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**Noureddine Bouteraa**

# EXISTENCE AND UNIQUENESS OF ZAKHAROV–KUZNETSOV–BURGERS EQUATION WITH CAPUTO–FABRIZIO FRACTIONAL DERIVATIVE

**Abstract.** In this work, we discuss the existence and uniqueness results of a general class of Zakharov–Kuznetsov–Burgers equation. We suggest the generalization via the Caputo–Fabrizio fractional derivative. We introduce some conditions for the existence and uniqueness of solutions and to obtain them, we utilize the concept of the fixed-point theorem.

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**Key words and phrases.** Generalized Zakharov–Kuznetsov–Burgers equation, existence, fractional derivative, Banach fixed point.

**რეზიუმე.** ნაშრომში განხილულია ზოგადი კლასის ზახაროვ-კუზნეცოვ-ბურგერის განტოლების არსებობისა და ერთადერთობის შედეგები. შემოთავაზებულია განზოგადება კაპუტო-ფაბრიციოს წილადი წარმოებულის გამოყენებით. შემოტანილია ამონახსნების არსებობისა და ერთადერთობის ზოგიერთი პირობა და მათ მისაღებად გამოყენებულია უძრავი წერტილის თეორემის კონცეფცია.

# 1 Introduction

In this paper, we are interested in the existence and uniqueness of solutions for a general class of Zakharov–Kuznetsov–Burgers (ZKB) equation

$$\begin{aligned} {}_0^{CF}D_t^{\alpha}u(x,y,t) + au(x,y,t)u_x(x,y,t) + bu_{xxx}(x,y,t) + cu_{xyy}(x,y,t) \\ = du_{xx}(x,y,t) + eu_{yy}(x,y,t), \quad u(x,y,0) = \varphi(x,y), \;\; u(0,0,t) = g(t), \end{aligned} \tag{1.1}$$

where ${}_0^{CF}D_t^{\alpha}$ denotes the Caputo–Fabrizio derivative, $\alpha \in (0,1)$, $t > 0$, $(a,b,c,d) \in \mathbb{R}^4$. It is known that many famous nonlinear differential equations (NDEs) are the special cases of equation (1.1). For example, if $\alpha = 1$, $c = 0$, then equation (1.1) is the Korteweg-de Vries (KdV) equation. If $\alpha = 1$, then (1.1) is the Zakharov–Kuznetsov (ZK) equation [19].

One of the most important partial differential equations which has a vast application in solitary wave's theory is the ZKB equation, also it plays an important role in electromagnetics and describes the propagation of Langmuir waves in an ionized plasma. Some of its modified forms illustrate the interactions of small amplitude, high frequency waves with acoustic waves. There are many useful articles for finding the solitary waves solutions (especially, for ZKB equation) (see, e.g., [7, 13, 14]).

Recently, fractional derivative operators became significant research topics due to their wide applications in various areas including mathematical, physical, life sciences and engineering problems. Furthermore, the recent decade shows that the fractional-order derivatives work better in modeling real phenomena than integer-order derivatives and include the system of internal memory. For example, in [13], the authors studied the mathematical model for AH1N1/09 influenza transmission by using the Caputo–Fabrizio fractional derivative. To cite only a few of this operator's applications, we refer to [2–4] and the references therein. Taking $\alpha = 1$, Zakharov and Kuznetsov established the non-linear evolution equation which is related to nonlinear ion-acoustic waves in magnetized plasma including cold ions and hot isothermal electrons. We can see some useful papers in the literature to study the applications of this equation (see [14, 19] for more details). This equation by omitting the details of derivatives can be written as

$$u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy} = 0,$$

where the constant quantities which involve the physical quantities $x$, $y$ and $t$ are independent variables and $u(x,y,t)$ is the dependent variable indicating the wave profile. El-Bedwehy and Moslem acquired the ZKB equation from an electron-positron-ion plasma [5].

It is an interesting and meaningful subject to find exact solutions of NDEs. During the past few years, there has been extraordinary progress in constructing explicit solutions of NDEs, for instance, the sine-cosine method [16], the modified simple equation method [8], the Lie group method [6, 11]. etc. A modified type of this equation is used in various functional approaches. Another modification requires replacing of ordinary differential equations by fractional differential equations (FDE). It is worth being pointed out that nowadays a great attention is devoted to develop extensions and generalizations of fractional differential equations and establish the existence results for such problems. Our motivation comes from the fact that very little is known about general class of the Zakharov–Kuznetsov–Burgers equation. Furthermore, as far as we know, the study of the existence of solutions of this kind of problems of time fractional (Caputo–Fabrizio) ZKB equation has not yet been studied. In our investigation, we focus on the fractional differential equation containing Caputo–Fabrizio fractional derivative (1.1).

