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Mesfin Mekuria Woldaregay, Gemechis File Duressa

NUMERICAL TREATMENT OF SINGULARLY PERTURBED PARABOLIC DIFFERENTIAL DIFFERENCE EQUATIONS

Abstract. This paper deals with the numerical treatment of singularly perturbed parabolic differential-difference equations. The considered equations contain a small perturbation parameter $\varepsilon \in (0,1]$ multiplied by the highest order derivative term, and shift parameters attached with the nonderivative terms. The solution of the equations exhibits an exponential boundary layer due to the presence of the perturbation parameter ε. Classical numerical methods fail to give relevant approximate solutions when the perturbation parameter approaches zero. We propose numerical schemes that converge uniformly irrespective of the parameter ε. The numerical schemes are formulated by using the Crank Nicolson method in temporal discretization, and the midpoint upwind non-standard finite difference method on uniform mesh and Shishkin mesh for spatial discretization. The schemes satisfy the discrete maximum principle and the uniform stability estimate. The uniform convergence of the schemes is proved with the second order of convergence in the temporal direction and with the first order of convergence in the spatial direction. Numerical test examples are considered for validating the theoretical findings and analysis of the schemes.

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Key words and phrases. Midpoint upwind, singularly perturbed, differential difference, uniform convergence, Shishkin mesh.

რეზიუმე. ნაშრომი ეხება სინგულარულად შეშფოთებული პარაბოლური დიფერენციალურ-სხვაობიანი განტოლებების რიცხვით ანალიზს. განხილული განტოლებები შეიცავს მცირე შემაშფოთებელ პარამეტრს $\varepsilon \in (0,1]$ გამრავლებულს უმაღლესი რიგის წარმოებულის შემცველ წევრზე და წანაცვლების პარამეტრს, გამრავლებულს წარმოებულის არშემცველ წევრზე. შეშფოთების ε პარამეტრის არსებობის გამო განტოლებების ამონახსნებს გააჩნია ექსპონენციალური სასაზღვრო ფენა. კლასიკური რიცხვითი მეთოდები ვერ იძლევიან რელევანტურ მიახლოებით ამონახსნებს, როდესაც შეშფოთების პარამეტრი ნულს უახლოვდება. შემოთავაზებულია რიცხვითი სქემები, რომლებიც თანაბრად კრებადია ε პარამეტრისგან დამოუკიდებლად. რიცხვითი სქემები აგებულია დროითი დისკრეტიზაციის კრენკ-ნიკოლსონის მეთოდის გამოყენებით, შუა წერტილის ქარის საწინააღმდეგო მიმართულების არასტანდარტული სასრული სხვაობის მეთოდით ერთიან ბადეზე და სივრცითი დისკრეტიზაციის შიშკინის ბადით. სქემები აკმაყოფილებენ დისკრეტულ მაქსიმუმის პრინციპს და თანაბარი მდგრადობის შეფასებას. დამტკიცებულია სქემების თანაბარი მეორე რიგის კრებადობა დროის მიმართულებით და პირველი რიგის კრებადობა სივრცითი მიმართულებით. თეორიული დასკვნების შესამოწმებლად და სქემების გასაანალიზებლად განხილულია რიცხვითი ტესტური მაგალითები.

1 Introduction

We consider a singularly perturbed parabolic differential equations with deviating arguments of the form

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + L_{\varepsilon,\delta,\eta}\Big) u(x,t) = f(x,t), \quad (x,t) \in D = \Omega \times \Lambda = (0,1) \times (0,T], \\ u(x,0) = u_0(x), \quad x \in D_0 = \big\{(x,0) : \ x \in \overline{\Omega} = [0,1]\big\}, \\ u(x,t) = \phi(x,t), \quad (x,t) \in D_L = \big\{(x,t) : \ (x,t) \in [-\delta,0] \times \Lambda\big\}, \\ u(x,t) = \psi(x,t), \quad (x,t) \in D_R = \big\{(x,t) : \ (x,t) \in [1,1+\eta] \times \Lambda\big\}, \end{cases} \tag{1.1}$$

where

$$L_{\varepsilon,\delta,\eta} u(x,t) = -\varepsilon^2 u_{xx}(x,t) + a(x) u_x(x,t) + \alpha(x) u(x-\delta,t) + \beta(x) u(x,t) + \omega(x) u(x+\eta,t),$$

ε $(0 < \varepsilon \ll 1)$ is the perturbation parameter and δ, η are the shift parameters assumed to satisfy $\delta, \eta < \varepsilon$. The coefficients a, α, β, ω, and the functions f, u_0, ϕ and ψ are assumed to be sufficiently smooth and bounded for guaranteeing unique solution. The coefficient functions α, β and ω are assumed to satisfy

$$\alpha(x) + \beta(x) + \omega(x) \geq b^* > 0$$

for some constant b^*. The existence of a unique solution of (1.1) can be established by assuming that the data are Hölder continuous and by imposing appropriate compatibility conditions at the corner points $(0,0), (1,0)$ and $(-\delta,0)$. The required compatibility condition are stated as follows.

Let $u_0(x) \in C^2[0,1]$, $\phi \in C^{2,1}([-\delta,0] \times [0,T])$ and $\psi \in C^{2,1}([1,1+\eta] \times [0,T])$ by imposing the compatibility conditions $u_0(0) = \phi(0,0)$, $u_0(1) = \psi(1,0)$ and

$$\begin{aligned} \frac{\partial \phi(0,0)}{\partial t} - \varepsilon \frac{\partial^2 u_0(0)}{\partial x^2} + a(0) \frac{\partial u_0(0)}{\partial x} + \alpha(0)\phi_0(-\delta,0) + \beta(0) u_0(0) + \omega(0)\phi(\delta,0) = f(0,0), \\ \frac{\partial \psi(1,0)}{\partial t} - \varepsilon \frac{\partial^2 u_0(1)}{\partial x^2} + a(1) \frac{\partial u_0(1)}{\partial x} + \alpha(1) u_0(1-\delta) + \beta(1) u_0(1) + \omega(1)\phi(1+\eta,0) = f(1,0), \end{aligned}$$

so that the data match at the two corners $(0,0)$ and $(1,0)$. Let $a(x)$, $\alpha(x)$, $\beta(x)$, $\omega(x)$ and $f(x,t)$ be continuous on D, then problem (1.1) has a unique solution $u \in C^{2,1}(D)$. In the case when the compatibility conditions are not satisfied, a unique solution may still exist, but may not be differentiable on $\partial D = \overline{D} - D$, where $\overline{D} = \overline{\Omega} \times \overline{\Lambda} = [0,1] \times [0,T]$.

Most of the standard numerical methods developed for solving regular problems do not treat singularly perturbed problems [23]. That is, due to the smoothness, the solution deteriorates and forms a boundary layer [25]. If one wants to solve singularly perturbed problems by using the standard numerical methods in the collocation method, finite difference method (FDM) and finite element method (FEM), a very large number of mesh points are required as the perturbation parameter approaches zero. It is not practical due to limited computer storage and processing ability, even for simple singularly perturbed ODEs [26].

There is a vast literature on the numerical solution of singularly perturbed problems. Interested readers may refer to [2–13, 20, 27] and the references therein. We focuse our review only on numerical schemes developed for solving singularly perturbed parabolic differential equations with deviating arguments. Rao and Chakravarthy [24] used the fitted operator FDM. Ramesh and Kadalbajoo [25] used the upwind and midpoint upwind FDM on the Shishkin mesh. Kumar and Kadalbajoo [18] used the B-Spline collocation method on the Shishkin mesh. Shivehare et al. [28] used the quadratic B-Spline collocation method on an exponentially graded mesh. Gupta et al. [14] developed a hybrid type FDM on the Shishkin meshes and applied the Richardson extrapolation technique. Kumar [17] developed a scheme using the midpoint upwind FDM on the Shishkin mesh. Bansal and Sharma [1] used the θ-method for temporal discretization with the non-standard FDM for the spatial discretization for the problem involving large deviating arguments. In [30–32], Woldaregay and Duressa studied uniform convergence analysis for the singularly perturbed differential-difference equations using the fitted mesh techniques or the fitted operator methods.

The non-standard FDM has better accuracy and order of convergence than the equivalent standard FDM on the Shishkin mesh. But the non-standard FDM loses the boundary layer resolving behavior (there is no sufficient number of mesh points in the boundary layer region) [1, 16]. Furthermore, the convergence analysis of the non-standard FDM was restricted to uniform mesh discretization [21, 22, 33]. Recently, He and Wang [15] developed a new form of non-standard FDM for stationary singularly perturbed problems by using infinite Taylor's series expansion. The authors conclude without proof that their scheme works on the Shishkin mesh. Kumar et al. [19] extended the work in [15] for the time fractional singularly perturbed parabolic problem. Motivated by the works in [15, 19], we propose a midpoint upwind non-standard FDM and prove its uniform convergence. We applied the Crank Nicolson method for the temporal discretization and the midpoint upwind non-standard FDM on a uniform mesh and the Shishkin mesh for the spatial discretization. Moreover, we discuss the uniform convergence analysis of the schemes.

Notation. The norm $\|\cdot\|$ is denoted for the maximum/suprimum norm; the symbols M and N are denoted for the number of mesh intervals in temporal and spatial discretization; C is denotes the positive constant independent of ε and N.

