Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2025, 1–18

Nanasaheb Phatangare, Krishnat Masalkar, Subhash Kendre

LIMIT CYCLES OF PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS WITH NILPOTENT CENTER AND LINEAR SADDLE

Abstract. In this paper, we study a number of limit cycles of the piecewise smooth differential systems separated by one or two parallel straight lines and formed by a nilpotent center, or degenerate center and linear saddle. Piecewise linear differential systems separated by one or two parallel straight lines, one of whose subsystems is of nilpotent center type and the other is of linear saddle type, can have at most two limit cycles, and there are systems in these classes having one limit cycle. The limit cycle, in particular, consists of saddle separatrices of subsystems.

2020 Mathematics Subject Classification. 37G05, 37G10, 37G15, 37E05, 37G35, 37H20, 37J20.

Key words and phrases. Piecewise linear differential system, Hamiltonian, limit cycle.

1 Introduction

Limit cycles are isolated periodic orbits of the differential system. The study of limit cycles has long history (see, e.g., [23]). Many real-life phenomena are related to the existence of limit cycles (for details, see [19, 26–28]). Piecewise linear differential systems separated by a straight line appear in mechanics, electrical circuits, economics, control theory, etc. (see, e.g., [4, 24, 25]). The study of such a system goes back to the mid-twentieth century (see, e.g., [2]). Continuous piecewise linear differential systems (PLDS) separated by one or two parallel straight lines appear in the control theory (see, e.g., [3, 10, 13, 18]). Planar continuous piecewise linear vector fields with two zones are studied in [8, 21]. A canonical form is obtained and different techniques for the formation of periodic orbits are discussed. Global properties of continuous piecewise linear vector fields in $\mathbb{R}^2$ are studied in [16, 17]. Using the higher order averaging theory, in [12], it is shown that the discontinuous quadratic and cubic polynomial perturbations of a linear center have more limit cycles than those of continuous and discontinuous linear perturbations. Limit cycles of piecewise Hamiltonian systems with boundary perturbation are discussed in [20].

Discontinuous PLDS formed by two linear differential systems separated by a straight line may have three limit cycles (see, e.g., [5, 6, 9, 11]). In [22], normal forms for piecewise smooth systems of the types saddle-saddle and focus-focus are obtained and the upper bounds for a number of limit cycles, bifurcated from the period annulus, are discussed.

In this paper, we discuss the limit cycles of piecewise differential systems (PDS) placed in two zones and systems in three zones. Limit cycles placed in two or three zones can be either sliding limit cycles or crossing limit cycles. The paper is organized as follows. In Section 1, normal forms of nilpotent center and linear saddle are presented. Section 2 discusses limit cycles of piecewise smooth systems in two and three zones. In Section 3, the number and location of limit cycles of piecewise smooth systems in two and three zones formed by the integrable degenerate center and Hamiltonian saddle are discussed. Section 4 is devoted to the piecewise systems separated by rays.

In [14], it is proved that the continuous PDS separated by one straight line and composed of two linear saddles does not have limit cycles. Also, continuous PDS separated by two parallel straight lines and composed of three linear saddles does not have limit cycles.

Here, we state the results from [15].

Proposition 1.1. *For piecewise linear differential systems, the following statements hold:*

(1) *A continuous PLDS or discontinuous PLDS formed by one center and one linear Hamiltonian saddle and separated by one straight line has no limit cycles.*

(2) *A continuous PLDS formed by two centers and one linear Hamiltonian saddle and separated by two parallel straight lines has no limit cycles.*

(3) *A discontinuous PLDS formed by two centers and one linear Hamiltonian saddle and separated by two parallel straight lines may have at most one limit cycle.*

(4) *A continuous PLDS formed by one center and two Hamiltonian saddles and separated by two parallel straight lines has no limit cycles.*

(5) *A discontinuous PLDS formed by one center and two Hamiltonian saddles and separated by two parallel straight lines may have at most one limit cycle.*

Here, we discuss the limit cycles of PDS located in two zones and three zones and formed by the global nilpotent center and linear saddles.

The normal forms of a nilpotent center at the origin are mentioned in the following result from [7].