The present paper is organized as follows. In Section 2, we present basic definitions, lemmas and preliminary results that will be needed in the sequel. In Section 3, we set up the existence and uniqueness for problem (1.1).

# 2 Preliminaries

In this section, we introduce the following lemmas and definitions which will be used in the sequel.

**Definition 2.1** ([11])**.** Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$. The Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$ is defined by

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{1-\alpha} \int\limits_a^t f'(u) e^{-\alpha \frac{t-u}{1-\alpha}}\, du, $$

where $t \geq 0$, $M(\alpha)$ is a normalization function and $M(0) = M(1) = 1$ (see [1]).

If $f \notin H^1(a,b)$ and $\alpha \in (0,1)$, then this derivative for $f \in L^1(-\infty, b)$ is given by

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{\alpha M(\alpha)}{1-\alpha} \int\limits_{-\infty}^b (f(t) - f(u)) e^{-\alpha \frac{t-u}{1-\alpha}}\, du. \tag{2.1} $$

**Remark 2.1** ([1])**.** If $\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty]$, $\alpha = \frac{1}{1+\sigma} \in [0,1]$. Then equation (2.1) becomes

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{N(\sigma)}{\sigma} \int\limits_a^{\alpha} f'(u) e^{-\alpha \frac{t-u}{1-\alpha}}\, du, \quad N(0) = N(\infty) = 1. \tag{2.2} $$

Furthermore,

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \lim_{\sigma \to 0} \frac{1}{\sigma} e^{-\frac{t-u}{\sigma}} = \delta(u-t). $$

The corresponding anti-derivative turned out to be important. An integral connected to the Caputo–Fabrizio derivative of fractional order was suggested by Nieto and Losada [1] (see the definition below).

**Definition 2.2** (Caputo–Fabrizio fractional integral)**.** The Caputo–Fabrizio fractional integral of order $\alpha \in (0,1)$ of a function $f$ is defined by

$$ {}^{CF}I_t^{\alpha}(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int\limits_0^t f(s)\, ds, \quad t \geq 0. $$

**Remark 2.2.** The remainder occurring in the above definition of the fractional integral of Caputo type of the function of order $\alpha \in (0,1)$ is a mean between the function $f$ and its integral of order one. This, consequently, enforces

$$ \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1, $$

where

$$ M(\alpha) = \frac{2\alpha}{2-\alpha}, \quad \alpha \in [0,1], $$

so, Nieto and Losada noticed that the definition of the Caputo–Fabrizio derivative of order $\alpha \in (0,1)$ can be reformulated by

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{1}{1-\alpha} \int\limits_a^{\alpha} f'(u) e^{-\alpha \frac{t-u}{1-\alpha}}\, du. $$

**Theorem 2.1** ([1])**.** *Let $0 < \alpha < 1$. For the new Caputo–Fabrizio derivative of fractional order, if the function $f(t)$ is such that $f^{(s)}(a) = 0$, then we have*

$$ {}_0^{CF}D_t^{\alpha}\big({}_0^{CF}D_t^{n} f(t)\big) = {}_0^{CF}D_t^{n}\big({}_0^{CF}D_t^{\alpha} f(t)\big). $$

## 3 Main results

In this section, we focus on the time fractional (Caputo–Fabrizio derivative) Zakharov–Kuznetsov–Burgers equation (1.1) and show the existence of exact solution. For simplicity, let

$$G(x,y,t,u) = -au(x,y,t)u_x(x,y,t) - bu_{xxx}(x,y,t) - cu_{xyy}(x,y,t) + du_{xx}(x,y,t) + eu_{yy}(x,y,t). \quad (3.1)$$

Integrating (1.1) in the sense of Definition 2.2 and taking into account (3.1), we obtain

$$\begin{aligned} u(x,y,t) - u(x,y,0) &= {}^{CF}I_0^{\alpha} G(x,y,t,u) \\ &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t G(x,y,s,u)\, ds, \quad t \geq 0, \end{aligned} \quad (3.2)$$

where the initial condition is $u(x,y,0)$.

