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FEEDBACK STABILIZATION OF BILINEAR SYSTEMS IN $\mathbb{R}^3$

Abstract. In this paper, we provide sufficient condition under which the considered system is globally asymptotically stabilizable by a homogeneous feedback.

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რეზიუმე. ნაშრომში მოცემულია საკმარისი პირობა, რომლის დროსაც განსახილველი სისტემა გლობალურად ასიმპტოტურად სტაბილიზირებადია ერთგვაროვანი უკუკავშირის საშუალებით.

1 Introduction

Homogeneous systems appear naturally as local approximations of nonlinear systems. In order to make use of this approximation property in the design of locally stabilizing feedback for nonlinear systems, the main idea lies in the construction of homogeneous feedback, i.e., feedback laws that preserve the homogeneity of the resulting closed-loop systems. These laws can be shown to be locally stabilizing also for the approximated nonlinear system. Regarding the existence of stabilizing continuous feedback laws, it was shown in [12] that for general controllable homogeneous systems, the existence of stabilizing feedback does not necessarily imply the existence of a homogeneous stabilizing feedback. Homogeneous bilinear systems are a second-order approximation of the nonlinear system $\dot{x} = f(x) + ug(x)$ with $f(0) = 0$ and $g(0) = 0$. These systems can be written in the form

$$\dot{x} = Ax + uBx, \quad A, B \in \mathcal{M}_n(\mathbb{R}), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}. \tag{1.1}$$

In addition to the rich mathematical structure that motivates the problem, we can consider this as a stabilization problem for systems with a first-order singularity at an equilibrium point. Vector fields Ax and Bx are the first-order approximations of the state and the input vector fields. We illustrate this aspect by considering a numerical example. More precisely, we investigate the stabilization of a class of the bilinear system (1.1) in three dimensions. We suppose that the surface containing invariant straight lines is a submanifold with a special algebraic equation.

Many researchers are evolved in the above-mentioned subject [1, 3, 6, 7]. Bacciotti–Boieri [2] give a complete classification of such systems in the plane. In dimension two, a complete classification was given by Chabour, Sallet and Vivalda [4]. In particular, they introduced homogeneous feedbacks of zero degree to stabilize these systems. In [11], the authors consider a class of bilinear systems in dimension three and study the stabilization problem by continuous feedback and by homogeneous feedback of degree zero. The stabilizing feedback is explicitly given. In [9], the authors consider a class of bilinear systems in dimension three which can be an extension of another one in dimension two. They prove that there exists some homogeneous feedback of degree zero stabilizing the considered class if and only if these feedbacks are constants.

In [10], the authors prove that in the case of bilinear systems in $\mathbb{R}^2$ or $\mathbb{R}^3$, the stabilization by a constant feedback is analytically solved by determining the eigenvalues of the matrix $A + \alpha B$ with $\alpha \in \mathbb{R}$.

In the present work, we consider the problem of explicitly constructing of a feedback law $u(x)$ which is homogeneous of degree zero and asymptotically stabilizes system (1.1) under the following assumption:

$$\det(Ax, Bx, x) = R(x_1, x_2) - x_3 q(x_1, x_2).$$

The fundamental idea is that if a vector field X of the closed-loop system by a homogeneous feedback (defined on $\mathbb{R}^3$)

$$\dot{x} = Ax + u(x)Bx$$

is homogeneous, then it induces a dynamical system on a lower-dimensional space: the unit sphere $\mathbf{S}^2$. The main tool of this paper is the theorem of Coleman [5] who gives the necessary and sufficient conditions for global asymptotic stability (G.A.S.) of a homogeneous system in 3-space.

2 Preliminaries

Next, let us consider the system

$$\dot{x} = X(x), \tag{2.1}$$

where $x \in \mathbb{R}^3$ and X is a homogeneous vector field (not necessarily polynomial) of odd degree.

In the following, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the usual two-norm and standard inner product on $\mathbb{R}^n$ (i.e., for $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$; $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$).

Let $x \in \mathbb{R}^3 \setminus \{0\}$, $r = \|x\|$ and $y = \frac{x}{r}$. Differentiating $x = ry$ and denoting $r^{m-1}(t)dt = d\tau$, we obtain a system on the unit sphere $\mathbf{S}^2$ by writing the equation satisfied by $\dot{y}$:

$$\dot{y} = X(y) - \langle X(y), y\rangle y. \tag{2.2}$$

Coleman proved the following theorem.

