7}^{(2)})^\top,$$
$$E_\tau^{(1)}(V_2^{(1)},\overline{V}_2^{(1)}) = \mathcal{E}(v_2^{(1)},\overline{v}_2^{(1)}) + \rho_1\tau^2|v_2^{(1)}|^2 - \tau\beta_0^{(1)}v_{2,4}^{(1)}\operatorname{div}\overline{v}_2^{(1)}$$
$$+ k^{(1)}|\operatorname{grad} v_{2,4}^{(1)}|^2 + \tau\beta_0^{(1)}\operatorname{div} v_2^{(1)}\overline{v}_{2,4}^{(1)} + \tau^2 a^{(1)}|v_{2,4}^{(1)}|^2,$$
$$\mathcal{E}(v_2^{(1)},\overline{v}_2^{(1)}) = (\mu^{(1)}+\varkappa^{(1)})|\operatorname{grad} v_2^{(1)}|^2 + (\lambda^{(1)}+\mu^{(1)})|\operatorname{div} v_2^{(1)}|^2.$$

Here and in what follows, $a\cdot b$ denotes the scalar product of two, in general, complex-valued vectors

$$a\cdot b = \sum_{k=1}^N a_k\overline{b}_k, \quad a,b\in\mathbb{C}^N;$$

$\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the spaces $[H^{-\frac12}(S)]^N$ and $[H^{\frac12}(S)]^N$, which extends the usual L_2 inner product for complex-valued vector functions

$$\langle f,g\rangle_S = \int\limits_S f(x)\cdot g(x)\,dx \ \text{ for } \ f,g\in[L_2(S)]^N,$$

where S is a closed surface in $\mathbb{R}^3$. If S is an open surface, then $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the spaces $[H^{-\frac12}(S)]^N$ and $[\widetilde{H}^{\frac12}(S)]^N$.

Obviously, $\mathcal{E}(v_2^{(1)},\overline{v}_2^{(1)}) > 0$,

$$E_\tau^{(2)}(W_2^{(2)},\overline{W}_2^{(2)}) = B(v^{(2)},\overline{v}^{(2)}) + 2i\lambda_1^{(2)}\varepsilon_{ijk}\operatorname{Im}(\partial_k w_{2,9}^{(2)}\partial_i\overline{w}_{2,j+4}^{(2)})$$
$$+ 2i\lambda_2^{(2)}\operatorname{Im}(\partial_j w_{2,8}^{(2)}\,\partial_j\overline{w}_{2,9}^{(2)}) + 2i\nu_3^{(2)}\operatorname{Im}(\partial_j w_{2,4}^{(2)}\partial_j\overline{w}_{2,9}^{(2)}) + 2i\tau\beta_0^{(2)}\operatorname{Im}(\partial_j w_{2,j}^{(2)}\overline{w}_{2,4}^{(2)})$$
$$+ 2i\tau c_0^{(2)}\operatorname{Im}(w_{2,8}^{(2)}\overline{w}_{2,4}^{(2)}) + \tau^2\Big(\rho_2|w_2^{(2)}|^2 + I_0^{(2)}|\phi_2^{(2)}|^2 + j_0^{(2)}|w_{2,8}^{(2)}|^2 + a^{(2)}|w_{2,4}^{(2)}|^2\Big);$$

here, we assume that $B(v^{(2)},\overline{v}^{(2)})$ is a positive definite with respect to the vector

$$v^{(2)} = (e_{ij}, \varkappa_{ij}, \zeta_j, \varphi, T, \vartheta_i, E_i), \quad B(v^{(2)},\overline{v}^{(2)}) > 0 \ \ \forall\, v^{(2)}\neq 0,$$

where

$$e_{ij} = \partial_i w_{2,j}^{(2)} + \varepsilon_{jik}w_{2,k+4}^{(2)}, \quad \varkappa_{ij} = \partial_i w_{2,j+4}^{(2)},$$
$$\zeta_j = \partial_j w_{2,8}^{(2)}, \quad \varphi = w_{2,8}^{(2)}, \quad T = \tau w_{2,4}^{(2)}, \quad \vartheta_i^{(2)} = \partial_i w_{2,4}^{(2)}, \quad E_i = -\partial_i w_{2,9}^{(2)}$$

(for the definition of this form see [5, formula (2.19)]).

Adding Green's formulas (4.4) and (4.5) and taking into account the fact that $(V_2^{(1)}, W_2^{(2)}) \in [H^1(\Omega_{\Sigma_1})]^4\times[H^1(\Omega_{\Sigma_2})]^9$ is a solution to the boundary transmission Problem 4.2, we get