**Lemma 3.1.** *The function $G$ satisfies the Lipschitz condition.*

*Proof.* Let $u$ and $v$ be two bounded functions. We have

$$\begin{aligned} \|G(x,y,t,u) - G(x,y,t,v)\| = \Big\| & av(x,y,t) - au(x,y,t)u_x(x,y,t) \\ & + bv_{xxx}(x,y,t) - bu_{xxx}(x,y,t) + cu_{xyy}(x,y,t) - cv_{xyy}(x,y,t) \\ & + du_{xx}(x,y,t) - dv_{xx}(x,y,t) + eu_{yy}(x,y,t) - ev_{yy}(x,y,t)\Big\|. \end{aligned}$$

Applying the triangular inequality to the above equation, we obtain

$$\begin{aligned} \|G(x,y,t,u) - G(x,y,t,v)\| \leq & \big\|av(x,y,t)v_x(x,y,t) - au(x,y,t)u_x(x,y,t)\big\| \\ & + \big\|bv_{xxx}(x,y,t) - bu_{xxx}(x,y,t)\big\| + \big\|cu_{xyy}(x,y,t) - cv_{xyy}(x,y,t)\big\| \\ & + \big\|du_{xx}(x,y,t) - dv_{xx}(x,y,t)\big\| + \big\|eu_{yy}(x,y,t) - ev_{yy}(x,y,t)\big\|. \end{aligned} \quad (3.3)$$

We will estimate all the terms appearing in inequality (3.3) step by step:

$$\begin{aligned} \big\|av(x,y,t)v_x(x,y,t) - au(x,y,t)u_x(x,y,t)\big\| &\leq \frac{a}{2}\,\big\|\partial_x((v(x,y,t))^2 - (u(x,y,t))^2)\big\| \\ &\leq \frac{a\rho_1}{2}\,\|v(x,y,t) - u(x,y,t)\|\,\Big\|\sum_{j=0}^{1} C_1^j v^j(x,y,t)u^{1-j}(x,y,t)\Big\|, \end{aligned} \quad (3.4)$$

where $C_1^j = \frac{j!}{1!(j-1)!}$. Since the two functions are bounded, there exist two positive numbers $\eta$ and $\mu$ such that for all $(x,y,t)$

$$\begin{aligned} & \frac{a\rho_1}{2}\,\|v(x,y,t) - u(x,y,t)\|\,\Big\|\sum_{j=0}^{1} C_1^j v^j(x,y,t)u^{1-j}(x,y,t)\Big\| \\ & \qquad < \frac{a\rho_1}{2}\,\|v-u\|\,\Big\|\sum_{j=0}^{1} C_1^j \eta^j \mu^{1-j}\Big\| = \frac{a\rho_1}{2}\,\|v-u\|(\eta+\mu). \end{aligned} \quad (3.5)$$

Thus, from (3.4) and (3.5), we have

$$\big\|av(x,y,t)v_x(x,y,t) - au(x,y,t)u_x(x,y,t)\big\| \leq \frac{a\rho_1\lambda_1}{2}\,\|u-v\|, \quad (3.6)$$

where $\lambda_1 = \eta + \mu$.

$$\|bv_{xxx}(x,y,t) - bu_{xxx}(x,y,t)\| = b\|\partial_{xxx}(v(x,y,t) - u(x,y,t))\| \leq b\rho_2\|u-v\|, \quad (3.7)$$
$$\|du_{xx}(x,y,t) - dv_{xx}(x,y,t)\| = d\|\partial_{xx}(u(x,y,t) - v(x,y,t))\| \leq d\rho_3\|u-v\|, \quad (3.8)$$
$$\|eu_{yy}(x,y,t) - ev_{yy}(x,y,t)\| = d\|\partial_{yy}(u(x,y,t) - v(x,y,t))\| \leq e\rho_4\|u-v\|, \quad (3.9)$$

and

$$\|cu_{xyy}(x,y,t) - cv_{xyy}(x,y,t)\| = c\|\partial_{xyy}(u(x,y,t) - v(x,y,t))\| \le c\rho_5\|u - v\|. \tag{3.10}$$

Substituting (3.4)–(3.10) into (3.3), we obtain

$$\|G(x,y,t,u) - G(x,y,t,v)\| \le L\|u - v\|, \tag{3.11}$$

where

$$L = \frac{a\lambda_1\rho_1}{2} + b\rho_2 + c\rho_5 + d\rho_3 + e\rho_4.$$

This completes the proof. □

Now, taking into account the function $G$ considered here as a nonlinear kernel, equation (3.2) can be converted to the following recursive formula (Picard's repetitive series):

$$\begin{cases} u_n(x,y,t) = \dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \big(G(x,y,t,u_{n-1}) - G(x,y,t,u_{n-2})\big) \\ \qquad\qquad + \dfrac{2\alpha}{(2-\alpha)M(\alpha)} \displaystyle\int_0^t \big(G(x,y,t,u_{n-1}) - G(x,y,t,u_{n-2})\big)\, ds, \\ u_0(x,y,t) = u(x,y,0). \end{cases}$$

