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ON PERIODIC SOLUTIONS OF FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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In the present note, we establish sufficient conditions for the existence and uniqueness of a periodic solution of the differential equation

$$\frac{dx(t)}{dt}=f\Big(t,x(\tau_1(t)),\dots,x(\tau_m(t))\Big), \tag{1}$$

where $f: R\times R^m\to R$ is a function satisfying the local Carathéodory conditions and $\tau_k: R\to R$ $(k=1,\dots,m)$ are measurable functions. In what follows, the function f is assumed to be periodic in the first argument with the period $\omega>0$, i.e., on $R\times R^m$ the equality

$$f(t+\omega,x_1,\dots,x_m)=f(t,x_1,\dots,x_m). \tag{2}$$

is fulfilled. As for the functions τ_k $(k=1,2,\dots)$, they are such that

$$\tau_k(t+\omega)=\mu_k(t)\omega+\tau_k(t) \quad \text{for} \quad t\in R \ \ (k=1,\dots,m), \tag{3}$$

where μ_k $(k=1,\dots,m)$ are the functions admitting only integer values.

For any $k\in\{1,\dots,m\}$ and $t\in R$, we denote by $\nu_k(t)$ the integer part of the number $\frac{\tau_k(t)}{\omega}$ and assume that

$$\tau_{0k}(t)=\tau_k(t)-\nu_k(t)\omega \quad (k=1,\dots,m).$$

Then

$$0\le\tau_{0k}(t)<\omega \quad \text{for} \quad t\in R \ \ (k=1,\dots,m).$$

On the other hand, due to the conditions (2) and (3), the set of restrictions of all ω-periodic solutions of the equation (1) on the segment $[0,\omega]$ coincides with the set of solutions of the boundary value problem

$$\frac{dx(t)}{dt}=f\Big(t,x(\tau_{01}(t)),\dots,x(\tau_{0m}(t))\Big), \tag{4}$$

$$x(0)=x(\omega). \tag{5}$$

Hence the equation (1) has at least one ω-periodic (a unique ω-periodic) solution if and only if the problem (4), (5) is solvable (uniquely solvable).

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Theorem 1. *Let for some $\sigma \in \{-1, 1\}$ the equalities*

$$\left| f(t, x_1, \dots, x_m) - \sum_{k=1}^{m} p_k(t, x_1, \dots, x_m) x_k \right| \le q(t), \tag{6}$$

$$\sigma \sum_{k=1}^{m} p_k(t, x_1, \dots, x_m) \ge \alpha(t), \tag{7}$$

$$\left| p_k(t, x_1, \dots, x_m) \right| \le \beta_k(t) \quad (k = 1, \dots, m) \tag{8}$$

be fulfilled on the set $[0, \omega] \times R^m$, where $p_k : [0, \omega] \times R^m \to R$ $(k = 1, \dots, m)$ are functions satisfying the local Carathéodory conditions, while q, α and $\beta_k : [0, \omega] \to [0, +\infty[$ $(k = 1, \dots, m)$ are summable functions. Let, moreover, α be different from zero on a set of positive measure and let there exist $\delta \in]0, a[$ such that on $[0, \omega]$ the inequality

$$\sum_{i,k=1}^{m} \beta_i(t) \left| \int_t^{\tau_{0k}(t)} \beta_k(s) ds \right| \le \delta \alpha(t) \tag{9}$$

is fulfilled. Then the equation (1) *has at least one ω-periodic solution.*

Proof. Due to the above remark, in order to prove the theorem it suffices to establish the solvability of the problem (4), (5). On the other hand, by Theorem 1.1 in [1] and the conditions (6)–(8), the problem (4), (5) is solvable if for any summable functions $p_{0k} : [0, \omega] \to R$ $(k = 1, \dots, m)$ satisfying

$$\sigma \sum_{k=1}^{m} p_{0k}(t) \ge \alpha(t), \tag{10}$$

$$|p_{0k}(t)| \le \beta_k(t) \quad (k = 1, \dots, m), \tag{11}$$

the problem

$$\frac{dy(t)}{dt} = \sum_{k=1}^{m} p_{0k}(t) y(\tau_{0k}(t)), \quad y(0) = y(\omega) \tag{12}$$

has only the trivial solution.