Theorem 1.1 ([7]). *Every planar Hamiltonian polynomial vector field of degree three with a global nilpotent center at the origin, symmetric to the x-axis and with all infinite singular points being non-degenerated hyperbolic sectors, after a linear change of variables, can be written in one of the following* *forms:*

$$(\dot{x}, \dot{y}) = (y, -x^3), \tag{1.1}$$

$$(\dot{x}, \dot{y}) = (y + y^3, -x^3), \tag{1.2}$$

$$(\dot{x}, \dot{y}) = (y + x^2y + ay^3, -x^3 - xy^2) \ \ with \ \ a \geq 0, \tag{1.3}$$

$$(\dot{x}, \dot{y}) = (y - x^2y + ay^3, -x^3 + xy^2) \ \ with \ \ a \geq 1, \tag{1.4}$$

$$\begin{aligned}(\dot{x}, \dot{y}) = (y + 2xy + ax^2y + by^3, -x^3 - y^2 - axy^2) \ \ &with \ \ b > 0 \ \ and\ either \ \ a \geq 1, \\ or \ \ a < 1 \ \ with \ \ 4(a-1)^2(a^3 - a^2 - ab - 8a) - 27b^2 > 0.&\end{aligned} \tag{1.5}$$

The normal form of the planar linear Hamiltonian saddle is proved in [14]. Here, we state the result.

Theorem 1.2 ([14])**.** *Let α, β, γ, δ and μ be the constants with $\Delta = \alpha\delta - \beta^2 < 0$, $u = \beta\mu + \delta\gamma < 0$, and $v = \alpha\mu + \beta\gamma$. A Hamiltonian planar linear system having a saddle is topologically conjugate to*

$$\dot{X} = \begin{cases} -\beta x - \delta y + \mu, \\ \alpha x + \beta y + \gamma, \end{cases} \tag{1.6}$$

where $\alpha = 0$ or 1. Further, if $\alpha = 0$, then $\gamma = 0$, $\beta \neq 0$, and if $\alpha = 1$, then $\delta < \beta^2$. Moreover, if the saddle point for system (1.6) is (x_0, y_0), then the points of intersection of its separatrices with the y-axis are $(0, A)$ and $(0, B)$, where

$$x_0 = -\frac{u}{\Delta}, \ \ y_0 = \frac{v}{\Delta}, \quad A = y_0 + \frac{\beta + \sqrt{-\Delta}}{\delta}\, x_0, \quad B = y_0 + \frac{\beta - \sqrt{-\Delta}}{\delta}\, x_0. \tag{1.7}$$

Remark 1.1. Two separatrices of system (1.6) intersect the y-axis on the opposite side of the origin if and only if A and B have the opposite signs. Observe that A and B have opposite signs if and only if $AB < 0$, this amounts to saying that $\frac{y_0}{x_0} < \beta^2 - \alpha\delta + \frac{\beta}{\delta}$.

Hamiltonians for systems (1.1)–(1.6) are given, respectively, by

$$F_1(x, y) = \frac{y^2}{2} + \frac{x^4}{4}\,, \tag{1.8}$$

$$F_2(x, y) = \frac{y^2}{2} + \frac{y^4}{4} + \frac{x^4}{4}\,, \tag{1.9}$$

$$F_3(x, y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^2}{2} + \frac{ay^4}{4}\,, \tag{1.10}$$

$$F_4(x, y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + \frac{y^2}{2} + \frac{ay^4}{4}\,, \tag{1.11}$$

$$F_5(x, y) = \frac{x^4}{4} + a\,\frac{x^2y^2}{2} + xy^2 + \frac{y^2}{2} + \frac{by^4}{4}\,, \tag{1.12}$$

$$H(x, y) = -\frac{1}{2}\,\alpha x^2 - \frac{1}{2}\,\delta y^2 - \beta xy - \gamma x + \mu y. \tag{1.13}$$

Now, consider a piecewise smooth differential system separated by a straight line:

$$\dot{X} = (\dot{x}, \dot{y}) = \begin{cases} \big(F_y(x, y), -F_x(x, y)\big) & \text{if } \ x < 0, \\ \big(H_y(x, y), -H_x(x, y)\big) & \text{if } \ x > 0, \end{cases} \tag{1.14}$$

where $F_x = \frac{\partial F}{\partial x}$, $F_y = \frac{\partial F}{\partial y}$ with $F = F_i$ for $i = 1, 2, 3, 4, 5$.