1.1 Bounds and properties of the solution

For the case of $\delta, \eta < \varepsilon$, it is appropriate to use Taylor's approximation for the terms with deviating argument [29] as

$$\begin{aligned} u(x-\delta,t) &\approx u(x,t) - \delta u_x(x,t) + \Big(\frac{\delta^2}{2}\Big)u_{xx}(x,t) + O(\delta^3), \\ u(x+\eta,t) &\approx u(x,t) + \eta u_x(x,t) + \Big(\frac{\eta^2}{2}\Big)u_{xx}(x,t) + O(\eta^3). \end{aligned} \tag{1.2}$$

Using the approximations (1.2) to (1.1), we get

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + L_{c_\varepsilon}\Big)u(x,t) = f(x,t), \quad (x,t) \in D, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = \phi(0,t), \quad t \in \Lambda, \\ u(1,t) = \psi(1,t), \quad t \in \Lambda, \end{cases} \tag{1.3}$$

where

$$\begin{gathered} L_{c_\varepsilon}u(x,t) = -c_\varepsilon(x)u_{xx}(x,t) + p(x)u_x(x,t) + b(x)u(x,t), \\ c_\varepsilon(x) = \varepsilon^2 - \Big(\frac{\delta^2}{2}\Big)\alpha(x) - \Big(\frac{\eta^2}{2}\Big)\omega(x), \\ p(x) = a(x) - \delta\alpha(x) + \eta\omega(x) \ \text{ and } \ b(x) = \alpha(x) + \beta(x) + \omega(x). \end{gathered}$$

For small values of δ, η, (1.1) and (1.3) are asymptotically equivalent, since the difference between the two is $O(\delta^3, \eta^3)$. We assume that

$$0 < c_\varepsilon(x) \le \varepsilon - \frac{\delta^2}{2}\alpha - \frac{\eta^2}{2}\omega = c_\varepsilon,$$

where $\alpha(x) \ge \alpha$ and $\omega(x) \ge \omega$. We also assume that $p(x) \ge p^* > 0$, which implies the occurrence of the boundary layer on the right side of the spatial domain [14, 23]. The boundary layer is maintained for sufficiently small parameters $\delta, \eta \ne 0$. For the large delay problems the interested reader may refer to [1, 7].

Lemma 1.1 ([26])**.** *For $0 < \varepsilon \ll 1$, there exists a constant C independent of c_ε such that the solution $u(x,t)$ satisfies*

$$|u(x,t) - u_0(x)| \le Ct \ \textit{ and } \ |u(x,t) - \phi(0,t)| \le C(1-x), \quad (x,t) \in \overline{D}.$$

Remark 1.1. Since the layer occurs near $x = 1$, there does not exist a constant C such that

$$|u(x,t) - \psi(1,t)| \leq Cx.$$

The problem obtained by setting $c_\varepsilon = 0$ in (1.3) is called a reduced problem and is given as

$$\begin{cases} u_t^0(x,t) + p(x)u_x^0(x,t) + b(x)u^0(x,t) = f(x,t), & (x,t) \in D, \\ u^0(x,0) = u_0(x), & x \in \overline{\Omega}, \\ u^0(0,t) = \phi(0,t), & t \in \overline{\Lambda}, \\ u^0(1,t) \neq \psi(1,t), & t \in \overline{\Lambda}. \end{cases} \tag{1.4}$$

In (1.4), we have $u^0(1,t) \neq \psi(1,t)$, $t \in \overline{\Lambda}$, that is because the reduced problem does not satisfy the boundary condition in the layer region. For small values of c_ε, the solution $u(x,t)$ of (1.3) is very close to the solution $u^0(x,t)$ of (1.4).

The solution of (1.3) can be decomposed into regular $v(x,t)$ and singular component $w(x,t)$ as

$$u(x,t) = v(x,t) + w(x,t), \ \ (x,t) \in \overline{D}.$$

The regular component satisfies the non-homogeneous problem

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + L_{c_\varepsilon}\Big)v(x,t) = f(x,t), & (x,t) \in D, \\ v(x,0) = u_0(x), & x \in \overline{\Omega}, \\ v(0,t) = u(0,t), & t \in \overline{\Lambda}, \\ v(1,t) \neq u(1,t), & t \in \overline{\Lambda}, \end{cases}$$

and the singular component satisfies the homogeneous problem

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + L_{c_\varepsilon}\Big)w(x,t) = 0, & (x,t) \in D, \\ w(x,0) = 0, & x \in \Omega, \\ w(0,t) = 0, & t \in \overline{\Lambda}, \\ w(1,t) = u(1,t) - v(1,t), & t \in \overline{\Lambda}. \end{cases}$$

Lemma 1.2 ([4])**.** *Derivatives of the regular components solution satisfy the bound*

$$\Big|\frac{\partial^k v(x,t)}{\partial x^k}\Big| \leq C, \ \ k = 0,1,2,3,4,$$

and derivatives of the singular components solution satisfy the bound

$$\Big|\frac{\partial^k w(x,t)}{\partial x^k}\Big| \leq C c_\varepsilon^{-k} \exp\Big(\frac{-p^*(1-x)}{c_\varepsilon}\Big), \ \ k = 0,1,2,3,4,$$

where p^* is the lower bound of $p(x)$.

Lemma 1.3 ([4])**.** *Derivatives of the solution of* (1.3) *satisfy the bound*

$$\Big|\frac{\partial^k \partial^l u(x,t)}{\partial x^k \partial t^l}\Big| \leq C\Big(1 + c_\varepsilon^{-k} \exp\Big(\frac{-p^*(1-x)}{c_\varepsilon}\Big)\Big), \ \ 0 \leq k \leq 4, \ \ 0 \leq l \leq 2.$$

2 Numerical schemes

2.1 Temporal semi-discretization

Let the time domain $[0,T]$ be divided into $M-1$ equal intervals using grid points $t_0 = 0$, $t_j = j\Delta t$, $j = 1,2,\dots,M-1$, where $\Delta t = T/(M-1)$. Let $u_{j+1}(x)$ denote the approximation of $u(x,t_{j+1})$ at the

$(j+1)th$ time level discretization. Using the Crank Nicolson method, we semi-discretize the problem in (1.3) as

$$\begin{cases} \Big(1+\frac{\Delta t}{2}\,L^{\Delta t}_{c_\varepsilon}\Big)u_{j+1}(x)=\Big(1-\frac{\Delta t}{2}\,L^{\Delta t}_{c_\varepsilon}\Big)u_j(x)+\Delta t f(x,t_{j+1/2}),\ \ j=0,1,2,\dots,M-1,\\ u_{j+1}(0)=\phi(0,t_{j+1}),\\ u_{j+1}(1)=\psi(1,t_{j+1}),\end{cases} \tag{2.1}$$

where

$$L^{\Delta t}_{c_\varepsilon}u_{j+1}(x)=-c_\varepsilon u''_{j+1}(x)+p(x)u'_{j+1}(x)+b(x)u_{j+1}(x).$$

The semi-discrete scheme in (2.1) satisfies the maximum principle which is stated as follows.

Lemma 2.1 (Semi-discrete maximum principle)**.** *Let u_{j+1} be a smooth function on $\overline{\Omega}$. If*

$$u_{j+1}(0)\ge 0,\ \ u_{j+1}(1)\ge 0\ \ \textit{and}\ \ \Big(1+\frac{\Delta t}{2}\,L^{\Delta t}_{c_\varepsilon}\Big)u_{j+1}(x)\ge 0,\ \ x\in\Omega,$$

then

$$u_{j+1}(x)\ge 0,\ \ x\in\overline{\Omega}.$$

Proof. Suppose there exists $x^*\in[0,1]$ such that

$$u_{j+1}(x^*)=\min_{x\in\overline{\Omega}}u_{j+1}(x)<0.$$

From the assumption it is clear that $x^*\notin\{0,1\}$ implies that $x^*\in(0,1)$. Applying the property of extrema values in calculus, we have $u'_{j+1}(x^*)=0$ and $u''_{j+1}(x^*)\ge 0$. This gives that

$$\Big(1+\frac{\Delta t}{2}\,L^{\Delta t}_{c_\varepsilon}\Big)u_{j+1}(x^*)<0,$$

which contradicts

$$\Big(1+\frac{\Delta t}{2}\,L^{\Delta t}_{c_\varepsilon}\Big)u_{j+1}(x^*)\ge 0,\ \ \forall\, x\in\Omega.$$

Therefore, we conclude that $u_{j+1}(x)\ge 0$, $x\in\overline{\Omega}$. Hence the semi-discrete scheme satisfies the maximum principle. □

Lemma 2.2 (Error bound of semi-discrete scheme)**.** *The global error estimate up to the t_{j+1} time step is bounded as*

$$\|E_{j+1}\|\le C_2(\Delta t)^2,\ \ j=1,2,\dots,M-1,$$

where Δt is the mesh length in a temporal discretization.

Proof. Using Taylor's series approximation for $u(x,t_j)$ and $u(x,t_{j+1})$ centring at $t_{j+1/2}$, we obtain

$$\begin{aligned} u(x,t_j)&=u(x,t_{j+1/2})-\frac{\Delta t}{2}\,u_t(x,t_{j+1/2})+\frac{(\Delta t)^2}{8}\,u_{tt}(x,t_{j+1/2})+O((\Delta t)^3),\\ u(x,t_{j+1})&=u(x,t_{j+1/2})+\frac{\Delta t}{2}\,u_t(x,t_{j+1/2})+\frac{(\Delta t)^2}{8}\,u_{tt}(x,t_{j+1/2})+O((\Delta t)^3).\end{aligned} \tag{2.2}$$

From the approximation in (2.2), we obtain

$$\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}=u_t(x,t_{j+1/2})+O((\Delta t)^2).$$

Using the approximation in (1.3), we obtain

$$\begin{aligned}&\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}\\ &\qquad=c_\varepsilon u_{xx}(x,t_{j+1/2})-p(x)u_x(x,t_{j+1/2})-b(x)u(x,t_{j+1/2})+f(x,t_{j+1/2})+O((\Delta t)^2),\end{aligned}$$

where

$$u(x, t_{j+1/2}) = \frac{u(x, t_{j+1}) + u(x, t_j)}{2} \text{ and } f(x, t_{j+1/2}) = \frac{f(x, t_{j+1}) + f(x, t_j)}{2}.$$

Since the local pointwise error

$$e_{j+1}(x) =: u(x, t_{j+1}) - u_{j+1}(x)$$

satisfies the semi-discrete differential operator, we get

$$\Big(1 + \frac{\Delta t}{2} L^{\Delta t}_{c_\varepsilon}\Big) e_{j+1}(x) = O((\Delta t)^3), \quad e_{j+1}(0) = 0 = e_{j+1}(1).$$

By applying the maximum principle, we obtain

$$\|e_{j+1}\| \le C_1(\Delta t)^3. \tag{2.3}$$

Using the local error estimate in (2.3) up to the $(j+1)$th time steps, we obtain the global error estimate at $(j+1)$th time step as

$$\|E_{j+1}\| \le C_2(\Delta t)^2, \;\; j = 1, 2, \ldots, M-1.$$ □

2.2 Spatial discretization via midpoint upwind non-standard FDM on uniform mesh

In this subsection, we approximate the spatial derivatives by using the midpoint upwind non-standard FDM on a uniform mesh. Moreover, we prove the uniform convergence of the scheme.