We consider the variance between the two consecutive terms

$$\begin{aligned} V_n(x,y,t) &= u_n(x,y,t) - u_{n-1}(x,y,t) \\ &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \big(G(x,y,t,u_{n-1}) - G(x,y,t,u_{n-2})\big) \\ &\qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \big(G(x,y,s,u_{n-1}) - G(x,y,t,u_{n-2})\big)\, ds. \end{aligned}$$

Thanks to Lemma 3.1, we have

$$\begin{aligned} \|V_n(x,y,t)\| &= \|u_n(x,y,t) - u_{n-1}(x,y,t)\| \\ &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \big\|G(x,y,t,u_{n-1}) - G(x,y,t,u_{n-2})\big\| \\ &\qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \big\|G(x,y,s,u_{n-1}) - G(x,y,t,u_{n-2})\big\|\, ds \\ &\le \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} \|V_{n-1}\| + \frac{2L\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|V_{n-1}\|\, ds. \end{aligned} \tag{3.12}$$

**Lemma 3.2.** *Under the Lipschitz condition, we have the following estimate:*

$$\|V_n(x,y,t)\| \le u(x,y,0)\left[\Big(\frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)}\Big)^n + \Big(\frac{2LT\alpha}{(2-\alpha)M(\alpha)}\Big)^n\right]. \tag{3.13}$$

*Proof.* From (3.12) and by induction, we get (3.13). □

Using the Picard–Lindelof approach and the Banach fixed-point theorem, we prove the existence of exact solution.

**Theorem 3.1.** *Under assumptions (3.11), equation (3.1) has the exact solution.*

*Proof.* We assume that the exact solution is given by

$$\begin{aligned} u(x,y,t) &= u_n(x,y,t) - P_n(x,y,t) \\ &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u-P_n(x,y,t)) \\ &\qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t G(x,y,s,u-P_n(x,y,t))\, ds. \end{aligned} \tag{3.14}$$

In this case, the function $P_n(x,y,t)$ tends to zero for large $n$. From (3.14) and (3.2), we have

$$\begin{aligned} &u(x,y,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t G(x,y,s,u)\, ds \\ &\quad = P_n(x,y,t) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u) \\ &\qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \big(G(x,y,s,u-P_n(x,y,t)) - G(x,y,t,u)\big)\, ds. \end{aligned}$$

Then

$$\begin{aligned} &\left\| u(x,y,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t G(x,y,s,u)\, ds \right\| \\ &\qquad \le \|P_n(x,y,t)\| + \Big(\frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2LT\alpha}{(2-\alpha)M(\alpha)}\Big)\|P_n(x,y,t)\|. \end{aligned}$$

Taking the limit as $n$ tends to infinity, we obtain

$$\left\| u(x,y,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t G(x,y,s,u)\, ds \right\| = 0,$$

which implies

$$u(x,y,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, G(x,y,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t G(x,y,s,u)\, ds.$$

This completes the proof. □

Now, we prove the uniqueness of the exact solution of equation (3.2).

**Theorem 3.2.** *If the condition*

$$\Big(1 - \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2LT\alpha}{(2-\alpha)M(\alpha)}\Big) \neq 0$$

*holds, then equation (3.2) has a unique exact solution.*

*Proof.* We assume that there exists another solution of equation (3.2), say $v(x,y,t)$. We would have

$$\begin{aligned} u(x,y,t) - v(x,y,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, (G(x,y,t,u) - G(x,y,t,v)) \\ &\qquad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (G(x,y,s,u) - G(x,y,s,v))\, ds. \end{aligned}$$

Thus

$$\|u(x,y,t)-v(x,y,t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \|G(x,y,t,u)-G(x,y,t,v)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|G(x,y,s,u)-G(x,y,s,v)\|\, ds.$$

Due to Lemma 3.1, we have

$$\|u-v\| \leq \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} \|u-v\| + \frac{2LT\alpha}{(2-\alpha)M(\alpha)} \|u-v\|.$$

This leads to

$$\|u-v\|\Big(1-\frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)}+\frac{2LT\alpha}{(2-\alpha)M(\alpha)}\Big) \leq 0.$$

That is,

$$\|u-v\|=0, \quad \text{i.e.,} \quad u=v.$$

This completes the proof. □

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**Author's addresses:**

**Noureddine Bouteraa**

1. Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1 – Ahmed Benbella, Es Senia, Algeria

2. Oran Graduate School of Economics, Bir El Djir, Algeria

*E-mails:* bouteraa-27@hotmail.fr, bouteraanooreddine@gmail.com