2 Piecewise smooth Hamiltonian systems formed by the nilpotent center and linear saddle

Limit cycles of a piecewise differential system placed in two zones and one in three zones formed by the linear center and Hamiltonian saddle are discussed in [14,15]. In this section, we discuss a number of limit cycles of a piecewise smooth Hamiltonian differential system separated by one straight line formed by a nilpotent center and a Hamiltonian saddle and a system separated by two straight lines formed by a nilpotent center and two saddles.

Theorem 2.1. *Let $k = a$ or b.*

(1) *System (1.14) has a period annulus around the origin if and only if $\mu = 0$ and $k \geq 0$.*

(2) *System (1.14) has no limit cycle if and only if $-\frac{\delta^2}{\mu^2} < k < 0$.*

(3) *System (1.14) has one limit cycle if and only if $-\frac{\delta^2}{\mu^2} = k$.*

(4) *System (1.14) has two limit cycles if and only if $k < -\frac{\delta^2}{\mu^2}$.*

Proof. If system (1.14) has a periodic solution passing through $(0, y_1)$ and $(0, y_2)$ with $y_2 < 0 < y_1$, then $(0, y_1)$ and $(0, y_2)$ lie on the same level curve of the first integrals $F = F(x, y)$ and $H = H(x, y)$. Hence, we have $F(0, y_1) = F(0, y_2)$ and $H(0, y_1) = H(0, y_2)$.

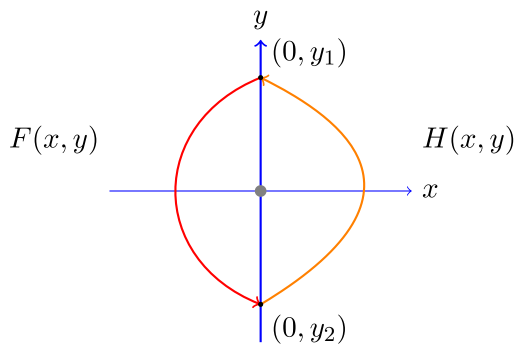


Figure 1: Periodic solution of system (1.14) with center-saddle

Therefore,

$$(y_1 - y_2)(y_1 + y_2)(ky_1{}^2 + ky_2{}^2 + 2) = 0, \quad (y_1 - y_2)(\delta\, y_1 + \delta y_2 - 2\,\mu) = 0. \tag{2.1}$$

Since $y_1 \neq y_2$, we have

$$(y_1 + y_2)(ky_1{}^2 + ky_2{}^2 + 2) = 0, \quad y_1 + y_2 = \frac{2\mu}{\delta}. \tag{2.2}$$

If $k \geq 0$, then $y_1 = -y_2$ and $\mu = 0$. Hence, system (1.14) has a period annulus around the origin, which is bounded by the separatrices of (1.6).

Now, assume that $k < 0$ and $\mu \neq 0$. Eliminating y_2 from the equations in (2.2), we have

$$\frac{2\mu}{\delta}\left(y_1^2 + \left(\frac{2\mu}{\delta} - y_1\right)^2 + \frac{2}{k}\right) = 0. \tag{2.3}$$

Solving (2.3) for y_1, we get

$$y_1 = \frac{k\mu \pm \sqrt{-k^2\mu^2 - k\delta^2}}{\delta k}.$$

Hence, if $-\frac{\delta^2}{\mu^2} < k < 0$, then system (1.14) has no limit cycle, if $-\frac{\delta^2}{\mu^2} = k$, then system (1.14) has one limit cycle, and if $k < -\frac{\delta^2}{\mu^2}$, then system (1.14) has two limit cycles. □

Example 2.1. Consider the particular cases of system (1.14) when $\mu = 0$ and $k \geq 1$.

(1) $F = F_1$:

$$\dot{X} = \begin{cases} (y, -x^3) & \text{if } x < 0, \\ (x + y, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.4}$$

(2) $F = F_2$:

$$\dot{X} = \begin{cases} (y + y^3, -x^3) & \text{if } x < 0, \\ (x + y, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.5}$$

(3) $F = F_3, a = 1$:

$$\dot{X} = \begin{cases} (y + x^2y + y^3, -x^3 - xy^2) & \text{if } x < 0, \\ (x + y, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.6}$$