$$\int\limits_{\Omega_{\Sigma_1}} \Psi^{(1)} \cdot V_2^{(1)}\,dx + \int\limits_{\Omega_{\Sigma_2}} \Psi^{(2)} \cdot W_2{}^{(2)}\,dx + \int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(V_2^{(1)}, \overline{V}_2{}^{(1)})\,dx + \int\limits_{\Omega_{\Sigma_2}} E_\tau^{(2)}(W_2^{(2)}, W_2^{(2)})\,dx$$
$$= \Big\langle \sum_{j=1}^{4} \big\{T^{(1)}V_2^{(1)}\big\}_j^+, \{V_2^{(1)}\}_j^+ \Big\rangle_{S_1} + \Big\langle \sum_{j=1}^{4} \big\{T^{(2)}W_2^{(2)}\big\}_j^+, \{W_2{}^{(2)}\}_j^+ \Big\rangle_{S_1} + \big\langle q_2^{*(N)}, \{W_2^{(2)}\}^+ \big\rangle_{S_2^{(N)}}$$
$$+ \sum_{j=1}^{3} \Big\langle \big(F_j^{*(1),+} - F_j^{*(1),-}\big), \{V_2^{(1)}\}_j^+ \Big\rangle_{\Sigma_1} + \big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+ \big\rangle_{\Sigma_1}$$
$$+ \sum_{j=1,\ j\neq 4}^{7} \Big\langle \big(F_j^{*(2),+} - F_j^{*(2),+}\big), \{W_2^{(2)}\}_j^+ \Big\rangle_{\Sigma_2} + \sum_{j=4,8,9} \big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+ \big\rangle_{\Sigma_2}$$
$$= \sum_{j=1}^{4} \Big\langle \Big(\big\{T^{(1)}V_2^{(1)}\big\}_j^+ + \big\{T^{(2)}W_2^{(2)}\big\}_j^+ \Big), \{W_2{}^{(2)}\}_j^+ \Big\rangle_{S_1} + \big\langle q_2^{*(N)}, \{W_2^{(2)}\}^+ \big\rangle_{S_2^{(N)}}$$
$$+ \sum_{j=1}^{3} \Big\langle \big(F_j^{*(1),+} - F_j^{*(1),-}\big), \{V_2^{(1)}\}_j^+ \Big\rangle_{\Sigma_1} + \big\langle F_4^{*(1)}, \{V_2^{(1)}\}^+ \big\rangle_{\Sigma_1}$$
$$+ \sum_{j=1,\ j\neq 4}^{7} \Big\langle \big(F_j^{*(2),+} - F_j^{*(2),+}\big), \{W_2^{(2)}\}_j^+ \Big\rangle_{\Sigma_2} + \sum_{j=4,8,9} \big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+ \big\rangle_{\Sigma_2}$$
$$= \sum_{j=1}^{4} \big\langle G_j, \{W_2^{(2)}\}_j^+ \big\rangle_{S_1} + \big\langle q_2^{*(N)}, \{W_2^{(2)}\}^+ \big\rangle_{S_2^{(N)}} + \sum_{j=1}^{3} \Big\langle \big(F_j^{*(1,+)} - F_j^{*(1,-)}\big), \{V_2^{(1)}\}_j^+ \Big\rangle_{\Sigma_1}$$
$$+ \big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+ \big\rangle_{\Sigma_1} + \sum_{j=1,\ j\neq 4}^{7} \Big\langle \big(F_j^{*(2),+} - F_j^{*(2),-}\big), \{W_2^{(2)}\}_j^+ \Big\rangle_{\Sigma_2} + \sum_{j=4,8,9} \big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+ \big\rangle_{\Sigma_2},$$

where

$$\{W_2^{(2)}\}_j^+ = \{w_{2,j}^{(2)}\}^+, \ \ j = \overline{1,9}, \quad \{V_2^{(1)}\}_j^+ = \{v_{2,j}^{(1)}\}^+, \ \ j = \overline{1,4}.$$

Therefore, we obtain

$$\int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(V_2^{(1)}, \overline{V}_2{}^{(1)})\,dx + \int\limits_{\Omega_{\Sigma_2}} E_\tau^{(2)}(W_2^{(2)}, \overline{W}_2{}^{(2)})\,dx$$
$$= \sum_{j=1}^{4} \big\langle G_j, \{W_2^{(2)}\}_j^+ \big\rangle_{S_1} + \big\langle q_2^{*(N)}, \{W_2^{(2)}\}^+ \big\rangle_{S_2^{(N)}} + \sum_{j=1}^{3} \Big\langle \big(F_j^{*(1),+} - F_j^{*(1),+}\big), \{V_2^{(1)}\}_j^+ \Big\rangle_{\Sigma_1}$$
$$+ \big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+ \big\rangle_{\Sigma_1} + \sum_{j=1,\ j\neq 4}^{7} \Big\langle \big(F_j^{*(2),+} - F_j^{*(2),-}\big), \{W_2^{(2)}\}_j^+ \Big\rangle_{\Sigma_2}$$
$$+ \sum_{j=4,8,9} \big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+ \big\rangle_{\Sigma_2} - \int\limits_{\Omega_{\Sigma_1}} \Psi^{(1)} \cdot V_2^{(1)}\,dx - \int\limits_{\Omega_{\Sigma_2}} \Psi^{(2)} \cdot W_2{}^{(2)}\,dx. \tag{4.6}$$