Theorem 2.1 ([5]). *The origin is an asymptotically stable equilibrium point for system (2.1) if and only if the following conditions are satisfied:*

(a) $\langle X(y), y\rangle < 0$ *for all equilibrium points of (2.2).*

(b) $\int_0^{\varrho} \langle X(y(s)), y(s)\rangle\, dt < 0$ *for any periodic solution $y(s)$ of system (2.2) (ϱ denotes the period of $y(s)$).*

Definition. Let $\mathcal{N}$ be a subset of $\mathbb{R}^3$, we say that $\mathcal{N}$ is an invariant set by the trajectories of system (2.1) if:

for all $y_0 \in \mathcal{N}$, one has $c_t(y_0) \in \mathcal{N}$ for all $t \in \mathbb{R}$ ($c_t(y_0)$ is the solution of equation $\dot{x} = X(x)$ and $c_0(y_0) = y_0$).

Remark 2.1. If $\mathcal{N}$ is a submanifold of $\mathbb{R}^3$, then: $\mathcal{N}$ is an invariant set by the trajectories of system (2.1) if

for all $y_0 \in \mathcal{N}$, one has $X(y) \in \mathcal{T}_y\mathcal{N}$ ($\mathcal{T}_y\mathcal{N}$ is the tangent space of $\mathcal{N}$ at the point y).

3 Main results

We consider the bilinear system

$$\dot{x} = Ax + uBx, \quad A, B \in \mathcal{M}_3(\mathbb{R}), \quad x \in \mathbb{R}^3 \text{ and } u \in \mathbb{R}. \tag{3.1}$$

In any basis of $\mathbb{R}^3$, the matrices A and B take the following forms:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

The closed-loop system (3.1) by a homogeneous feedback of degree zero $v(x)$ is

$$\dot{x} = Ax + v(x)Bx = X(x).$$

The vector field X can be written as

$$X(x) = \begin{pmatrix} X_1(x) \\ X_2(x) \\ X_3(x) \end{pmatrix} = \begin{pmatrix} A_1(x) + v(x)B_1(x) \\ A_2(x) + v(x)B_2(x) \\ A_3(x) + v(x)B_3(x) \end{pmatrix},$$

where

$$A_i(x) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \text{ and } B_i(x) = b_{i1}x_1 + b_{i2}x_2 + b_{i3}x_3 \text{ for } i = 1, 2, 3.$$

Since X is homogeneous of degree one, we can associate with it a vector field Y defined on the unit sphere $\mathbf{S}^2$ by

$$Y(y) = X(y) - \langle X(y), y\rangle y.$$

If y_0 is an equilibrium point of Y, then $Y(y_0) = 0$, so $X(y_0) = \langle X(y_0), y_0\rangle y_0$. The following lemma gives a localization of the equilibrium points of the system

$$\dot{y} = Y(y). \tag{3.2}$$

Lemma 3.1. *The equilibrium points of Y are contained in the surface*

$$\mathcal{M} = \left\{x \in \mathbb{R}^3 : \ \det(Ax, Bx, x) = 0\right\}.$$

Proof. It is clear that if y is an equilibrium point of Y, then $Y(y) = 0$. So,

$$X(y) = \langle X(y), y\rangle y.$$

Taking into account the form of X, we get

$$Ay + v(y)By = \langle X(y), y\rangle y.$$

Finally, the vectors $\{Ay, By, y\}$ are linearly dependant and $\det(Ay, By, y) = 0$. □

Next, we introduce the following homogeneous functions: for $x \in \mathbb{R}^3$, for $i = 1, 2, 3$, $j = 1, 2, 3$ and $i \neq j$,

$$\mathcal{G}_{ij}(x) := \det\begin{pmatrix} B_i(x) & x_i \\ B_j(x) & x_j \end{pmatrix}, \quad \mathcal{H}_{ij}(x) := \det\begin{pmatrix} A_i(x) & x_i \\ A_j(x) & x_j \end{pmatrix}, \quad \mathcal{F}_{ij}(x) := \det\begin{pmatrix} A_i(x) & B_i(x) \\ A_j(x) & B_j(x) \end{pmatrix}.$$

It is clear that

$$\Phi_{ij}(x) = \det\begin{pmatrix} X_i(x) & x_i \\ X_j(x) & x_j \end{pmatrix} = \mathcal{H}_{ij}(x) + v_{ij}(x)\mathcal{G}_{ij}(x).$$