For the problem in (2.1), to construct an exact finite difference scheme, we follow the techniques developed by Mickens in [22]. We consider a constant coefficient sub-equations of (2.1),

$$-c_\varepsilon u''_{j+1}(x) + p^* u'_{j+1}(x) + b^* u_{j+1}(x) = 0, \tag{2.4}$$

$$-c_\varepsilon u''_{j+1}(x) + p^* u'_{j+1}(x) = 0, \tag{2.5}$$

where $p(x) \ge p^*$ and $b(x) \ge b^*$. Thus (2.4) has two independent solutions, namely, $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$, where

$$\lambda_{1,2} = \frac{-p^* \pm \sqrt{(p^*)^2 + 4c_\varepsilon b^*}}{-2c_\varepsilon}.$$

We consider uniform grid points $\{x_i = x_0 + ih\}_{i=1}^N$, $x_0 = 0$, $x_N = 1$, $h = \frac{1}{N}$, where N is the number of mesh intervals. The objective is to calculate a difference equation that has the same general solution as the differential equation in (2.1) at the mesh point x_i. The solution is given by

$$U_{i,j+1} = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i).$$

Using the theory of difference equations for the second order linear difference equations, we get

$$\begin{vmatrix} U_{i-1,j+1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ U_{i,j+1} & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ U_{i+1,j+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{vmatrix} = 0.$$

Substituting the values of $\lambda_{1,2}$, we obtain

$$\exp\Big(\frac{p^* h}{2c_\varepsilon}\Big) U_{i-1,j+1} - 2\cosh\Big(\frac{h\sqrt{(p^*)^2 + 4c_\varepsilon b^*}}{2c_\varepsilon}\Big) U_{i,j+1} + \exp\Big(-\frac{p^* h}{2c_\varepsilon}\Big) U_{i+1,j+1} = 0$$

which is an exact difference scheme for (2.5). Simplifying, we obtain

$$-c_\varepsilon \frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{\frac{hc_\varepsilon}{p^*}(\exp(\frac{p^* h}{c_\varepsilon}) - 1)} + p^* \frac{U_{i,j+1} - U_{i-1,j+1}}{h} = 0.$$

The denominator function for the second derivative term discretization is obtained as

$$\gamma^2 = \frac{hc_\varepsilon}{p^*}\Big(\exp\Big(\frac{hp^*}{c_\varepsilon}\Big) - 1\Big).$$

Using Taylor's series representation of $\exp\big(\frac{hp^*}{c_\varepsilon}\big)$, we find that

$$\gamma^2 = \frac{hc_\varepsilon}{p^*}\left(\exp\left(\frac{hp^*}{c_\varepsilon}\right) - 1\right) = \frac{hc_\varepsilon}{p^*}\left(1 + \frac{hp^*}{c_\varepsilon} + \frac{1}{2!}\left(\frac{hp^*}{c_\varepsilon}\right)^2 + \frac{1}{3!}\left(\frac{hp^*}{c_\varepsilon}\right)^3 + \cdots - 1\right) = h^2 + O(h^3).$$

We adopt γ^2 for the variable coefficient problem written as

$$\gamma_i^2 = \frac{hc_\varepsilon}{p(x_i)}\left(\exp\left(\frac{hp(x_i)}{c_\varepsilon}\right) - 1\right). \tag{2.6}$$

Using the denominator function γ_i^2 in (2.6) and applying the midpoint upwind finite difference discretization, the proposed mid-point upwind non-standard finite difference scheme becomes

$$\begin{cases} \Big(1 + \dfrac{\Delta t}{2} L^{h,\Delta t}_{c_\varepsilon,um}\Big)U_{i,j+1} = \Big(1 - \dfrac{\Delta t}{2} L^{h,\Delta t}_{c_\varepsilon,um}\Big)U_{i,j} + \Delta t f(x_{i-1/2}, t_{j+1/2}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, N-1, \\ U_{0,j+1} = \phi(0, t_{j+1}), \\ U_{N,j+1} = \psi(1, t_{j+1}), \end{cases} \tag{2.7}$$

where

$$L^{h,\Delta t}_{c_\varepsilon,um}U_{i,j+1} = -c_\varepsilon \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\gamma_i^2} + p(x_{i-1/2})\frac{U_{i,j+1} - U_{i-1,j+1}}{h} + b(x_{i-1/2})U_{i,j+1},$$

$$L^{h,\Delta t}_{c_\varepsilon,um}U_{i,j} = -c_\varepsilon \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\gamma_i^2} + p(x_{i-1/2})\frac{U_{i,j} - U_{i-1,j}}{h} + b(x_{i-1/2})U_{i,j}.$$

The notation $p(x_{i-1/2})$, $b(x_{i-1/2})$ and $f(x_{i-1/2}, t_j)$ stands for

$$\frac{p(x_i) + p(x_{i-1})}{2}, \quad \frac{b(x_i) + b(x_{i-1})}{2} \quad \text{and} \quad \frac{f(x_i, t_j) + f(x_{i-1}, t_j)}{2},$$

respectively.

2.2.1 Stability analysis on a uniform mesh

Here, we want to show that the discrete scheme in (2.7) satisfies the maximum principle, uniform stability estimates and uniform convergence.

Lemma 2.3 (Discrete maximum principle)**.** *Let $U_{i,j+1}$ be any mesh function satisfying $U_{0,j+1} \geq 0$, $U_{N,j+1} \geq 0$. Then*

$$\Big(1 + \frac{\Delta t}{2} L^{h,\Delta t}_{c_\varepsilon,um}\Big)U_{i,j+1} \geq 0, \quad i = 1, 2, \ldots, N-1,$$

implies that

$$U_{i,j+1} \geq 0, \quad \forall\, i = 0, 1, \ldots, N.$$

Proof. Suppose there exist $k \in \{0, 1, \ldots, N\}$ such that

$$U_{k,j+1} = \min_{0 \leq i \leq N} U_{i,j+1}.$$

Suppose that $U_{k,j+1} < 0$ which implies $k \neq 0, N$. So, we have

$$U_{k+1,j+1} - U_{k,j+1} > 0 \ \text{ and } \ U_{k,j+1} - U_{k-1,j+1} < 0.$$

So, we obtain

$$\Big(1 + \frac{\Delta t}{2} L^{h,\Delta t}_{c_\varepsilon,um}\Big)U_{k,j+1} < 0 \ \text{ for } \ k = 1, 2, 3, \ldots, N-1.$$

Thus the supposition $U_{i,j+1} < 0$ for $i = 0, 1, \ldots, N$ is wrong. Hence we obtain $U_{i,j+1} \geq 0$, $\forall\, i = 0, 1, \ldots, N$. □

Using the discrete maximum principle, we have to prove that the discrete scheme in (2.7) satisfies the uniform stability result.

Lemma 2.4. *The solution $U_{i,j+1}$ of the discrete scheme in (2.7) satisfies the bound*

$$|U_{i,j+1}| \leq \frac{\|(1+\frac{\Delta t}{2}\, L^{h,\Delta t}_{c_\varepsilon,um})U_{i,j+1}\|}{1+\frac{\Delta t}{2}\, b^*} + \max\big\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\big\}.$$

Proof. Let

$$Q = \frac{\|(1+\frac{\Delta t}{2}\, L^{h,\Delta t}_{c_\varepsilon,um})U_{i,j+1}\|}{1+\frac{\Delta t}{2}\, b^*} + \max\big\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\big\}$$

and define barrier functions $\vartheta^{\pm}_{i,j+1}$ by $\vartheta^{\pm}_{i,j+1} = Q \pm U_{i,j+1}$. At the boundary points, we have

$$\vartheta^{\pm}_{0,j+1} = Q \pm U_{0,j+1} \geq 0, \quad \vartheta^{\pm}_{N,j+1} = Q \pm U_{N,j+1} \geq 0.$$

On the discretized spatial domain x_i, $0 < i < N$, we obtain

$$\begin{aligned}
&\Big(1+\frac{\Delta t}{2}\, L^{h,\Delta t}_{c_\varepsilon,um}\Big)\vartheta^{\pm}_{i,j+1}\\
&\quad = Q \pm U_{i,j+1} - c_\varepsilon\,\frac{\Delta t}{2}\left(\frac{Q\pm U_{i+1,j+1} - 2(Q\pm U_{i,j+1}) + Q\pm U_{i-1,j+1}}{\gamma_i}\right)\\
&\qquad + p(x_{i-1/2})\,\frac{\Delta t}{2}\left(\frac{Q\pm U_{i,j+1} - Q\pm U_{i-1,j+1}}{h}\right) + b(x_{i-1/2})\,\frac{\Delta t}{2}\,(Q\pm U_{i,j+1})\\
&\quad = \Big(1+\frac{\Delta t}{2}\, b(x_{i-1/2})\Big)\left(\frac{\|(1+\frac{\Delta t}{2}\, L^{h,\Delta t}_{c_\varepsilon,um})U_{i,j+1}\|}{1+\frac{\Delta t}{2}\, b^*} + \max\big\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\big\}\right)\\
&\qquad \pm \Big(1+\frac{\Delta t}{2}\, L^{h,\Delta t}_{c_\varepsilon,um}\Big)U_{i,j+1} \geq 0, \ \text{ since }\ b(x_{i-1/2}) \geq b^*.
\end{aligned}$$