Similarly, we get (see [5])

$$\int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(V_2^{(1)}, \overline{V_2}{}^{(1)})\,dx + \int\limits_{\Omega_2} \widetilde{E}_\tau^{(2)}(W_2^{(2)}, \overline{W_2}{}^{(2)})\,dx$$
$$= \sum_{j=1}^{4} \big\langle G_j, \{W_2^{(2)}\}_j^+ \big\rangle_{S_1} + \sum_{j=1}^{8} \langle q_{2,j}^{*(N)}, \{\omega_{2,j}^{(2)}\}^+ \rangle_{S_2^{(N)}} + \overline{\langle q_{2,9}^{*(N)}, \{\omega_{2,9}^{(2)}\}^+ \rangle}_{S_2^{(N)}}$$
$$+ \sum_{j=1}^{3} \Big\langle \big(F_j^{*(1),+} - F_j^{*(1),+}\big), \{V_2^{(1)}\}_j^+ \Big\rangle_{\Sigma_1} + \big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+ \big\rangle_{\Sigma_1}$$

$$+\sum_{j=1,\, j\neq 4}^{7}\Big\langle\big(F_j^{*(2),+}-F_j^{*(2),-}\big),\{W_2^{(2)}\}_j^+\Big\rangle_{\Sigma_2}+\sum_{j=4,8}\langle F_j^{*(2)},\{W_2^{(2)}\}_j^+\rangle_{\Sigma_2}$$

$$+\overline{\langle F_9^{*(2)},\{W_2^{(2)}\}_9^+\rangle}_{\Sigma_2}-\int\limits_{\Omega_{\Sigma_1}}\Psi^{(1)}\cdot V_2^{(1)}\,dx-\sum_{j=1}^{8}\int\limits_{\Omega_{\Sigma_2}}\Psi_j^{(2)}\cdot\omega_{2,j}{}^{(2)}\,dx-\int\limits_{\Omega_{\Sigma_2}}\overline{\Psi_9^{(2)}\cdot\omega_{2,9}{}^{(2)}}\,dx, \quad (4.7)$$

where

$$q_2^{*(N)}=(q_{2,1}^{*(N)},\dots,q_{2,9}^{*(N)}),\quad \Psi^{(2)}=(\Psi_1^{(2)},\dots,\Psi_9^{(2)}),$$
$$\{V_2^{(1)}\}_j^+=\{v_{2,j}^{(1)}\}^+,\quad j=\overline{1,4},\quad \{W_2^{(2)}\}_j^+=\{\omega_{2,j}^{(2)}\}^+,\quad j=\overline{1,9},$$
$$\widetilde{E}_\tau^{(2)}(W_2^{(2)},\overline{W}_2^{(2)}):=B^{(2)}(v^{(2)},\overline{v}^{(2)})+2i\tau\beta_0^{(2)}\operatorname{Im}(\partial_j w_{2,j}^{(2)}\,\overline{w}_{2,4}^{(2)})$$
$$+2i\tau c_0^{(2)}\operatorname{Im}(w_{2,8}^{(2)}\,\overline{w}_{2,4}^{(2)})+\tau^2\Big(\rho_2|w_2^{(2)}|^2+I_0^{(2)}|\phi_2^{(2)}|^2+j_0^{(2)}|w_{2,8}^{(2)}|^2+a^{(2)}|w_{2,4}^{(2)}|^2\Big).$$

Now, let us take first the real part of equality (4.7), and then the imaginary part, where

$$\tau=\sigma+i\omega,\quad \tau^2=(\sigma^2-\omega^2)+2i\sigma\omega,\ \ \sigma>\sigma_0>0,\ \ \omega\in\mathbb{R}.$$

Thus we obtain the following integral equalities:

$$\int\limits_{\Omega_{\Sigma_1}}\Big[\mathcal{E}(v_2^{(1)},\overline{v}_2^{(1)})+(\sigma^2-\omega^2)\rho_1|v_2^{(1)}|^2$$
$$-2\beta_0^{(1)}\omega\operatorname{Im}(\overline{v}_{2,4}^{(1)}\operatorname{div}v_2^{(1)})+k^{(1)}|\operatorname{grad}v_{2,4}^{(1)}|^2+a^{(1)}(\sigma^2-\omega^2)|v_{2,4}^{(1)}|^2\Big]\,dx$$
$$+\int\limits_{\Omega_{\Sigma_2}}\Big[B(v^{(2)},\overline{v}^{(2)})-2\omega\beta_0^{(2)}\operatorname{Im}(\overline{w}_{2,4}^{(2)}\operatorname{div}w_2^{(2)})-2\omega c_0^{(2)}\operatorname{Im}(w_{2,8}^{(2)}\overline{w}_{2,4}^{(2)})$$
$$+(\sigma^2-\omega^2)\Big(\rho_2|w_2^{(2)}|^2+I_0^{(2)}|\phi_2^{(2)}|^2+j_0^{(2)}|w_{2,8}^{(2)}|^2+a^{(2)}|w_{2,4}^{(2)}|^2\Big)\Big]\,dx$$
$$=\operatorname{Re}\sum_{j=1}^{4}\langle G_j,\{W_2{}^{(2)}\}_j^+\rangle_{S_1}+\operatorname{Re}\langle q_2^{*(N)},\{W_2^{(2)}\}^+\rangle_{S_2^{(N)}}$$
$$+\operatorname{Re}\sum_{j=1}^{3}\Big\langle\big(F_j^{*(1),+}-F_j^{*(1),+}\big),\{V_2^{(1)}\}_j^+\Big\rangle_{\Sigma_1}+\operatorname{Re}\langle F_4^{*(1)},\{V_2^{(1)}\}_4^+\rangle_{\Sigma_1}$$
$$+\operatorname{Re}\sum_{j=1,\, j\neq 4}^{7}\Big\langle\big(F_j^{*(2),+}-F_j^{*(2),-}\big),\{W_2^{(2)}\}_j^+\Big\rangle_{\Sigma_2}+\operatorname{Re}\sum_{j=4,8,9}\langle F_j^{*(2)},\{W_2^{(2)}\}_j^+\rangle_{\Sigma_2}$$
$$-\operatorname{Re}\int\limits_{\Omega_{\Sigma_1}}\Psi^{(1)}\cdot V_2^{(1)}\,dx-\operatorname{Re}\int\limits_{\Omega_{\Sigma_2}}\Psi^{(2)}\cdot W_2{}^{(2)}\,dx, \quad (4.8)$$