Remark 3.1. Without loss of generality, for $i = 1$, $j = 2$ we get the functions $\mathcal{G}$, $\mathcal{H}$, $\mathcal{F}$ and Φ defined as follows:

$$\mathcal{G}(x) := \det\begin{pmatrix} B_1(x) & x_1 \\ B_2(x) & x_2 \end{pmatrix}, \quad \mathcal{H}(x) := \det\begin{pmatrix} A_1(x) & x_1 \\ A_2(x) & x_2 \end{pmatrix}, \quad \mathcal{F}(x) := \det\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix}.$$

It is clear that

$$\Phi(x) = \det\begin{pmatrix} X_1(x) & x_1 \\ X_2(x) & x_2 \end{pmatrix} = \mathcal{H}(x) + v(x)\mathcal{G}(x).$$

The functions $\mathcal{G}$, $\mathcal{H}$, Φ and $\mathcal{F}$ play an important role in determining the invariant lines of system (3.1), hence, the equilibrium points of system (3.2) and the construction of the feedback v.

Lemma 3.2. *Let $a \in \mathbb{R}^3 \setminus \{0\}$ be a point such that $\mathcal{F}(a) \neq 0$ and $(a_1, a_2) \neq (0, 0)$. $y = \frac{a}{\|a\|}$ is an equilibrium point of Y if and only if*

$$y \in \mathcal{M} \cap \mathbf{S}^2 \ \text{ and } \ \Phi(a) = 0.$$

Proof. If y is an equilibrium point of Y, then by Lemma 3.1, the point y is in the set $\mathcal{M} \cap \mathbf{S}^2$, and we have $X(y) = \nu y$, $\nu \in \mathbb{R}$ ($\langle y\rangle$ is an invariant straight line associate with the system $\dot{x} = X(x) = Ax + v(x)Bx$),

$$X(y) = \nu y \Longrightarrow \begin{pmatrix} X_1(y) \\ X_2(y) \\ X_3(y) \end{pmatrix} = \begin{pmatrix} \nu y_1 \\ \nu y_2 \\ \nu y_3 \end{pmatrix}.$$

It is clear that

$$\Phi(y) = \det\begin{pmatrix} X_1(y) & y_1 \\ X_2(y) & y_2 \end{pmatrix} = 0 \Longrightarrow \Phi(a) = 0.$$

Inversely, if a satisfies

$$y \in \mathcal{M} \cap \mathbf{S}^2 \ \text{ and } \ \Phi(a) = 0,$$

then there exists $(\alpha_1, \alpha_2) \neq (0, 0)$ such that

$$\alpha_1 \begin{pmatrix} X_1(a) \\ X_2(a) \end{pmatrix} + \alpha_2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the assumption $(a_1, a_2) \neq (0,0)$, we can deduce that $\alpha_1 \neq 0$. The point a satisfies the condition

$$\mathcal{F}(a) := \det \begin{pmatrix} A_1(a) & B_1(a) \\ A_2(a) & B_2(a) \end{pmatrix} \neq 0,$$

then the family

$$\left\{ \begin{pmatrix} A_1(a) \\ A_2(a) \end{pmatrix}, \begin{pmatrix} B_1(a) \\ B_2(a) \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^2$.

If we suppose that $\alpha_2 = 0$, we obtain

$$\alpha_1 \begin{pmatrix} X_1(a) \\ X_2(a) \end{pmatrix} = \alpha_1 \begin{pmatrix} A_1(a) \\ A_2(a) \end{pmatrix} + \alpha_1 v(a) \begin{pmatrix} B_1(a) \\ B_2(a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is absurd, thus $\alpha_2 \neq 0$.