Using the maximum principle in Lemma 2.3, we obtain $\vartheta^{\pm}_{i,j+1} \geq 0$, $\forall\, x_i \in \overline{\Omega}^N$. Hence the required bound is satisfied. □

2.2.2 Convergence analysis on a uniform mesh

Now, we have to prove uniform convergence of the discrete scheme in (2.7). Let us denote the forward and backward finite differences operators in the spatial variable as

$$D^+ z_{j+1}(x_i) = \frac{z_{j+1}(x_{i+1}) - z_{j+1}(x_i)}{h}\,, \quad D^- z_{j+1}(x_i) = \frac{z_{j+1}(x_i) - z_{j+1}(x_{i-1})}{h}\,,$$

respectively, and the second order finite difference operator as

$$D^+D^- z_{j+1}(x_i) = \frac{D^+ z_{j+1}(x_i) - D^- z_{j+1}(x_i)}{h}\,.$$

Lemma 2.5 ([33])**.** *For a fixed mesh N and $m = 1,2,3,\ldots$ as $\varepsilon \to 0$, there hold*

$$\lim_{c_\varepsilon\to 0}\,\max_{1\leq i\leq N-1} c_\varepsilon^{-m}\exp\left(\frac{-p^* x_i}{c_\varepsilon}\right) = 0 \ \ \text{and}\ \ \lim_{c_\varepsilon\to 0}\,\max_{1\leq i\leq N-1} c_\varepsilon^{-m}\exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right) = 0,$$

where $x_i = ih$, $h = N^{-1}$, $\forall\, i = 1,2,\ldots,N-1$.

Theorem 2.1. *The spatial discretization by using the midpoint upwind non-standard FDM satisfies the truncation error bound*

$$\big|L^{h,\Delta t}_{c_\varepsilon,um}(U_{j+1}(x_i) - U_{i,j+1})\big| \leq Ch\Big(1+\max_i \frac{\exp\big(-p^*(1-x_i)/c_\varepsilon\big)}{c_\varepsilon^3}\Big).$$

Proof. Considering the difference between the exact and the approximate solutions in discrete operators, we obtain

$$\begin{aligned}
&\left|L^{h,\Delta t}_{c_\varepsilon,um}(U_{j+1}(x_i)-U_{i,j+1})\right| \\
&\qquad \le C\Big|-c_\varepsilon\Big(\frac{d^2}{dx^2}-\frac{D_x^+D_x^-h^2}{\gamma_i}\Big)U_{j+1}(x_i)\Big| + \Big|p(x_{i-1/2})\Big(\frac{d}{dx}-D_x^-\Big)U_{j+1}(x_i)\Big| \\
&\qquad \le Cc_\varepsilon\Big|\Big(\frac{d^2}{dx^2}-D_x^+D_x^-\Big)U_{j+1}(x_i)\Big| + Cc_\varepsilon\Big|\Big(\frac{h^2}{\gamma_i}-1\Big)D_x^+D_x^-U_{j+1}(x_i)\Big| \\
&\qquad\qquad + Ch\Big|\frac{d^2}{dx^2}U_{j+1}(x_i)\Big|.
\end{aligned}$$

Let us define

$$\rho = p(x_i)\,\frac{h}{c_\varepsilon}\,,\quad \rho\in(0,\infty).$$

Then, using the expression for γ_i, we obtain

$$c_\varepsilon\Big|\frac{h^2}{\gamma_i}-1\Big| = p(x_i)h\Big|\frac{1}{\exp(\rho)-1}-\frac{1}{\rho}\Big| =: p(x_i)hQ(\rho), \tag{2.8}$$

where

$$Q(\rho)=\frac{\exp(\rho)-1-\rho}{\rho(\exp(\rho)-1)}$$

which satisfies the bound

$$\lim_{\rho\to 0}Q(\rho)=\frac{1}{2}\,,\qquad \lim_{\rho\to\infty}Q(\rho)=0. \tag{2.9}$$

Therefore, $Q(\rho)$ is bounded for all $\rho\in(0,\infty)$. So, we can write $Q(\rho)\le C_2$, $\rho\in(0,\infty)$, where C_2 is a positive constant. Hence from (2.8) and (2.9) the estimate $c_\varepsilon|\frac{h^2}{\gamma_i}-1|\le Ch$ follows. So, the truncation error bound becomes

$$\left|L^{h,\Delta t}_{c_\varepsilon,um}(U_{j+1}(x_i)-U_{i,j+1})\right| \le Cc_\varepsilon h^2\Big|\frac{d^4}{dx^4}U_{j+1}(x_i)\Big| + Ch\Big|\frac{d^2}{dx^2}U_{j+1}(x_i)\Big|. \tag{2.10}$$

Using the bound of Lemma 1.2 in (2.10), we obtain

$$\begin{aligned}
&\left|L^{h,\Delta t}_{c_\varepsilon,um}(U_{j+1}(x_i)-U_{i,j+1})\right| \\
&\qquad \le Cc_\varepsilon h^2\Big|1+c_\varepsilon^{-4}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big| + Ch\Big|1+c_\varepsilon^{-2}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big| \\
&\qquad \le Ch^2\Big|c_\varepsilon+c_\varepsilon^{-3}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big| + Ch\Big|1+c_\varepsilon^{-2}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big| \\
&\qquad \le Ch\Big(1+\max_i c_\varepsilon^{-3}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big),\quad \text{since}\quad c_\varepsilon^{-3}\ge c_\varepsilon^{-2}.
\end{aligned}$$

□

Theorem 2.2. *The error due to the spatial discretization of the midpoint upwind non-standard FDM satisfies the bound*

$$|U_{j+1}(x_i)-U_{i,j+1}|\le Ch.$$

Proof. Using the results of Lemma 2.5, we obtain

$$\left|L^{h,\Delta t}_{c_\varepsilon,um}(U_{j+1}(x_i)-U_{i,j+1})\right|\le Ch.$$

By applying the discrete maximum principle, the error bound is given as

$$|U_{j+1}(x_i)-U_{i,j+1}|\le Ch.$$

□

Theorem 2.3. *Let u and U be the exact and computed solution of problem in* (1.3)*, then the discrete scheme in* (2.7) *satisfies the error bound*

$$\|u-U\|\le C(N^{-1}+(\Delta t)^2).$$

Proof. Using the error bound for the temporal and spatial discretization given in Lemma 2.2 and Theorem 2.2, we obtain the required bound. □

2.3 Spatial discretization via midpoint upwind non-standard FDM on Shishkin mesh

In this subsection, we approximate the spatial derivatives by using the midpoint upwind non-standard FDM on Shishkin mesh. Moreover, we prove the uniform convergence of the scheme.

Let $\{x_i\}_{i=0}^{N}$ be the discretized domain of the spatial variable, where N is the number of grid points in the domain being an even positive integer. For each $i \geq 1$, we define $h_i = x_i - x_{i-1}$. Since the considered problem exhibits the right boundary layer, we set a mesh transition parameter τ which divides the domain $[0, 1]$ into the outer layer region $\Omega_1 = [0, 1 - \tau]$ and inner (boundary) layer region $\Omega_2 = [1 - \tau, 1]$. The mesh transition parameter τ is defined as

$$\tau = \min\left\{\frac{1}{2}, \frac{\sigma c_\varepsilon}{b^*}\ln N\right\}, \tag{2.11}$$

where N is the number of mesh points in spatial discretization and σ denotes a constant that represents the order of the scheme and b^* is the lower bound of $b(x)$. Now, we set a uniform mesh $N/2$ in Ω_1 with mesh spacing $H = \frac{2(1-\tau)}{N}$ and, similarly, a uniform mesh $N/2$ is placed in Ω_2 with the mesh spacing $h = \frac{2\tau}{N}$. So, the mesh point x_i is given as

$$x_i = \begin{cases} \dfrac{2(1-\tau)}{N}\, i & \text{for } i = 0, 1, 2, \dots, \dfrac{N}{2}, \\ 1 - \tau + \dfrac{2\tau}{N}\left(i - \dfrac{N}{2}\right) & \text{for } i = \dfrac{N}{2}, \dots, N. \end{cases} \tag{2.12}$$

Here, we introduce a numerical method that is constructed by using the denominator function of the non-standard FDM on a piecewise uniform Shishkin mesh. We use the mesh transition parameter defined in (2.11) and the piecewise uniform mesh defined in (2.12). It is known that the fitted mesh methods have a layer resolving behavior (which means that $N/2$ mesh points are computed in the boundary layer region). This is the drawback of the non-standard FDM on a uniform mesh. The proposed scheme uses the denominator function in (2.6) together with the fitted mesh technique. The non-standard FDM is traditionally applied on uniform meshes, while using it on a fitted mesh requires the following modifications:

$$\gamma_i = \frac{c_\varepsilon}{p(x_i)}\left(\exp\left(\frac{h_i p(x_i)}{c_\varepsilon}\right) - 1\right), \quad \gamma_{i+1} = \frac{c_\varepsilon}{p(x_i)}\left(\exp\left(\frac{h_{i+1} p(x_i)}{c_\varepsilon}\right) - 1\right). \tag{2.13}$$

Note that $\gamma_i \approxeq h_i + O(h_i^2)$ and $\gamma_{i+1} \approxeq h_{i+1} + O(h_{i+1}^2)$ are the denominator functions for the first derivative backward and forward difference approximations, respectively.