and

$$\int\limits_{\Omega_{\Sigma_1}}\Big[2\sigma\omega\rho_1|v_2^{(1)}|^2+2a^{(1)}\sigma\omega|v_{2,4}^{(1)}|^2+2\beta_0^{(1)}\sigma\operatorname{Im}(\overline{v}_{2,4}^{(1)}\operatorname{div}v_2^{(1)})\Big]\,dx$$
$$+\int\limits_{\Omega_{\Sigma_2}}\Big[2\sigma\beta_0^{(2)}\operatorname{Im}(\overline{w}_{2,4}^{(2)}\operatorname{div}w_2^{(2)})+2\sigma c_0^{(2)}\operatorname{Im}(w_{2,8}^{(2)}\overline{w}_{2,4}^{(2)})$$
$$+2\sigma\omega\Big(\rho_2|w_2^{(2)}|^2+I_0^{(2)}|\phi_2^{(2)}|^2+j_0^{(2)}|w_{2,8}^{(2)}|^2+a^{(2)}|w_{2,4}^{(2)}|^2\Big)\Big]\,dx$$
$$=\operatorname{Im}\sum_{j=1}^{4}\langle G_j,\{W_2^{(2)}\}_j^+\rangle_{S_1}+\sum_{j=1}^{8}\operatorname{Im}\langle q_{2,j}^{*(N)},\{\omega_{2,j}^{(2)}\}^+\rangle_{S_2^{(N)}}+\operatorname{Im}\overline{\langle q_{2,9}^{*(N)},\{\omega_{2,9}^{(2)}\}^+\rangle}_{S_2^{(N)}}$$

$$+ \operatorname{Im}\sum_{j=1}^{3}\Big\langle\big(F_j^{*(1),+} - F_j^{*(1),+}\big), \{V_2^{(1)}\}_j^+\Big\rangle_{\Sigma_1} + \operatorname{Im}\big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+\big\rangle_{\Sigma_1}$$

$$+ \sum_{j=1,\, j\neq 4}^{7}\operatorname{Im}\Big\langle\big(F_j^{*(2),+} - F_j^{*(2),-}\big), \{W_2^{(2)}\}_j^+\Big\rangle_{\Sigma_2} + \operatorname{Im}\sum_{j=4,8}\big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+\big\rangle_{\Sigma_2}$$

$$+ \operatorname{Im}\overline{\big\langle F_9^{*(2)}, \{W_2^{(2)}\}_9^+\big\rangle}_{\Sigma_2} - \operatorname{Im}\int\limits_{\Omega_{\Sigma_1}}\Psi^{(1)}\cdot V_2^{(1)}\,dx$$

$$- \sum_{j=1}^{8}\operatorname{Im}\int\limits_{\Omega_{\Sigma_2}}\Psi_j^{(2)}\cdot\omega_{2,j}{}^{(2)}\,dx - \operatorname{Im}\int\limits_{\Omega_{\Sigma_2}}\overline{\Psi_9^{(2)}\cdot\omega_{2,9}{}^{(2)}}\,dx. \tag{4.9}$$

Multiplying (4.9) by $\frac{\omega}{\sigma}$ and adding equality (4.8), we get

$$\int\limits_{\Omega_{\Sigma_1}}\Big[\mathcal{E}(v_2^{(1)}, \overline{v}_2^{(1)}) + (\sigma^2+\omega^2)\rho_1|v_2^{(1)}|^2 + k^{(1)}|\operatorname{grad} v_{2,4}^{(1)}|^2 + a^{(1)}(\sigma^2+\omega^2)|v_{2,4}^{(1)}|^2\Big]\,dx$$

$$+ \int\limits_{\Omega_{\Sigma_2}}\Big[B(v^{(2)}, \overline{v}^{\,(2)}) + (\sigma^2+\omega^2)\Big(\rho_2|w_2^{(2)}|^2 + I_0^{(2)}|\phi_2^{(2)}|^2 + j_0^{(2)}|w_{2,8}^{(2)}|^2 + a^{(2)}|w_{2,4}^{(2)}|^2\Big)\Big]\,dx$$