Using the fact that $a \in \mathcal{M}$, we can deduce that there exists $(\beta_1, \beta_2, \beta_3) \neq (0,0,0)$ such that $\beta_1 Aa + \beta_2 Ba + \beta_3 a = 0$. It is clear that

$$\beta_1 \begin{pmatrix} A_1(a) \\ A_2(a) \end{pmatrix} + \beta_2 \begin{pmatrix} B_1(a) \\ B_2(a) \end{pmatrix} + \beta_3 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using the argument that $\left\{ \begin{pmatrix} A_1(a) \\ A_2(a) \end{pmatrix}, \begin{pmatrix} B_1(a) \\ B_2(a) \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$ and $(a_1, a_2) \neq (0,0)$, we can deduce that $\beta_3 \neq 0$ and we have

$$\alpha_1 \begin{pmatrix} A_1(a) \\ A_2(a) \end{pmatrix} + \alpha_1 v(a) \begin{pmatrix} B_1(a) \\ B_2(a) \end{pmatrix} + \alpha_2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We deduce that

$$\frac{\beta_1}{\beta_3} = \frac{\alpha_1}{\alpha_2} \quad \text{and} \quad \frac{\beta_2}{\beta_3} = v(a)\, \frac{\alpha_1}{\alpha_2},$$

and

$$Aa + v(a) Ba + \frac{\alpha_2}{\alpha_1}\, a = 0.$$

Finally, one has $Y(y) = -\frac{\alpha_2}{\alpha_1}\, y$ and y is an equilibrium point of Y. □

Proposition 3.1. *Suppose that the equilibrium points of Y satisfy $F(y) \neq 0$. The vector field Y satisfies the first condition of Coleman's theorem if and only if*

$$\Phi(y) = 0 \ \ \textit{and} \ \ \frac{\mathcal{F}(y)}{\mathcal{G}(y)} > 0.$$

Proof. Let $a \in \mathbb{R}^3 \setminus \{0\}$ and $y = \frac{a}{\|a\|}$. Notice that if y is an equilibrium point of Y, then $X(x) = \nu x$ ($\nu = \langle X(x), x \rangle$) and

$$\begin{pmatrix} A_1(a) \\ A_2(a) \end{pmatrix} + v(a) \begin{pmatrix} B_1(a) \\ B_2(a) \end{pmatrix} = \nu \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} A_1(a) & B_1(a) \\ A_2(a) & B_2(a) \end{pmatrix} \begin{pmatrix} 1 \\ v(a) \end{pmatrix} = \nu \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

and by the fact that $F(y) \neq 0$, one has

$$\begin{pmatrix} 1 \\ v(a) \end{pmatrix} = \nu \begin{pmatrix} A_1(a) & B_1(a) \\ A_2(a) & B_2(a) \end{pmatrix}^{-1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Thus $1 = -\nu \frac{\mathcal{F}(a)}{\mathcal{G}(a)}$. Finally,

$$\nu = \langle X(a), a\rangle = \frac{-\mathcal{G}(a)}{\mathcal{F}(a)} .$$

The vector field Y satisfies the first condition of Coleman's theorem, then $\nu = \langle X(a), a\rangle < 0$, which is equivalent to

$$\frac{\mathcal{F}(a)}{\mathcal{G}(a)} > 0. \qquad \square$$

The main contribution of the paper is the following statement.

Theorem 3.1. *We suppose that $P = \{x \in \mathbb{R}^3 : \langle x, Bx\rangle = 0\}$ is not invariant by the trajectories of $Y(x)$ and the surface $\mathcal{M}$ is not invariant by the trajectories of Bx. If there exists a homogeneous feedback of degree zero v which stabilizes the system $\dot{x} = Ax + v(x)Bx = X(x)$ in all invariant straight lines, then for $n > 0$ large enough, the homogeneous feedback $u(x) = v(x) + w(x)$ with*

$$w(x) = -n\, \frac{det(Ax, Bx, x)^2 \langle x, Bx\rangle}{(x_1^2 + x_2^2 + x_3^2)^4}$$

makes system (3.1) globally asymptotically stable at the origin.

Proof. We denote

$$X_n(x) = \frac{1}{n}\, X(x) + \frac{1}{n}\, v(x)Bx.$$

The feedback v is chosen such that Y is asymptotically stable in all invariant straight lines. Since $w(x) = 0$ for all $x \in \mathcal{M}$, X_n satisfies the first condition of Coleman's theorem.