Using the mid-point upwind finite difference discretization with the denominator functions in (2.13), the proposed discrete scheme on Shishkin mesh is given as

$$\begin{cases} \left(1 + \dfrac{\Delta t}{2} L_{c_\varepsilon, sm}^{\Delta t, N}\right) U_{i,j+1} = \left(1 - \dfrac{\Delta t}{2} L_{c_\varepsilon, sm}^{\Delta t, N}\right) U_{i,j} + \Delta t f(x_{i-1/2}, t_{j+1/2}), \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \dots, N-1, \\ U_{0,j+1} = \phi(0, t_{j+1}), \\ U_{N,j+1} = \psi(1, t_{j+1}), \end{cases} \tag{2.14}$$

where

$$L_{c_\varepsilon, sm}^{\Delta t, N} U_{i,j+1} = -c_\varepsilon\left(\frac{U_{i+1,j+1} - U_{i,j+1}}{\gamma_{i+1}(h_i + h_{i+1})} - \frac{U_{i,j+1} - U_{i-1,j+1}}{\gamma_i(h_i + h_{i+1})}\right) + p(x_{i-1/2})\,\frac{U_{i,j+1} - U_{i-1,j+1}}{h_i} + b(x_{i-1/2}) U_{i,j+1}.$$

2.3.1 Stability analysis on Shishkin mesh

We assume that

$$\tau = \min\left\{\frac{1}{2}, \frac{\sigma c_\varepsilon}{b^*}\ln N\right\}.$$

Now, we assign to each of the subintervals $[0,1-\tau]$ and $[1-\tau,1]$ with $N/2$ the number of equidistant grid points. Let H be the mesh width in the subinterval $[0,1-\tau]$ and h be the mesh width in $[1-\tau,1]$. These mesh widths satisfy

$$N^{-1} \le H \le 2N^{-1}, \;\; h \le N^{-1} \;\text{ and }\; h = \frac{\sigma c_\varepsilon}{b^*} N^{-1} \ln N.$$

Lemma 2.6 (Discrete maximum principle.)**.** *Let $U_{i,j+1}$ be any mesh function satisfying $U_{0,j+1} \ge 0$, $U_{N,j+1} \ge 0$. Then*

$$\Big(1 + \frac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) U_{i,j+1} \ge 0, \;\; i = 1,2,\ldots,N-1,$$

implies that $U_{i,j+1} \ge 0$, $\forall\, i = 0,1,\ldots,N$.

Proof. The proof is similar to that of Lemma 2.3. □

Lemma 2.7. *The solution $U_{i,j+1}$ of the discrete scheme in (2.14) satisfies the bound*

$$|U_{i,j+1}| \le \frac{\|(1 + \frac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}) U_{i,j+1}\|}{1 + \frac{\Delta t}{2} b^*} + \max\big\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\big\}.$$

Proof. Let

$$Q = \frac{\|(1 + \frac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}) U_{i,j+1}\|}{1 + \frac{\Delta t}{2} b^*} + \max\big\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\big\}$$

and define the barrier functions $\vartheta^{\pm}_{i,j+1}$ by $\vartheta^{\pm}_{i,j+1} = Q \pm U_{i,j+1}$. At the boundary points, we have

$$\vartheta^{\pm}_{0,j+1} = Q \pm U_{0,j+1} \ge 0, \quad \vartheta^{\pm}_{N,j+1} = Q \pm U_{N,j+1} \ge 0.$$

On the discretized spatial domain x_i, $0 < i < N$, we obtain

$$\begin{aligned}
&\Big(1 + \frac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) \vartheta^{\pm}_{i,j+1} \\
&\quad = Q \pm U_{i,j+1} - c_\varepsilon \frac{\Delta t}{2} \Big(\frac{Q \pm U_{i+1,j+1} - (Q \pm U_{i,j+1})}{\gamma_{i+1}(h_i + h_{i+1})} - \frac{Q \pm U_{i,j+1} - (Q \pm U_{i-1,j+1})}{\gamma_i(h_i + h_{i+1})} \Big) \\
&\qquad + p(x_{i-1/2}) \frac{\Delta t}{2} \Big(\frac{Q \pm U_{i,j+1} - Q \pm U_{i-1,j+1}}{h} \Big) + b(x_{i-1/2}) \frac{\Delta t}{2} (Q \pm U_{i,j+1}) \\
&\quad = \Big(1 + \frac{\Delta t}{2} b(x_{i-1/2})\Big) \Big(\frac{\|(1 + \frac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}) U_{i,j+1}\|}{1 + \frac{\Delta t}{2} b^*} + \max\big\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\big\} \Big) \\
&\qquad \pm \Big(1 + \frac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) U_{i,j+1} \ge 0, \;\; \text{since } \; b(x_{i-1/2}) \ge b^*.
\end{aligned}$$

Using the maximum principle in Lemma 2.6, we obtain $\vartheta^{\pm}_{i,j+1} \ge 0$, $\forall\, x_i \in \overline{\Omega}^N$. Hence the required bound is satisfied. □

For the mesh function $U_{j+1}(x_i)$ at the grid points x_i we denote the approximation of the first and second derivative as

$$D^-_x U_{j+1}(x_i) = \frac{U_{i,j+1} - U_{i-1,j+1}}{h_i}$$

and

$$(D^+_x D^-_x)_\gamma U_{j+1}(x_i) = \frac{2}{h_i + h_{i+1}} \Big(\frac{U_{i+1,j+1} - U_{i,j+1}}{\gamma_{i+1}} - \frac{U_{i,j+1} - U_{i-1,j+1}}{\gamma_i} \Big).$$

2.3.2 Uniform convergence on Shishkin mesh

Decomposition of the discrete solution: Here, we decompose the numerical solution $U_{i,j+1}$ into regular and singular components in a way similar to the continuous case:

$$U_{i,j+1} = V_{i,j+1} + W_{i,j+1},$$

where the regular part satisfies the non-homogeneous equation

$$\begin{cases} \Big(1 + \dfrac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) V_{i,j+1} = \Big(1 - \dfrac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) V_{i,j} + \Delta t f(x_{i-1/2}, t_{j+1/2}), \\ V_{j+1}(0) = v_{j+1}(0), \\ V_{j+1}(1) = v_{j+1}(1), \end{cases}$$

and the singular component satisfies the homogeneous equation

$$\begin{cases} \Big(1 + \dfrac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) W_{i,j+1} = \Big(1 - \dfrac{\Delta t}{2} L^{\Delta t,N}_{c_\varepsilon,sm}\Big) W_{i,j}, \\ W_{j+1}(0) = w_{j+1}(0), \\ W_{j+1}(1) = w_{j+1}(1). \end{cases}$$

The error in the numerical solution can also be decomposed as

$$U_{i,j+1} - U_{j+1}(x_i) = V_{i,j+1} - V_{j+1}(x_i) + W_{i,j+1} - W_{j+1}(x_i).$$

First, let us consider the case $\tau = 1/2$, which is a uniform mesh case, say the mesh size is h. The truncation error becomes

$$\big|L^{\Delta t,N}_{c_\varepsilon,sm}(U_{j+1}(x_i) - U_{i,j+1})\big| = \Big|c_\varepsilon\Big(\frac{d^2}{dx^2} - (D_x^+ D_x^-)_\gamma\Big)U_{j+1}(x_i)\Big| + \Big|p(x_{i-1/2})\Big(\frac{d}{dx} - D_x^-\Big)U_{j+1}(x_i)\Big|,$$

since

$$(D_x^+ D_x^-)_\gamma U_{j+1}(x_i) = \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\frac{hc_\varepsilon}{p(x_i)}\,(\exp(\frac{hp(x_i)}{c_\varepsilon}) - 1)}.$$

From the truncation error in a uniform mesh, we obtain

$$\big|L^{\Delta t,N}_{c_\varepsilon,sm}(U_{j+1}(x_i) - U_{i,j+1})\big| \le C c_\varepsilon N^{-2}\Big|\frac{d^4}{dx^4} U_{j+1}(x_i)\Big| + CN^{-1}\Big|\frac{d^2}{dx^2} U_{j+1}(x_i)\Big| \le CN^{-1}.$$

Using the discrete maximum principle, we obtain

$$|U_{j+1}(x_i) - U_{i,j+1}| \le CN^{-1}.$$

For the case

$$\tau = \frac{\sigma c_\varepsilon}{b^*} \ln N,$$

we estimate the error in the regular and singular solution separately. Here,

$$\frac{\sigma c_\varepsilon}{b^*} \ln N \le 0.5$$

implies that $c_\varepsilon^{-1} \le C \ln N$.