$$= \operatorname{Re}\sum_{j=1}^{4}\big\langle G_j, \{W_2{}^{(2)}\}_j^+\big\rangle_{S_1} + \operatorname{Re}\big\langle q_2^{*(N)}, \{W_2^{(2)}\}^+\big\rangle_{S_2^{(N)}} + \operatorname{Re}\sum_{j=1}^{3}\Big\langle\big(F_j^{*(1),+} - F_j^{*(1),-}\big), \{V_2^{(1)}\}_j^+\Big\rangle_{\Sigma_1}$$

$$+ \operatorname{Re}\big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+\big\rangle_{\Sigma_1} + \operatorname{Re}\sum_{j=1,\, j\neq 4}^{7}\Big\langle\big(F_j^{*(2),+} - F_j^{*(2),-}\big), \{W_2^{(2)}\}_j^+\Big\rangle_{\Sigma_2}$$

$$+ \operatorname{Re}\sum_{j=4,8,9}\big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+\big\rangle_{\Sigma_2} - \operatorname{Re}\int\limits_{\Omega_{\Sigma_1}}\Psi^{(1)}\cdot V_2^{(1)}\,dx - \operatorname{Re}\int\limits_{\Omega_{\Sigma_2}}\Psi^{(2)}\cdot W_2{}^{(2)}\,dx$$

$$+ \frac{\omega}{\sigma}\operatorname{Im}\sum_{j=1}^{4}\big\langle G_j, \{W_2{}^{(2)}\}_j^+\big\rangle_{S_1} + \frac{\omega}{\sigma}\sum_{j=1}^{8}\operatorname{Im}\big\langle q_{2,j}^{*(N)}, \{\omega_{2,j}^{(2)}\}^+\big\rangle_{S_2^{(N)}} + \frac{\omega}{\sigma}\operatorname{Im}\overline{\big\langle q_{2,9}^{*(N)}, \{\omega_{2,9}^{(2)}\}^+\big\rangle}_{S_2^{(N)}}$$

$$+ \frac{\omega}{\sigma}\operatorname{Im}\sum_{j=1}^{3}\Big\langle\big(F_j^{*(1),+} - F_j^{*(1),-}\big), \{V_2^{(1)}\}_j^+\Big\rangle_{\Sigma_1} + \frac{\omega}{\sigma}\operatorname{Im}\big\langle F_4^{*(1)}, \{V_2^{(1)}\}_4^+\big\rangle_{\Sigma_1}$$

$$+ \frac{\omega}{\sigma}\operatorname{Im}\sum_{j=1,\, j\neq 4}^{7}\Big\langle\big(F_j^{*(2),+} - F_j^{*(2),-}\big), \{W_2^{(2)}\}_j^+\Big\rangle_{\Sigma_2} + \frac{\omega}{\sigma}\operatorname{Im}\sum_{j=4,8}\big\langle F_j^{*(2)}, \{W_2^{(2)}\}_j^+\big\rangle_{\Sigma_2}$$

$$+ \frac{\omega}{\sigma}\operatorname{Im}\overline{\big\langle F_9^{*(2)}, \{W_2^{(2)}\}_9^+\big\rangle}_{\Sigma_2} - \frac{\omega}{\sigma}\operatorname{Im}\int\limits_{\Omega_{\Sigma_1}}\Psi^{(1)}\cdot V_2^{(1)}\,dx$$

$$- \frac{\omega}{\sigma}\sum_{j=1}^{8}\operatorname{Im}\int\limits_{\Omega_{\Sigma_2}}\Psi_j^{(2)}\cdot\omega_{2,j}{}^{(2)}\,dx - \frac{\omega}{\sigma}\operatorname{Im}\int\limits_{\Omega_{\Sigma_2}}\overline{\Psi_9^{(2)}\cdot\omega_{2,9}{}^{(2)}}\,dx \tag{4.10}$$

From equality (4.10), when $|\tau|\to\infty$, we obtain following estimates:

$$c_1\Big(\int\limits_{\Omega_{\Sigma_1}}\mathcal{E}(v_2^{(1)}, \overline{v}_2^{(1)})\,dx + \|v_2^{(1)}\|^2_{[L_2(\Omega_{\Sigma_1})]^3} + \|v_{2,4}^{(1)}\|^2_{H^1(\Omega_{\Sigma_1})}\Big)$$

$$+ c_2\Big(\int\limits_{\Omega_{\Sigma_2}}B(v^{(2)}, \overline{v}^{\,(2)})\,dx + \|w_2^{(2)}\|^2_{[L_2(\Omega_{\Sigma_2})]^3} + \|w_{2,4}^{(2)}\|^2_{L_2(\Omega_{\Sigma_2})} + \|\phi_2^{(2)}\|^2_{[L_2(\Omega_{\Sigma_2})]^3} + \|w_{2,8}^{(2)}\|^2_{L_2(\Omega_{\Sigma_2})}\Big)$$