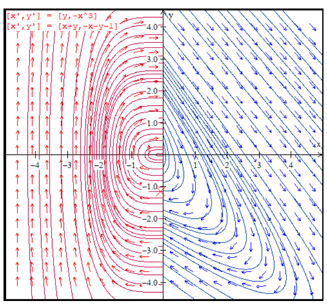
(4) $F = F_4, a = 1$:

$$\dot{X} = \begin{cases} (y - x^2y + y^3, -x^3 + xy^2) & \text{if } x < 0, \\ (x + y, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.7}$$

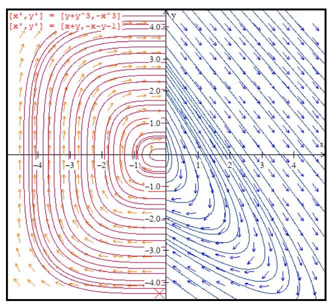
(5) $F = F_5, a = b = 1$:

$$\dot{X} = \begin{cases} (y + 2xy + x^2y + y^3, -x^3 - y^2 - xy^2) & \text{if } x < 0, \\ (x + y, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.8}$$

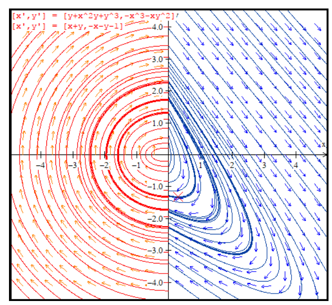
From part (1) of Theorem 2.1, each of systems (2.4)–(2.8) has a period annulus (see Fig. 2).



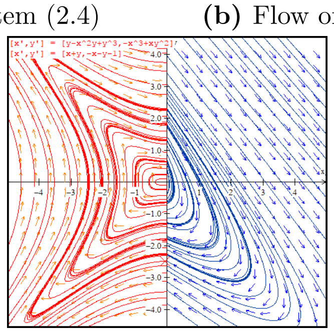
(a) Flow of system (2.4)



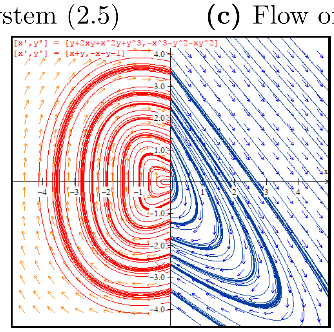
(b) Flow of System (2.5)



(c) Flow of system (2.6)



(d) Flow of system (2.7)



(e) Flow of system (2.8)

Figure 2: Nilpotent center-linear saddle system (1.14) when $\mu = 0$

Example 2.2. Consider system (1.14) with $k = a$ or b and $-\frac{\delta^2}{\mu^2} < k < 0$. In this case, $\delta \neq 0$ and $\mu \neq 0$.

(1) $F = F_3$, $a = -2$:

$$\dot{X} = \begin{cases} (y + x^2y - 2y^3, -x^3 - xy^2) & \text{if } x < 0, \\ (2x + 2y - 1, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.9}$$

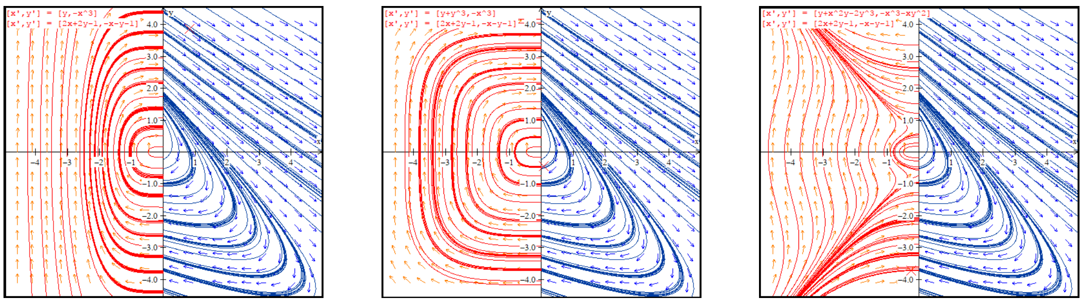
(2) $F = F_4$, $a = -2$:

$$\dot{X} = \begin{cases} (y - x^2y - 2y^3, -x^3 + xy^2) & \text{if } x < 0, \\ (2x + 2y - 1, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.10}$$