Theorem 2.4. *The error in the regular component satisfy the estimate*

$$|V_{i,j+1} - V_{j+1}(x_i)| \le CN^{-1}.$$

Proof. On an outer layer region $[0, 1-\tau]$, it is clear that the mesh is uniform, i.e., $h_i = h_{i+1} = H$, $i = 1, 2, \dots, N/2$. So, the truncation error in the regular component is given as

$$\begin{aligned}
&\big|L^{\Delta t,N}_{c_\varepsilon,sm}(V_{j+1}(x_i) - V_{i,j+1})\big| \\
&\qquad = -c_\varepsilon\Big(\frac{d^2}{dx^2} - (D_x^+ D_x^-)_\gamma\Big)V_{j+1}(x_i) + p(x_{i-1/2})\Big(\frac{d}{dx} - D_x^-\Big)V_{j+1}(x_i) \\
&\qquad \le Cc_\varepsilon N^{-2}\Big|\frac{d^4}{dx^4}\, V_{j+1}(\xi)\Big| + CN^{-1}\Big|\frac{d^2}{dx^2}\, V_{j+1}(\xi)\Big|, \ \text{ since } \ N^{-1} \le H \le 2N^{-1}, \\
&\qquad \le Cc_\varepsilon N^{-2} + CN^{-1} \le CN^{-1}.
\end{aligned}$$

In the layer region $[1-\tau, 1]$, the mesh is uniform, i.e., $h_i = h_{i+1} = h$, $i = N/2+1, \dots, N-1$. So, the truncation error is given as

$$\begin{aligned}
&\big|L^{\Delta t,N}_{c_\varepsilon,sm}(V_{j+1}(x_i) - V_{i,j+1})\big| \\
&\qquad = -c_\varepsilon\Big(\frac{d^2}{dx^2} - (D_x^+ D_x^-)_\gamma\Big)V_{j+1}(x_i) + p(x_{i-1/2})\Big(\frac{d}{dx} - D_x^-\Big)V_{j+1}(x_i) \\
&\qquad \le Cc_\varepsilon N^{-2}\Big|\frac{d^4}{dx^4}\, V_{j+1}(\xi)\Big| + CN^{-1}\Big|\frac{d^2}{dx^2}\, V_{j+1}(\xi)\Big|, \ \text{ since } \ h \le N^{-1} \\
&\qquad \le Cc_\varepsilon N^{-2} + CN^{-1} \le CN^{-1}.
\end{aligned}$$

Using the discrete maximum principle, we obtain

$$|V_{j+1}(x_i) - V_{i,j+1}| \le CN^{-1}. \qquad \square$$

Theorem 2.5. *The error in the singular component satisfies the bound*

$$|W_{i,j+1} - W_{j+1}(x_i)| \le CN^{-1}(\ln N)^2.$$

Proof. On an outer layer region $[0, 1-\tau]$, we have $h_i = h_{i+1} = H$, $i = 1, 2, \dots, N/2$. So, the truncation error becomes

$$\begin{aligned}
&\big|L^{\Delta t,N}_{c_\varepsilon,sm}(W_{j+1}(x_i) - W_{i,j+1})\big| \\
&\qquad = -\, c_\varepsilon\Big(\frac{d^2}{dx^2} - (D_x^+ D_x^-)_\gamma\Big)W_{j+1}(x_i) + p(x_{i-1/2})\Big(\frac{d}{dx} - D_x^-\Big)W_{j+1}(x_i) \\
&\qquad \le Cc_\varepsilon N^{-2}\Big|\frac{d^4}{dx^4}\, W_{j+1}(x_i)\Big| + CN^{-1}\Big|\frac{d^2}{dx^2}\, W_{j+1}(x_i)\Big|, \ \text{ since } \ H \le 2N^{-1} \\
&\qquad \le Cc_\varepsilon N^{-2}\Big(c_\varepsilon^{-4}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big) + CN^{-1}\Big(c_\varepsilon^{-2}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big) \\
&\qquad \le C(N^{-2}c_\varepsilon^{-3} + N^{-1}c_\varepsilon^{-2})\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big), \\
&\qquad \le C(N^{-2}c_\varepsilon^{-3} + N^{-1}c_\varepsilon^{-2})N^{-1}, \ \text{ since } \ \exp\Big(\frac{-p^*(1-\{x_i\})}{c_\varepsilon}\Big) \le CN^{-1}, \\
&\qquad \le CN^{-2}(\ln N)^2, \ \text{ since } \ c_\varepsilon^{-1} \le C\ln N.
\end{aligned}$$

In the layer region $[1-\tau, 1]$, the mesh is also uniform, i.e., $h_i = h_{i+1} = h$, $i = N/2+1, \dots, N-1$. Hence we have

$$\begin{aligned}
&\big|L^{\Delta t,N}_{c_\varepsilon,sm}(W_{j+1}(x_i) - W_{i,j+1})\big| \\
&\qquad = -\, c_\varepsilon\Big(\frac{d^2}{dx^2} - (D_x^+ D_x^-)_\gamma\Big)W_{j+1}(x_i) + p(x_{i-1/2})\Big(\frac{d}{dx} - D_x^-\Big)W_{j+1}(x_i) \\
&\qquad \le Cc_\varepsilon N^{-2}\Big|\frac{d^4}{dx^4}\, W_{j+1}(x_i)\Big| + CN^{-1}\Big|\frac{d^2}{dx^2}\, W_{j+1}(x_i)\Big|, \ \text{ since } \ h \le N^{-1} \\
&\qquad \le Cc_\varepsilon N^{-2}\Big(c_\varepsilon^{-4}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big) + CN^{-1}\Big(c_\varepsilon^{-2}\exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq C(N^{-2}c_\varepsilon^{-3} + N^{-1}c_\varepsilon^{-2}) \exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big),\\
&\leq CN^{-1}c_\varepsilon^{-2}, \ \text{ since } \ \max_i \exp\Big(\frac{-p^*(1-x_i)}{c_\varepsilon}\Big) \leq 1,\\
&\leq CN^{-1}(\ln N)^2, \ \text{ since } \ c_\varepsilon^{-1} \leq C\ln N.
\end{aligned}
$$

Thus

$$\big|L_{c_\varepsilon,sm}^{\Delta t,N}(W_{j+1}(x_i) - W_{i,j+1})\big| \leq \max\big\{CN^{-2}(\ln N)^2, CN^{-1}(\ln N)^2\big\}.$$

Using the discrete maximum principle, we obtain

$$|W_{j+1}(x_i) - W_{i,j+1}| \leq CN^{-1}(\ln N)^2. \qquad \square$$

Theorem 2.6. *The error due to the spatial discretization of the computed solution satisfies the estimate*

$$|U_{j+1}(x_i) - U_{i,j+1}| \leq CN^{-1}(\ln N)^2.$$

Proof. Combining the error estimate in the regular and singular components in Theorems 2.4 and 2.5, we get the result. $\square$

Theorem 2.7. *Let u and U be the solutions of* (1.3) *and* (2.14)*, respectively, then the discrete scheme satisfies the uniform error estimate*

$$\|u - U\| \leq C\big(N^{-1}(\ln N)^2 + (\Delta t)^2\big).$$

Proof. Using the error bound for the temporal and spatial discretization in Lemma 2.2 and Theorem 2.6, we obtain the required bound. $\square$

3 Numerical results and discussion

We consider numerical examples to illustrate the theoretical findings of the developed scheme.

Example 3.1. We consider the problem

$$\frac{\partial u}{\partial t} - \varepsilon^2\frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial u}{\partial x} + 2u(x-\delta,t) + (x-3)u(x,t) + u(x+\eta,t) = 10t^2\exp(-t)x(1-x)$$

with $T = 3$, subject to the initial condition $u(x,0) = 0$, $x \in [0,1]$, and the interval-boundary conditions $\phi(x,t) = 0$, $-\delta \leq x \leq 0$, $\psi(1,t) = 0$ on $t \in [0,3]$.

Example 3.2. We consider the problem

$$\frac{\partial u}{\partial t} - \varepsilon^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + (2-x^2)u(x-\delta,t) + (1+x^2)u(x,t) + \exp(x)u(x+\eta,t) = 50(x(1-x))^3$$

with $T = 2$, subject to the initial condition $u(x,0) = 0$, $x \in [0,1]$, and the interval-boundary conditions $\phi(x,t) = 0$, $-\delta \leq x \leq 0$, $\psi(1,t) = 0$ on $t \in [0,2]$.

In the considered examples, the exact solution to the problems is not known. So, we use the double mesh procedure to calculate the maximum pointwise absolute error. Let $U_{i,j}^{N,M}$ denote the computed solution of the problem for N, M number of mesh points in x and t direction, respectively, and let $U_{i,j}^{2N,2M}$ denote the computed solution on a double number of mesh points $2N$, $2M$ by including the midpoints

$$x_{i+1/2} = \frac{x_{i+1} + x_i}{2} \ \text{ and } \ t_{j+1/2} = \frac{t_{j+1} + t_j}{2}$$

into the mesh points. The maximum pointwise absolute error is given by

$$Error_{\varepsilon,\delta,\eta}^{N,M} = \max_{i,j}\big|U_{i,j}^{N,M} - U_{i,j}^{2N,2M}\big|.$$

Table 1: Example 3.1, maximum absolute error of the scheme on uniform mesh for $\delta = 0.6\varepsilon$, $\eta = 0.5\varepsilon$.

ε	$N = 2^5$	2^6	2^7	2^8	2^9
$\downarrow$	$M = 60$	120	240	480	960
2^{-6}	1.9732e-03	9.5545e-04	4.7020e-04	2.3328e-04	1.1617e-04
2^{-8}	1.9579e-03	9.4782e-04	4.6640e-04	2.3137e-04	1.1523e-04
2^{-10}	1.9541e-03	9.4594e-04	4.6546e-04	2.3090e-04	1.1499e-04
2^{-12}	1.9532e-03	9.4547e-04	4.6523e-04	2.3078e-04	1.1495e-04
2^{-14}	1.9529e-03	9.4535e-04	4.6517e-04	2.3075e-04	1.1493e-04
2^{-16}	1.9529e-03	9.4532e-04	4.6515e-04	2.3075e-04	1.1491e-04
2^{-18}	1.9529e-03	9.4532e-04	4.6515e-04	2.3075e-04	1.1491e-04
2^{-20}	1.9529e-03	9.4532e-04	4.6515e-04	2.3075e-04	1.1491e-04
$Error^{N,M}$	1.9529e-03	9.4532e-04	4.6515e-04	2.3075e-04	1.1491e-04
$rate^{N,M}$	1.0467	1.0231	1.0114	1.0058	1.0048

The ε-uniform error is calculated as

$$Error^{N,M} = \max_{\varepsilon,\delta,\eta} \left| Error^{N,M}_{\varepsilon,\delta,\eta} \right|.$$

The rate of convergence of the scheme is calculated by using the formula

$$rate^{N,M}_{\varepsilon,\delta,\eta} = \log_2 \frac{Error^{N,M}_{\varepsilon,\delta,\eta}}{Error^{2N,2M}_{\varepsilon,\delta,\eta}},$$

and the ε- uniform rate of convergence is calculated as

$$rate^{N,M} = \log_2 \frac{Error^{N,M}}{Error^{2N,2M}}.$$

The solution of the problems considered in Examples 3.1 and 3.2 exhibits a boundary layer on the right side of the spatial domain. As one observes in Figures 3(a–d), as the perturbation parameter ε gets small, the boundary layer formation is visible. In Figures 1(a) and 2(a), we have the computed solution of Examples 3.1 and 3.2 by using midpoint upwind non-standard FDM on a uniform mesh at $\varepsilon = 2^{-10}$ and $T = 3$. In these figures, we observe that there is no computed solution in the boundary layer region, this is the main drawback of the non-standard FDM on a uniform mesh and, in general, the fitted operator methods. In figures 1(b) and 2(b), we observe the computed solution of Examples 3.1 and 3.2 by using the scheme on a Shishkin mesh at $\varepsilon = 2^{-10}$ and $T = 3$. In these figures, one can observe a sufficient number of mesh points and computed solutions in the boundary layer region. This assures that the scheme on the Shishkin mesh has the layer resolving property. In Figure 4, the computed solution and the absolute error of the proposed scheme on the Shishkin mesh at $\varepsilon = 2^{-20}$ and $N = 2^8$, $M = 120$ are depicted. In these figures, one can observe that the absolute error is dominant in the boundary layer region, in direct agreement with the result we proved in the convergence analysis.