$$\leq \sum_{j=1}^{4} \|G_j\|_{H^{-\frac{1}{2}}(S_1)} \big\|\{W_2{}^{(2)}\}_j^+\big\|_{H^{\frac{1}{2}}(S_1)} + \|q_2^{*(N)}\|_{[H^{-\frac{1}{2}}(S_2^{(N)})]^9} \big\|\{W_2{}^{(2)}\}^+\big\|_{[H^{\frac{1}{2}}(S_2^{(N)})]^9}$$

$$+ \sum_{j=1}^{3} \big\|F_j^{*(1),+} - F_j^{*(1),-}\big\|_{H^{-\frac{1}{2}}(\Sigma_1)} \big\|\{V_2^{(1)}\}_j^+\big\|_{H^{\frac{1}{2}}(\Sigma_1)}$$

$$+ \|F_4^{*(1)}\|_{H^{-\frac{1}{2}}(\Sigma_1)} \big\|\{V_2{}^{(1)}\}_4^+\big\|_{H^{\frac{1}{2}}(\Sigma_1)} + \sum_{j=1,\ j\neq 4}^{7} \big\|F_j^{*(2),+} - F_j^{*(2),-}\big\|_{H^{-\frac{1}{2}}(\Sigma_2)} \big\|\{W_2^{(2)}\}_j^+\big\|_{H^{\frac{1}{2}}(\Sigma_2)}$$

$$+ \sum_{j=4,8,9} \|F_j^{*(2)}\|_{H^{-\frac{1}{2}}(\Sigma_2)} \big\|\{W_2^{(2)}\}_j^+\big\|_{H^{\frac{1}{2}}(\Sigma_2)}$$

$$+ \|\Psi^{(1)}\|_{[L_2(\Omega_{\Sigma_1})]^4} \|V_1{}^{(2)}\|_{[L_2(\Omega_{\Sigma_1})]^4} + \|\Psi^{(2)}\|_{[L_2(\Omega_{\Sigma_2})]^9} \|W_2{}^{(2)}\|_{[L_2(\Omega_{\Sigma_2})]^9}$$

$$+ \frac{|\omega|}{\sigma} \sum_{j=1}^{4} \|G_j\|_{H^{-\frac{1}{2}}(S_1)} \big\|\{W_2{}^{(2)}\}_j^+\big\|_{H^{\frac{1}{2}}(S_1)} + \frac{|\omega|}{\sigma} \|q_2^{*(N)}\|_{[H^{-\frac{1}{2}}(S_2^{(N)})]^9} \big\|\{W_2{}^{(2)}\}^+\big\|_{[H^{\frac{1}{2}}(S_2^{(N)})]^9}$$

$$+ \frac{|\omega|}{\sigma} \sum_{j=1}^{3} \big\|F_j^{*(1),+} - F_j^{*(1),-}\big\|^2_{H^{-\frac{1}{2}}(\Sigma_1)} \big\|\{V_2^{(1)}\}_j^+\big\|^2_{H^{\frac{1}{2}}(\Sigma_1)} + \frac{|\omega|}{\sigma} \|F_4^{*(1)}\|_{H^{-\frac{1}{2}}(\Sigma_1)} \big\|\{V_2{}^{(1)}\}_4^+\big\|_{H^{\frac{1}{2}}(\Sigma_1)}$$

$$+ \frac{|\omega|}{\sigma} \sum_{j=1,\ j\neq 4}^{7} \big\|F_j^{*(2),+} - F_j^{*(2),-}\big\|_{H^{-\frac{1}{2}}(\Sigma_2)} \big\|\{W_2^{(2)}\}_j^+\big\|_{H^{\frac{1}{2}}(\Sigma_2)}$$

$$+ \frac{|\omega|}{\sigma} \sum_{j=4,8,9} \|F_j^{*(2)}\|_{H^{-\frac{1}{2}}(\Sigma_2)} \big\|\{W_2^{(2)}\}_j^+\big\|_{H^{\frac{1}{2}}(\Sigma_2)}$$

$$+ \frac{|\omega|}{\sigma} \|\Psi^{(1)}\|_{[L_2(\Omega_{\Sigma_1})]^4} \|V_2^{(1)}\|_{[L_2(\Omega_{\Sigma_1})]^4} + \frac{|\omega|}{\sigma} \|\Psi^{(2)}\|_{[L_2(\Omega_{\Sigma_2})]^9} \|W_2{}^{(2)}\|_{[L_2(\Omega_{\Sigma_2})]^9}, \quad (4.11)$$

where $c_1 := \min\{\rho_1, k^{(1)}, a^{(1)}, 1\}$ and $c_2 := \min\{\rho_2, I_0^{(2)}, j_0^{(2)}, a^{(2)}, 1\}$.