Suppose that the second condition of Coleman's theorem is not satisfied, then there exists $N_0 > 0$ such that for all $n > N_0$, there exists a periodic solution $c_n(t)$ of the equation

$$\dot{y} = X_n(y) - \langle X_n(y), y\rangle y = \widetilde{X}_n(y), \quad y \in S^2, \tag{3.3}$$

satisfying

$$\int_0^{\varrho_n} \langle X_n(c_n(s)), c_n(s)\rangle \, ds \geq 0. \tag{3.4}$$

The periodic orbit is defined by $\Upsilon_n = \{c_n(t), t \in [0, \varrho_n]\}$ with $c_n(t)$. So, there exists a function $g(n)$ increasing on the set of integers $\mathbb{N}$ such that

$$\lim_{n\to+\infty} c_{g(n)}(t) = c(s)$$

($c_n(s)$ is defined in the united sphere S^2). Thus

$$\lim_{n\to+\infty} \int_0^{\varrho_{g(n)}} \left\langle X_{g(n)}(c_{g(n)}(s)), c_{g(n)}(s)\right\rangle ds = -\int_0^{\varrho} \det\left(Ac(s), Bc(s), c(s)\right)^2 \left\langle c(s), Bc(s)\right\rangle^2 ds \geq 0,$$

where

$$\lim_{n\to+\infty} \varrho_{g(n)} = \varrho \in \mathbb{R}_+ \cup \{+\infty\}.$$

Since $\varrho \geq 0$ and

$$\det(Ac(s), Bc(s), c(s))^2 \langle c(s), Bc(s)\rangle^2 \geq 0,$$

we can deduce that $\varrho = 0$, or $c(s) \in \mathcal{M} \cup P$.

First, if $\lim_{n\to+\infty} \varrho_{g(n)} = \varrho = 0$, then these periodic orbits Υ_n are around an equilibrium point y_0 of equation (3.3). The straight line $\mathcal{D} = \langle y_0\rangle$ is invariant by the trajectories of X_n and X. The length of Υ_n is defined by

$$L(\Upsilon_n) = \int_0^{\varrho_n} \|\dot{c}_n(t)\| \, dt.$$

It is clear that
$$L(\Upsilon_n) \le \varrho_n \sup_{w\in\Upsilon_n} \|\widetilde{X}_n(w)\|.$$
We deduce that
$$\lim_{n\to+\infty} L(\Upsilon_{g(n)}) = 0 \text{ and } c(t) = y_0 \ \ \forall t \in \mathbb{R}.$$
The feedback v is chosen to satisfy $\langle X(y_0), y_0\rangle < 0$. The vector field X is continuous, then for ε small enough, one has $\langle X_{g(n)}(x), x\rangle < 0$ for all $x \in B(y_0,\varepsilon) = \{x \in \mathbb{R}^3$ such that $\|x - y_0\| < \varepsilon\}$. Using the fact that $c(t) = y_0\ \forall t \in \mathbb{R}$, we get that for n large enough, $c_{g(n)}(t) \in B(y_0,\varepsilon)\ \forall t \in \mathbb{R}$. So, we can write
$$\int_0^{\varrho_{g(n)}} \left\langle X_{g(n)}(c_{g(n)}(s)), c_{g(n)}(s)\right\rangle ds < 0.$$
This contradicts hypothesis (3.4).

In the case $\varrho > 0$ and $c(s) \in \mathcal{M}\cup\mathcal{P}$, we can say that the trajectories of X_n are the same trajectories of the vector field $Z_n = \frac{1}{\|X_n\|}X_n$ and the periodic orbits of $\widetilde{X}_n$ are the same orbits of
$$\widetilde{Z}_n = Z_n(y) - \langle Z_n(y), y\rangle y.$$

It is clear that
$$\lim_{n\to+\infty} X_n(x) = \lim_{n\to+\infty} \frac{1}{n} X(x) + \frac{1}{n} v(x)Bx = -\frac{\det(Ax,Bx,x)^2\langle x,Bx\rangle}{(x_1^2+x_2^2+x_3^2)^4} Bx,$$
So,
$$Z(x) = \lim_{n\to+\infty} Z_n(x) = \begin{cases} \dfrac{-Bx}{\|Bx\|} & \text{if } \langle x, Bx\rangle > 0, \\ \dfrac{Bx}{\|Bx\|} & \text{if } \langle x, Bx\rangle < 0. \end{cases}$$
It follows that
$$\dot{y} = \widetilde{Z}(y) = \frac{-By}{\|By\|} + \left\langle \frac{By}{\|By\|}, y\right\rangle y$$
is a tangent to the trajectory $c(s)$ when $c(s) \notin P$, where $c(s)$ is the solution of $\dot{y} = \widetilde{Z}(y)$. Under the hypothesis that $\mathcal{M}$ is not invariant by the constant vector field Bx, we can write $Bx \notin \mathcal{T}_{c(s)}\mathcal{M}$ (with the exception of some points). It follows that $c(s) \in P$. But this is impossible because of the equality
$$\lim_{n\to+\infty} n\widetilde{X}_n(c(s)) = \widetilde{X}(c(s)).$$
We deduce that $\widetilde{X}(c(s))$ is a tangent to the trajectory $\{c(t), t \in \mathbb{R}\}$ and P is invariant by the trajectories of X. $\square$