In Tables 1 and 5, the maximum pointwise absolute error, ε-uniform error and the ε-uniform rate of convergence of Examples 3.1 and 3.2 are given by using the scheme on a uniform mesh. In Tables 2 and 6, the maximum pointwise absolute error, ε-uniform error and the ε-uniform rate of convergence of the scheme on the Shishkin mesh are given. The numerical results in Tables 1, 2, 5 and 6 show that the developed schemes are uniformly convergent (converge independent of the perturbation parameter as the perturbation parameter gets small) with linear convergence. In Tables 3 and 7, the maximum absolute error of the scheme on the Shishkin mesh for different values of delay and advance parameter is given. In Tables 4 and 8, one can observe the comparison of the scheme on the Shishkin mesh with the results of the papers [18, 24, 25]. So, we confirm that the scheme on the Shishkin mesh gives a more accurate result than some schemes available in the literature.

Table 2: Example 3.1, maximum absolute error of the scheme on Shishkin mesh for $\delta = 0.6\varepsilon$, $\eta = 0.5\varepsilon$.

ε $\downarrow$	N= 2^5 $M = 60$	2^6 120	2^7 240	2^8 480	2^9 960
2^{-6}	4.1988e-03	1.9687e-03	9.5357e-04	4.6940e-04	2.3283e-04
2^{-8}	4.1723e-03	1.9556e-03	9.4715e-04	4.6621e-04	2.3128e-04
2^{-10}	4.1650e-03	1.9520e-03	9.4533e-04	4.6530e-04	2.3083e-04
2^{-12}	4.1631e-03	1.9510e-03	9.4487e-04	4.6507e-04	2.3071e-04
2^{-14}	4.1626e-03	1.9508e-03	9.4475e-04	4.6501e-04	2.3068e-04
2^{-16}	4.1625e-03	1.9508e-03	9.4472e-04	4.6500e-04	2.3068e-04
2^{-18}	4.1625e-03	1.9508e-03	9.4471e-04	4.6499e-04	2.3067e-04
2^{-20}	4.1625e-03	1.9508e-03	9.4471e-04	4.6499e-04	2.3067e-04
$Error^{N,M}$	4.1625e-03	1.9508e-03	9.4471e-04	4.6499e-04	2.3067e-04
$rate^{N,M}$	1.0934	1.0461	1.0227	1.0114	1.0054

Table 3: Example 3.1, maximum absolute error of the non-standard FDM on Shishkin mesh for different values of δ and η for $\varepsilon = 2^{-10}$.

	$N = 2^5$ $M = 60$	2^6 120	2^7 240	2^8 480	2^9 960
$\delta \downarrow, \eta = 0.5\varepsilon$					
0	4.1607e-03	1.9498e-03	9.4425e-04	4.6476e-04	2.3059e-04
0.1ε	4.1614e-03	1.9502e-03	9.4443e-04	4.6485e-04	2.3063e-04
0.3ε	4.1628e-03	1.9509e-03	9.4479e-04	4.6503e-04	2.3085e-04
0.5ε	4.1643e-03	1.9516e-03	9.4515e-04	4.6521e-04	2.3092e-04
0.7ε	4.1657e-03	1.9523e-03	9.4551e-04	4.6539e-04	2.3411e-04
$\eta \downarrow, \delta = 0.6\varepsilon$					
0	4.1668e-03	1.9529e-03	9.4578e-04	4.6553e-04	2.3097e-04
0.1ε	4.1664e-03	1.9527e-03	9.4569e-04	4.6548e-04	2.3089e-04
0.3ε	4.1657e-03	1.9523e-03	9.4551e-04	4.6539e-04	2.3080e-04
0.5ε	4.1650e-03	1.9520e-03	9.4533e-04	4.6530e-04	2.3072e-04
0.7ε	4.1643e-03	1.9516e-03	9.4515e-04	4.6521e-04	2.3065e-04

Table 4: Example 3.1 ε-uniform error and ε-uniform rate of convergence of the proposed scheme in (2.14) and result in [18, 24, 25].

Schemes $\downarrow$	$N = 32$ $M = 60$	64 120	128 240	256 480	512 960
Proposed scheme in (2.7)	1.9529e-03	9.4532e-04	4.6515e-04	2.3075e-04	1.1491e-04
	1.0467	1.0231	1.0114	1.0058	1.0048
Proposed scheme in (2.14)	4.1625e-03	1.9508e-03	9.4471e-04	4.6499e-04	2.3067e-04
	1.0934	1.0461	1.0227	1.0114	1.0054
Upwind scheme on	1.6716e-02	9.2021e-03	4.9863e-03	2.6885e-03	1.4245e-03
Shishkin in [25]	0.8612	0.8840	0.8912	0.9163	0.9178
Fitted operator in [24]	6.0781e-03	3.3107e-03	1.7254e-03	8.8049e-04	4.4473e-04
	0.8765	0.9402	0.9705	0.9854	0.9927
B-Spline coloc.	7.5020e-03	4.4966e-03	2.4450e-03	1.2728e-03	6.4909e-04
on Shishkin in [18]	0.7384	0.8791	0.9418	0.9715	0.9859

Table 5: Example 3.2, maximum absolute error of the scheme on uniform mesh for $\delta = 0.6\varepsilon$, $\eta = 0.5\varepsilon$.

ε	$N = 2^5$	2^6	2^7	2^8	2^9
$\downarrow$	$M = 60$	120	240	480	960
2^{-6}	2.4592e-03	1.3224e-03	6.8778e-04	3.5109e-04	1.7737e-04
2^{-8}	2.4594e-03	1.3213e-03	6.8716e-04	3.5089e-04	1.7732e-04
2^{-10}	2.4613e-03	1.3210e-03	6.8704e-04	3.5084e-04	1.7730e-04
2^{-12}	2.4617e-03	1.3209e-03	6.8702e-04	3.5083e-04	1.7730e-04
2^{-14}	2.4619e-03	1.3209e-03	6.8701e-04	3.5083e-04	1.7730e-04
2^{-16}	2.4619e-03	1.3209e-03	6.8701e-04	3.5083e-04	1.7730e-04
2^{-18}	2.4619e-03	1.3209e-03	6.8701e-04	3.5083e-04	1.7730e-04
2^{-20}	2.4619e-03	1.3209e-03	6.8701e-04	3.5083e-04	1.7730e-04
$Error^{N,M}$	2.4619e-03	1.3209e-03	6.8701e-04	3.5083e-04	1.7730e-04
$rate^{N,M}$	0.8983	0.9431	0.9696	0.9846	0.9946

Table 6: Example 3.2, maximum absolute error of the scheme on Shishkin mesh for $\delta = 0.6\varepsilon$, $\eta = 0.5\varepsilon$.

ε	$N = 2^5$	2^6	2^7	2^8	2^9
$\downarrow$	$M = 60$	120	240	480	960
2^{-6}	4.7649e-03	2.5722e-03	1.3516e-03	6.9475e-04	3.5244e-04
2^{-8}	4.7612e-03	2.5725e-03	1.3525e-03	6.9552e-04	3.5296e-04
2^{-10}	4.7589e-03	2.5718e-03	1.3523e-03	6.9548e-04	3.5295e-04
2^{-12}	4.7582e-03	2.5716e-03	1.3523e-03	6.9546e-04	3.5294e-04
2^{-14}	4.7581e-03	2.5715e-03	1.3522e-03	6.9545e-04	3.5294e-04
2^{-16}	4.7580e-03	2.5715e-03	1.3522e-03	6.9545e-04	3.5294e-04
2^{-18}	4.7580e-03	2.5715e-03	1.3522e-03	6.9545e-04	3.5294e-04
2^{-20}	4.7580e-03	2.5715e-03	1.3522e-03	6.9545e-04	3.5294e-04
$Error^{N,M}$	4.7580e-03	2.5715e-03	1.3522e-03	6.9545e-04	3.5294e-04
$rate^{N,M}$	0.8878	0.9273	0.9593	0.9785	0.9889

Table 7: Example 3.2, maximum absolute error of the scheme on Shishkin mesh for different values δ and η with $\varepsilon = 2^{-10}$.