Now, in the left part of inequality (4.11) we use Korn's inequality, the positive-definiteness of the form $B(v^{(2)}, v^{(2)})$ and the Poincarè inequality, and in the right part of the same inequality we use the trace theorem and estimates (4.3), then we get

$$\Big(\|V_2^{(1)}\|_{[H^1[(\Omega_{\Sigma_1})]^4} + \|W_2^{(2)}\|_{H^1[(\Omega_{\Sigma_2})]^9}\Big)^2 \leq c|\tau|^{-M+3}\Big(\|V_2^{(1)}\|_{[H^1[(\Omega_{\Sigma_1})]^4} + \|W_2^{(2)}\|_{[H^1[(\Omega_{\Sigma_2})]^9}\Big).$$

Therefore, we obtain

$$\|V_2^{(1)}\|_{[H^1[(\Omega_{\Sigma_1})]^4} + \|W_2^{(2)}\|_{[H^1[(\Omega_{\Sigma_2})]^9} \leq c|\tau|^{-M+3} \ \text{ for } \ |\tau| \to \infty, \quad (4.12)$$

where c is a positive number, which does not depend on the complex parameter τ.

Thus, in view of (4.2) and (4.12), we conclude

$$\|\widetilde{U}^{(1)}\|_{[H^1[(\Omega_{\Sigma_1})]^4} \leq c|\tau|^{-M+3}, \quad \|\widetilde{U}^{(2)}\|_{[H^1[(\Omega_{\Sigma_2})]^9} \leq c|\tau|^{-M+3} \ \text{ for } \ |\tau| \to \infty. \quad (4.13)$$

In its turn, from estimates (4.13) with $M > m + 4$ and inverse Laplace transform

$$U^{(1)}(\,\cdot\,, t) = \frac{1}{2\pi i} \int\limits_{\alpha-i\infty}^{\alpha+\infty} e^{\tau t} \widetilde{U}^{(1)}(\,\cdot\,, \tau)\, d\tau, \quad \alpha > a,$$

$$U^{(2)}(\,\cdot\,, t) = \frac{1}{2\pi i} \int\limits_{\alpha-i\infty}^{\alpha+\infty} e^{\tau t} \widetilde{U}^{(2)}(\,\cdot\,, \tau)\, d\tau, \quad \alpha > a$$

of solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ of the boundary transmission pseudo-oscillation problem, we find that

$$U^{(1)} \in C_a^m([0, +\infty), [H^1(\Omega_{\Sigma_1})]^4),$$

$$U^{(2)} \in C_a^m([0,+\infty),[H^1(\Omega_{\Sigma_2})]^9)$$

with $m \geq 2$.

Therefore, we arrive at the following existence and uniqueness results for the original transmission dynamical problem $(TM)_{c,t}$.

Theorem 4.3. *Let $S_1, S_2 \in C^\infty$, and*

$$\begin{gathered}
\Phi_1 \in C_{0,a}^{m+5}([0,+\infty),[L_2(\Omega_{\Sigma_1})]^4), \quad \Phi_2 \in C_{0,a}^{m+5}([0,+\infty),[L_2(\Omega_{\Sigma_2})]^9),\\
f_j^{(1)} \in C_{0,a}^{m+7}([0,+\infty),H^{\frac12}(S_1)), \quad f_j^{(2} \in C_{0,a}^{m+7}([0,+\infty),H^{\frac12}(S_1)), \;\; j=\overline{1,4},\\
Q_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty),H^{\frac12}(S_1)), \;\; j=\overline{5,9},\\
p_2^{(D)} \in C_{0,a}^{m+7}([0,+\infty),[H^{\frac12}(S_2^{(D)})]^9), \quad q_2^{(N)} \in C_{0,a}^{m+7}([0,\infty),[H^{-\frac12}(S_N)]^9);\\
F_j^{(1),\pm} \in C_{0,a}^{m+7}([0,+\infty),H^{-\frac12}(\Sigma_1)), \;\; j=1,2,3,\\
F_j^{(2),\pm} \in C_{0,a}^{m+7}([0,+\infty),H^{-\frac12}(\Sigma_2)), \;\; j=1,2,3,5,6,7,\\
G_4^{(1)} \in C_{0,a}^{m+7}([0,+\infty),\widetilde H^{-\frac12}(\Sigma_1)), \quad F_4^{(1)} \in C_{0,a}^{m+7}([0,+\infty),\widetilde H^{-\frac12}(\Sigma_1)),\\
G_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty),\widetilde H^{-\frac12}(\Sigma_2)), \quad F_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty),\widetilde H^{-\frac12}(\Sigma_2)), \;\; j=4,8,9,
\end{gathered}$$

and let the compatibility conditions

$$\begin{aligned}
&F_j^{(1),+} - F_j^{(1),-} \in C_{0,a}^{m+7}([0,+\infty),\widetilde H^{-\frac12}(\Sigma_1)), \;\; j=1,2,3,\\
&F_j^{(2),+} - F_j^{(2),-} \in C_{0,a}^{m+7}([0,+\infty),\widetilde H^{-\frac12}(\Sigma_2)), \;\; j=1,2,3,5,6,7, \;\; m \geq 2,
\end{aligned}$$

be satisfied.