Remark 3.2. If there exists a feedback v stabilizing system (3.1) in all invariant straight lines, then for any homogeneous function of degree zero $\widetilde{v}$ such that $\|v - \widetilde{v}\| < \varepsilon$, the vector field $X = A + \widetilde{v}B$ satisfies condition 1 of Coleman's theorem. The assumption that P is invariant by the trajectories of X is equivalent to that P is invariant by the trajectories of both vector fields Ax and Bx.

4 Construction of the feedback function v

The homogeneous feedback v stabilizes system (3.1) in all invariant straight lines if the conditions of Proposition 3.1 are satisfied. Next, we try to construct the homogeneous function v such that all zeros of the equation
$$\Phi(x) = \mathcal{H}(x) + v(x)\mathcal{G}(x), \quad x \in \mathcal{M},$$

are contained in the set $\mathcal{K} = \{x \in \mathbb{R}^3 : \ \mathcal{F}(x)\mathcal{G}(x) > 0\}$. Our ability to solve this problem depends on the choice of the form of the surface $\mathcal{M}$. The candidate is the submanifold with the following algebraic equation:

$$\det(Ax, Bx, x) = R(x_1, x_2) - x_3 q(x_1, x_2), \tag{4.1}$$

where q is a definite quadratic form and R is a polynomial function of degree three. Under this assumption, it is clear that

$$x \in \mathcal{M} \text{ if and only if } x_3 = \frac{R(x_1, x_2)}{q(x_1, x_2)}.$$

We define the homogeneous function of degree 4 in dimension two as follows:

$$\mathcal{G}'(x_1, x_2) = q(x_1, x_2)\mathcal{G}\Big(x_1, x_2, \frac{R(x_1, x_2)}{q(x_1, x_2)}\Big).$$

For $x_2 \neq 0$, we can write

$$\mathcal{G}'(x_1, x_2) = \mathcal{G}'\Big(x_2\Big(\frac{x_1}{x_2}, 1\Big)\Big) = x_2^4 \mathcal{G}'(s, 1),$$

where $s = \frac{x_1}{x_2}$. It is clear that if $\mathcal{G}'(s, 1)$ is a polynomial of degree 4, then $\mathcal{G}'$ takes one of the following forms:

$$\begin{aligned}
\mathcal{G}'(x_1, x_2) &= (c_1x_1 - \widetilde{c}_1x_2)(c_2x_1 - \widetilde{c}_2x_2)q_1(x_1, x_2),\\
\mathcal{G}'(x_1, x_2) &= (c_1x_1 - \widetilde{c}_1x_2)(c_2x_1 - \widetilde{c}_2x_2)(c_3x_1 - \widetilde{c}_3x_2)(c_4x_4 - \widetilde{c}_4x_2),\\
\mathcal{G}'(x_1, x_2) &= q_1(x_1, x_2)q_2(x_1, x_2),
\end{aligned}$$

where q_i are definite positive quadratic forms. Since $\mathcal{H}$ is a homogeneous polynomial of degree 2, we have that

$$\mathcal{H}'(x_1, x_2) = q(x_1, x_2)\mathcal{H}\Big(x_1, x_2, \frac{R(x_1, x_2)}{q(x_1, x_2)}\Big)$$

is an homogeneous function of degree 4 in dimension two. Next, we define the following polynomial functions:

$$g(s) := \mathcal{G}'(s, 1), \quad h(s) := \mathcal{H}'(s, 1) \text{ and } f(s) := \mathcal{F}\Big(s, 1, \frac{R(s, 1)}{q(s, 1)}\Big).$$

Under these notation, we can easily prove that the conditions of Proposition 3.1 are equivalent to the following: the roots of the equation

$$\phi(s) = h(s) + v\Big(s, 1, \frac{R(s, 1)}{q(s, 1)}\Big)g(s) = 0, \tag{4.2}$$

are contained in the set $\{s \in \mathbb{R} : \ f(s)g(s) > 0\}$.