Let y be an arbitrary solution of the problem (12). Then almost everywhere on $[0, \omega]$ we have

$$\begin{aligned} \frac{dy(t)}{dt} &= \Big[\sum_{k=1}^{m} p_{0k}(t) \Big] y(t) + \sum_{k=1}^{m} p_{0k}(t) \int_t^{\tau_{0k}(t)} y'(s)\, ds = \\ &= p(t) y(t) + q(t), \end{aligned} \tag{13}$$

where

$$p(t) = \sum_{k=1}^{m} p_{0k}(t), \quad q(t) = \sum_{i,k=1}^{m} p_{0k}(t) \int_t^{\tau_{0k}(t)} p_{0i}(s) y(\tau_{0i}(s))\, ds.$$

The functions α, β_k $(k = 1, \dots, m)$, y, p and q will be assumed to be ω-periodically extended to the whole R. Then, because of (9)–(11), the following inequalities are fulfilled on R:

$$\sigma p(t) \ge \alpha(t) \ge 0, \quad |q(t)| \le \delta \sigma p(t) y_0, \tag{14}$$

where

$$y_0 = \max\Big\{|y(t)| : t \in [0,\omega]\Big\}. \tag{15}$$

Moreover,

$$\sigma \int_0^\omega p(t)\,dt \ge \int_0^\omega \alpha(t)\,dt > 0 \tag{16}$$

since α is different from zero on a set of positive measure.

Owing to (13) and (16),

$$y(t) = \Big[\exp\Big(-\int_0^\omega p(s)\,ds\Big) - 1\Big]^{-1} \int_t^{t+\omega} \exp\Big(\int_s^t p(\xi)\,d\xi\Big) q(s)\,ds,$$

whence, taking into account (14), (15) and ω-periodicity of p, we find

$$|y(t) \le \delta y_0 \Big|\exp\Big(-\int_0^\omega p(s)\,ds\Big) - 1\Big|^{-1} \sigma \int_t^{t+\omega} \exp\Big(\int_s^t p(\xi)\,d\xi\Big) p(\xi)\,d\xi = \delta y_0$$

and

$$y_0 \le \delta y_0.$$

Therefore $y_0 = 0$. Consequently, the problem (12) has only the trivial solution. $\square$

Theorem 2. *Let the function f in the last m variables have partial derivatives which on $[0,\omega] \times R^m$ satisfy both the local Carathéodory conditions and the inequalities*

$$\sigma \sum_{k=1}^m \frac{\partial f(t, x_1, \dots, x_m)}{\partial x_k} \ge \alpha(t), \tag{17}$$

$$\left|\frac{\partial f(t, x_1, \dots, x_m)}{\partial x_k}\right| \le \beta_k(t) \quad (k = 1, \dots, m), \tag{18}$$

where α and $\beta_k : [0,\omega] \to [0,+\infty[$ $(k = 1,\dots,m)$ are summable functions. Moreover, let α be different from zero on a set of positive measure and almost everywhere on $[0,\omega]$ the inequality (9) *be fulfilled. Then the equation* (1) *has one and only one ω-periodic solution.*

Proof. Suppose

$$f_k(t, x_1, \dots, x_m) = \frac{\partial f(t, x_1, \dots, x_m)}{\partial x_k} \quad (k = 1, \dots, m),$$

$$\overline{p}_k(t, x_1, \dots, x_m, y_1, \dots, y_m) = \int_0^1 f_k\Big(t, sx_1 + (1-s)y_1, \dots, sx_m + (1-s)y_m\Big)\,ds,$$

$$p_k(t, x_1, \dots, x_m) = \overline{p}_k(t, x_1, \dots, x_m, 0, \dots, 0) \quad (k = 1, \dots, m),$$

$$q(t) = \Big|f(t, 0, \dots, 0)\Big|.$$

Then, because of (17) and (18), conditions (6)–(8) are fulfilled. Therefore, by Theorem 1, the equation (1) has at least one ω-periodic solution.

To complete the proof of the theorem, it remains to show that the equation (1) has no more than one ω-periodic solution, i.e., the problem (4), (5) has no more than one solution. Let x and $\overline{x}$ be two arbitrary solutions of this problem. Then the function $y(t) = \overline{x}(t) - x(t)$ is a solution of the problem (12), where

$$p_{0k}(t) = \overline{p}_k\Big(t, \overline{x}(\tau_{01}(t)), \dots, \overline{x}(\tau_{0m}(t)), x(\tau_{01}(t)), \dots, x(\tau_{0m}(t))\Big) \quad (k = 1, \dots, m).$$

Moreover, the functions $p_{0k}(k = 1, \dots, m)$, as it follows from (17) and (18), satisfy the inequalities (10) and (11). However, as it is proved above, the conditions (9)–(11) and (16) guarantee for the problem (12) to have only the trivial solution. Consequently, $\overline{x}(t) \equiv x(t)$. $\square$

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References

1. I. T. Kiguradze and B. Puža, Conti–Opial type theorems for nonlinear functional differential equations. (Russian) *Differentsial'nye Uravneniya* **33**(1997), No. 2 (*to appear*).

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