Then the transmission dynamical problem $(TM)_{c,t}$ has a unique solution $(U^{(1)}, U^{(2)})$ in the space

$$C_a^m([0,+\infty),[H^1(\Omega_{\Sigma_1})]^4) \times C_a^m([0,+\infty),[H^1(\Omega_{\Sigma_2})]^9).$$

Let us introduce the notation

$$\begin{aligned}
C_{0,a}^m([0,+\infty),C^\infty(\overline\Omega_q)) &:= \bigcap_{k=1}^{\infty} C_{0,a}^m([0,+\infty),C^k(\overline\Omega_q)), \;\; q=1,2,\\
C_{0,a}^m([0,+\infty),C^\infty(S_q)) &:= \bigcap_{k=1}^{\infty} C_{0,a}^m([0,+\infty),C^k(S_q)), \;\; q=1,2,\\
C_{0,a}^m([0,+\infty),C^\infty(\overline\Sigma_q)) &:= \bigcap_{k=1}^{\infty} C_{0,a}^m([0,+\infty),C^k(\overline\Sigma_q)), \;\; q=1,2,
\end{aligned}$$

and denote by

$$C_{0,a}^m([0,+\infty),C_0^\infty(\overline\Sigma_q)) := C_{0,a}^m([0,+\infty),C^\infty(\overline\Sigma_q)), \;\; q=1,2,$$

the class of functions vanishing with all tangential (to Σ_q) derivatives at $\ell_c^{(q)} = \partial\Sigma_q$, $q=1,2$.

Using the approach developed in [6] and [8], we can obtain the estimates similar to [6] of the first coefficient and reminder terms of the asymptotic expansion of solutions with respect to the complex parameter τ. Then by using the inverse Laplace transform in the asymptotic expansion of solution we can obtain the following optimal regularity result for the transmission dynamical problem $(TM)_{c,t}$ near the lines $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$ and $\ell_c^{(q)} = \partial\Sigma_q$, $q=1,2$.

Theorem 4.4. *Suppose that $S_1, S_2 \in C^\infty$ and*

$$\begin{gathered}
\Phi_1 \in C_{0,a}^{m+5}([0,+\infty),[C^\infty(\overline\Omega_1)]^4), \quad \Phi_2 \in C_{0,a}^{m+5}([0,+\infty),[C^\infty(\overline\Omega_2)]^9),\\
f_j^{(1)} \in C_{0,a}^{m+7}([0,+\infty),C^\infty(S_1)), \quad f_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty),\in C^\infty(S_1)), \;\; j=\overline{1,4},
\end{gathered}$$

$$Q_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(S_1)), \;\; j = \overline{5,9},$$
$$p_2^{(D)} \in C_{0,a}^{m+7}([0,+\infty), [C^\infty(\overline{S}_2^{(D)})]^9), \quad q_2^{(N)} \in C_{0,a}^{m+7}([0,+\infty), [C^\infty(\overline{S}_2^{(N)})]^9),$$
$$F_j^{(1),\pm} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(\overline{\Sigma}_1)), \;\; j = 1,2,3,$$
$$F_j^{(2),\pm} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(\overline{\Sigma}_2)), \;\; j = 1,2,3,5,6,7,$$
$$G_4^{(1)} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(\overline{\Sigma}_1)), \quad F_4^{(1)} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(\overline{\Sigma}_1)),$$
$$G_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(\overline{\Sigma}_2)), \quad F_j^{(2)} \in C_{0,a}^{m+7}([0,+\infty), C^\infty(\overline{\Sigma}_2)), \;\; j = 4,8,9,$$
$$F_j^{(1),+} - F_j^{(1),-} \in C_{0,a}^{m+7}([0,+\infty), C_0^\infty(\overline{\Sigma}_1)), \;\; j = 1,2,3,$$
$$F_j^{(2),+} - F_j^{(2),-} \in C_{0,a}^{m+7}([0,+\infty), C_0^\infty(\overline{\Sigma}_2)), \;\; j = 1,2,3,5,6,7, \;\; m \geq 2.$$

Let $(U^{(1)}, U^{(2)})$ be the unique solution to the transmission dynamical problem $(TM)_{c,\tau}$.

Then $u^{(1)}$ and $U^{(2)}$ have $C_a^m([0,+\infty), [C^{\frac{1}{2}}]^3)$-smoothness and $C_a^m([0,+\infty), [C^{\frac{1}{2}}]^9)$-smoothness, respectively, in one-sided interior and exterior neighborhoods of the surfaces $S_0^{(1)}$ and $S_0^{(2)}$, respectively, and $\vartheta^{(1)}$ has the $C_a^m([0,+\infty), C^{\frac{3}{2}})$-smoothness in one-sided interior and exterior neighborhoods of the surface $S_0^{(1)}$. While

(1) *If $d < 0$, then the vector $U^{(2)}$ belongs to the class $C_a^m([0,+\infty), [C^{\gamma_1}]^9)$ in the neighborhood of the line $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$ where $\gamma_1 = \frac{1}{2} - \frac{1}{\pi} \operatorname{arctg} 2\sqrt{-d}$, γ_1 depends on the material constants, does not depend on the geometry of the exceptional line ℓ_m and may take any values from the interval $(0, \frac{1}{2})$;*

(2) *If $d \geq 0$, then the vector $U^{(2)}$ belongs to the class $C_a^m([0,+\infty), [C^{\frac{1}{2}}]^9)$ in the a neighborhood of the line ℓ.*

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(Received 21.07.2023; accepted 06.09.2023)

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