Theorem 4.1. *Let there exist a polynomial function ϕ of even degree $2n > 4$ and a definite polynomial P of degree $2n - 2$ satisfying the conditions:*

(A_1) a real root of the polynomial function $g(s)$ is also a root of $\phi - Ph$ with the same multiplicity,

(A_2) if s is a real root of ϕ, then $f(s)g(s) > 0$.

Then the feedback law

$$v(x) = \frac{x_2^{n_1} P_1(\frac{x_1}{x_2})}{x_2^{n_1} P_2(\frac{x_1}{x_2}) + (x_3 - \frac{R}{q}(x_1, x_2))^{n_1}}$$

is C^∞ on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, homogeneous of degree 0 and stabilizes system (3.1) in all invariant straight lines (with $\frac{P_1(s)}{P_2(s)} = \frac{\phi(s) - P(s)h(s)}{P(s)g(s)}$ is a polynomial fraction smooth on $\mathbb{R}$, $P_2(s) > 0$ and n_1 is the degree of P_2).

Proof. Suppose that there exist two polynomial functions P and ϕ of even degree such that the real roots of the polynomial function $g(s)$ are also the roots of $\phi(s) - P(s)h(s)$ with the same multiplicity, then $\frac{\phi(s)-P(s)h(s)}{P(s)g(s)}$ is a smoothly fraction in $\mathbb{R}$. We can write

$$\frac{P_1(s)}{P_2(s)} = \frac{\phi(s) - P(s)h(s)}{P(s)g(s)}$$

with P_2 having no real roots and the degree of P_1 being equal to the degree of P_2.

Using the fact that the function

$$x_2^{n_1} P_2\Big(\frac{x_1}{x_2}\Big) + \Big(x_3 - \frac{R}{q}\,(x_1, x_2)\Big)^{n_1}$$

is definite positive, we obtain that the feedback

$$v(x_1, x_2, x_3) = \frac{x_2^{n_1} P_1(\frac{x_1}{x_2})}{x_2^{n_1} P_2(\frac{x_1}{x_2}) + (x_3 - \frac{R}{q}\,(x_1, x_2))^{n_1}}$$

is C^∞on $\mathbb{R}^3 \setminus \{(0,0,0)\}$, homogeneous of degree 0. The function v is constructed to satisfy the following equalities:

$$v\Big(s, 1, \frac{R(s,1)}{q(s,1)}\Big) = \frac{P_1(s)}{P_2(s)} = \frac{\phi(s) - P(s)h(s)}{P(s)g(s)}.$$

Since ϕ satisfies condition (A_2), according to condition (4.2), the closed-loop system (3.1) is $G.A.S.$ in all straight-lines. $\square$

Remark 4.1. Suppose that $h(s) = \prod\limits_{i \in I} (s - s_i)\widetilde{h}(s)$ with s_i being the real common roots of h and g. The existence of two polynomial functions satisfying condition (A_2) is equivalent to the following assumption:

(Σ) If $\widetilde{h}(s_1)\widetilde{h}(s_2) < 0$ (s_1, s_2 are some real roots of $g(s)$ and not in $\{s_i, i \in I\}$), then there exists $m \in]s_1, s_2[$ such that $f(m)g(m) > 0$.

If condition (Σ) holds, we can find the construction of P and ϕ in [8].

5 Example

We consider the bilinear system

$$\dot{x} = Ax + uBx, \tag{5.1}$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, $u \in \mathbb{R}$ and

$$A = \begin{pmatrix} -\dfrac{67}{2} & \dfrac{71}{2} & 6 \\ \dfrac{79}{2} & -\dfrac{67}{2} & -6 \\ 1 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -3 \\ 11 & 3 & 1 \end{pmatrix}.$$

The class under consideration satisfies condition (4.1):

$$det(Ax, Bx, x) = -\frac{1}{2}\,(867x_1^3 + 233x_1^2x_2 - 783x_1x_2^2 - 217x_2^3) + 6x_3(23x_1^2 + x_2^2),$$

and the submanifold $\mathcal{M}$ is defined by

$$\mathcal{M} = \Big\{x \in \mathbb{R}^3 \ \text{ such that } \ x_3 = \frac{867x_1^3 + 233x_1^2x_2 - 783x_1x_2^2 - 217x_2^3}{12(23x_1^2 + x_2^2)}\Big\}.$$