	$N = 2^5$	2^6	2^7	2^8	2^9
	$M = 60$	120	240	480	960
$\delta \downarrow, \eta = 0.5\varepsilon$					
0	4.7557e-03	2.5705e-03	1.3518e-03	6.9520e-04	3.5282e-04
0.1ε	4.7562e-03	2.5707e-03	1.3519e-03	6.9525e-04	3.5287e-04
0.3ε	4.7573e-03	2.5711e-03	1.3520e-03	6.9534e-04	3.5295e-04
0.5ε	4.7584e-03	2.5716e-03	1.3522e-03	6.9543e-04	3.5307e-04
0.7ε	4.7594e-03	2.5720e-03	1.3524e-03	6.9553e-04	3.5315e-04
$\eta \downarrow, \delta = 0.6\varepsilon$					
0	4.7612e-03	2.5728e-03	1.3528e-03	6.9572e-04	3.5307e-04
0.1ε	4.7607e-03	2.5726e-03	1.3527e-03	6.9567e-04	3.5302e-04
0.3ε	4.7598e-03	2.5722e-03	1.3525e-03	6.9558e-04	3.5294e-04
0.5ε	4.7589e-03	2.5718e-03	1.3523e-03	6.9548e-04	3.5284e-04
0.7ε	4.7580e-03	2.5714e-03	1.3521e-03	6.9539e-04	3.5275e-04

Table 8: Example 3.2, ε-uniform error and ε-uniform rate of convergence of the proposed scheme in (2.14) and result in [14].

Schemes	$N = 64$	128	256	512	1024
↓	$M = 16$	32	64	128	256
Proposed scheme in (2.14)	3.6745e-03	1.1291e-03	5.6471e-04	3.1147e-04	1.6474e-04
	1.0724	0.9996	0.9593	0.91890	-
Result in [14]	7.4860e-03	4.6192e-03	2.6516e-03	1.4278 e-03	7.4242e-04
	0.6965	0.8008	0.8931	0.9435	-

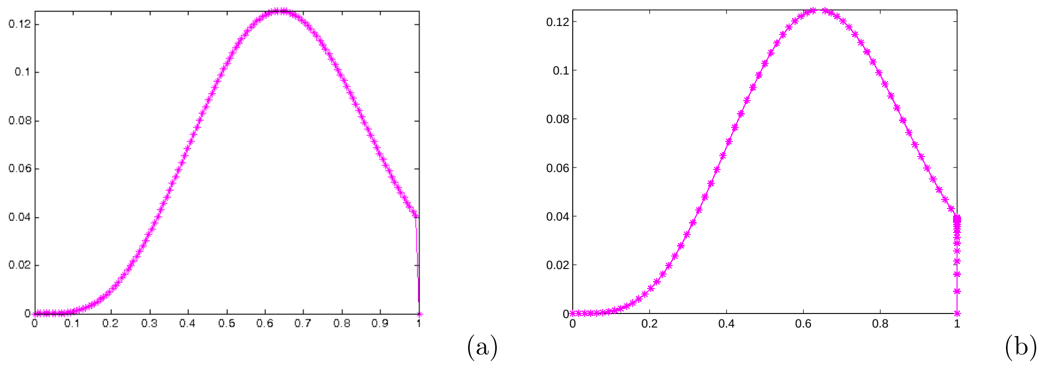

Figure 1: Example 3.2, the layer resolving property for $\varepsilon = 2^{-10}$: on (a) the scheme in (2.7), (b) the scheme in (2.14), at $T = 3$ and $N = 2^7$.

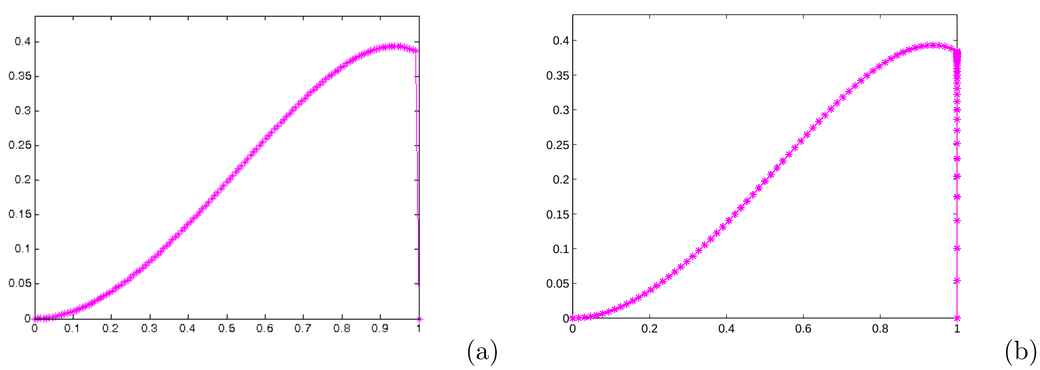

Figure 2: Example 3.1, the layer resolving property for $\varepsilon = 2^{-10}$: on (a) the scheme in (2.7), (b) the scheme in (2.14), at $T = 3$ and $N = 2^7$.

4 Conclusion

Numerical schemes are developed for solving singularly perturbed parabolic differential equations having deviating arguments on the spatial variable. The solution to the considered problem exhibits a boundary layer. The developed schemes use the Crank Nicolson method in temporal semi-discretization and midpoint upwind non-standard FDM for spatial discretization on a uniform mesh and a Shishkin mesh. The uniform stability of the schemes is investigated by using the barrier function and the maximum principle for the solution bound. The parameter uniform convergence of the schemes is proved. The applicability of the schemes is investigated by considering test examples. The effects of the perturbation parameter and the shift parameters on the solution are shown using figures and tables. The developed schemes are uniformly convergent with a linear order of convergence. The schemes give accurate and stable numerical results. In the future works, we extend these schemes for solving higher dimensional singularly perturbed problems.

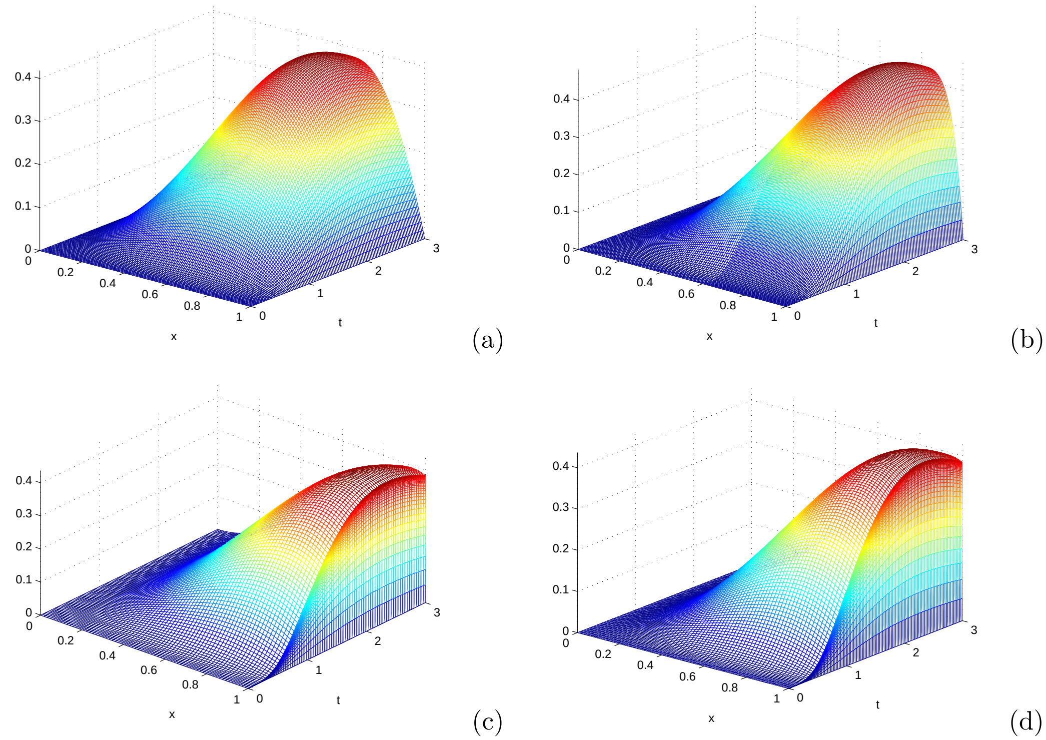

Figure 3: 3D view of solution of Example 3.1 with layer formation, on (a) $\varepsilon = 2^0$, (b) $\varepsilon = 2^{-2}$, (c) $\varepsilon = 2^{-10}$ and (d) $\varepsilon = 2^{-20}$.

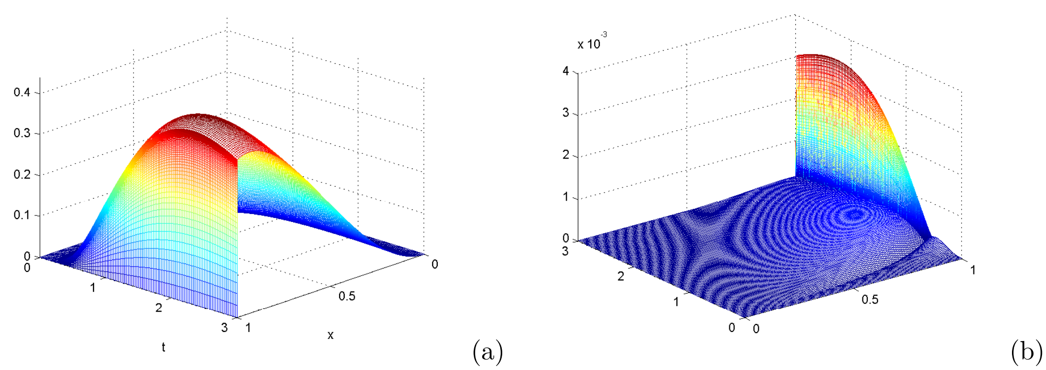

Figure 4: Example 3.2, on (a) computed solution, (b) absolute error of the scheme (2.14) for $\varepsilon = 2^{-10}$, $N = 2^8$, $M = 120$.

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Authors' addresses:

Mesfin Mekuria Woldaregay
Department of Applied Mathematics, Adama Science and Technology University, Adama, Ethiopia
E-mail: msfnmkr02@gmail.com

Gemechis File Duressa
Department of Mathematics, Jimma University, Jimma, Ethiopia
E-mail: gammeef@gmail.com