It is easy to see that for $x \in \mathcal{M}$, one has

$$\begin{aligned}
\mathcal{G}(x_1,x_2,x_3) &= \frac{75(31x_1^4+44x_1^3x_2-26x_1^2x_2^2-40x_2^3x_1-9x_2^4)}{12(23x_1^2+x_2^2)}\,,\\
\mathcal{H}(x_1,x_2,x_3) &= \frac{-150(38x_1^4-44x_1^3x_2-37x_1^2x_2^2+40x_2^3x_1+3x_2^4)}{12(23x_1^2+x_2^2)}\,,\\
\mathcal{F}(x_1,x_2,x_3) &= \frac{25}{2}\,(-9x_1+6x_1x_2+3x_2^2)\,.
\end{aligned}$$

It follows that $g(s) = 75(31s^4+44s^3-26s^2-40s-9)$ is a polynomial function of degree 4. The roots of g are $s_1 = -1,4885$, $s_2 = -0,6124$, $s_3 = -0,3185$ and $s_4 = 1$. We consider the polynomial function $h(s) = -150(38s^4-44s^3-37s^2+40s+3)$. We show that $h(s_1) = -1200$, $h(s_2) = 2988.3$, $h(s_1) = 1752.1$ and $h(s_4) = 0$. We are in the case where s_4 is a real common root of both g and h, it follows that s_4 is also a real root of ϕ. Due to the fact that $h(s_1)h(s_2) < 0$, condition (A_1) is equivalent to the existence of such m satisfying $f(m)g(m) > 0$. We choose $m = -1$, because $g(-1) = -600$ and $f(-1) = -150$. Since -1 is a common root of both g and h, we can write:

$$\begin{aligned}
g(s) &= 75(s-1)(31s^3+75s^2+49s+9) = 75(s-1)g_1(s),\\
h(s) &= 75(s-1)(-76s^3+12s^2+86s+6) = 75(s-1)h_1(s).
\end{aligned}$$

We can define the following constants:

$$\begin{aligned}
a_1 &= \frac{(1+s_1^2)h_1(s_1)}{(s_1-s_2)^2(s_1-s_3)^2(1+s_1)}\,,\\
a_2 &= \frac{(1+s_2^2)h_1(s_2)}{(s_1-s_2)^2(s_2-s_3)^2(1+s_2)}\,,\\
a_3 &= \frac{(1+s^3)h_1(s_3)}{(s_2-s_3)^2(s_1-s_3)^2(1+s_3)}\,.
\end{aligned}$$

These constants a_i are chosen such that

$$\phi_1(s) = a_1(s-s_2)^2(s-s_3)^2 + a_2(s-s_1)^2(s-s_3)^2 + a_3(s-s_2)^2(s-s_1)^2$$

is a definite polynomial function and

$$(1+s_i)\phi(s_i) = (1+s_i)h_1(s_i) \;\text{ for }\; i \in \{2,3,4\}.$$

One has

$$\begin{aligned}
\phi(s) &= 75(1+s)(s-1)\phi_1(s),\\
&= -75(s^2-1)(2537.2s^4+76070.8s^3+8304.9s^2+3546s+355.5)
\end{aligned}$$

and $P(s) = s^2+1$ satisfies condition (A_1). It is clear that

$$\frac{\phi(s)-P(s)h(s)}{P(s)g(s)} = \frac{-79.3945s^2-135.5620s+60.1675}{s^2+1}$$

and the feedback

$$u_1(x) = \frac{-79.3945x_1^2-135.5620x_1x_2+60.1675x_2^2}{x_1^2+x_2^2+(x_3-\frac{T}{q}\,(x_1,x_2))^2}$$

is homogeneous of degree 0. Since $u_1(N^T) = -137.562$ with $N^T = (1,1,1/3)$ and $AN+u_1(N^T)BN = -(271,271,1968)^T$, the feedback $u_1(x)$ stabilizes system (5.1) in the unique invariant straight line $D_1 = \langle x, (-1,1,-17/75)\rangle = 0$. Finally, the feedback

$$u(x) = u_1(x) - 10^3\Big(x_3 - \frac{R}{q}\,(x_1,x_2)\Big)^2\,\frac{2x_1^2+14x_1x_3+2x_2^2+x_3^2}{x_1^2+x_2^2+x_3^2}$$

stabilizes system (5.1).

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