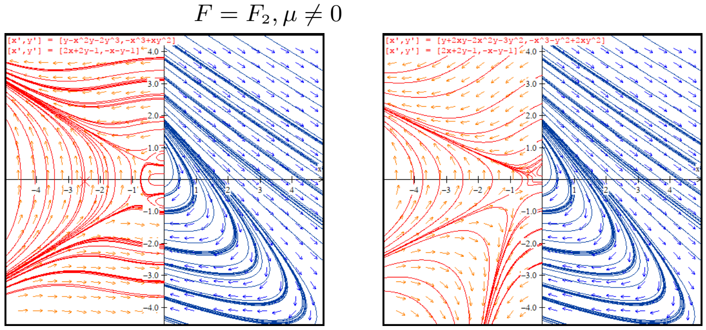
(3) $F = F_5$, $a = -2$, $b = -3$:

$$\dot{X} = \begin{cases} (y + 2xy - 2x^2y - 3y^3, -x^3 - y^2 + 2xy^2) & \text{if } x < 0, \\ (2x + 2y - 1, -x - y - 1) & \text{if } x > 0. \end{cases} \tag{2.11}$$

From part (2) of Theorem (1.1), systems (2.9)–(2.11) have no limit cycles (see Fig. 3).



(a) Flow of system (1.14), $F = F_1, \mu \neq 0$ **(b)** Flow of System (1.14), $F = F_2, \mu \neq 0$ **(c)** Flow of system (2.9)



(d) Flow of system (2.10) **(e)** Flow of system (2.11)

Figure 3: Nilpotent center-linear saddle system (1.14) when $\mu \neq 0$, $-\frac{\delta^2}{\mu^2} < k < 0$

Example 2.3. Consider system (1.14) with $k = -\frac{\delta^2}{\mu^2}$.

(1) $F = F_3$, $a = -4$:

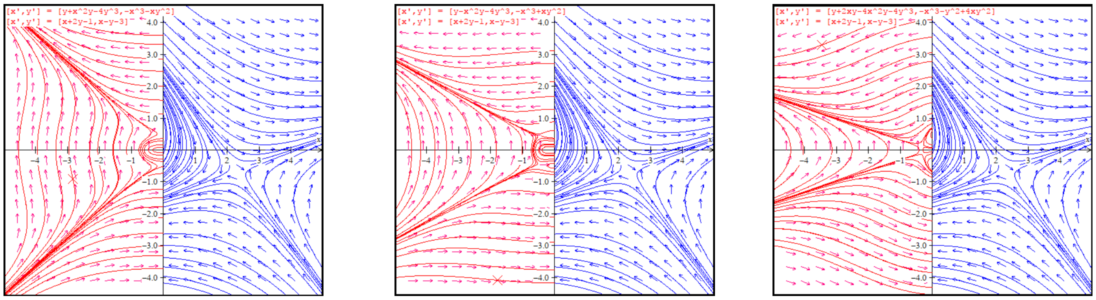
$$\dot{X} = \begin{cases} (y + x^2y - 4y^3, -x^3 - xy^2) & \text{if } x < 0, \\ (x + 2y - 1, x - y - 3) & \text{if } x > 0. \end{cases} \tag{2.12}$$

(2) $F = F_4$, $a = -4$:

$$\dot{X} = \begin{cases} (y - x^2y - 4y^3, -x^3 + xy^2) & \text{if } x < 0, \\ (x + 2y - 1, x - y - 3) & \text{if } x > 0. \end{cases} \tag{2.13}$$

(3) $F = F_5$, $a = -4$, $b = -4$:

$$\dot{X} = \begin{cases} (y + 2xy - 4x^2y - 4y^3, -x^3 - y^2 + 4xy^2) & \text{if } x < 0, \\ (x + 2y - 1, x - y - 3) & \text{if } x > 0. \end{cases} \tag{2.14}$$



(a) Flow of system (2.12) **(b)** Flow of system (2.13) **(c)** Flow of system (2.14)

Figure 4: Nilpotent center-linear saddle system (1.14) when $\mu \neq 0$, $-\frac{\delta^2}{\mu^2} = k < 0$

From part (3) of Theorem (1.1), systems (2.12)–(2.14) may have at most one limit cycle (see Fig. 4).

Example 2.4. Consider system (1.14) with $k < -\frac{\delta^2}{\mu^2}$.

(1) $F = F_3$, $a = -5$:

$$\dot{X} = \begin{cases} (y + x^2y - 5y^3, -x^3 - xy^2) & \text{if } x < 0, \\ (x + 2y - 1, x - y - 3) & \text{if } x > 0. \end{cases} \tag{2.15}$$

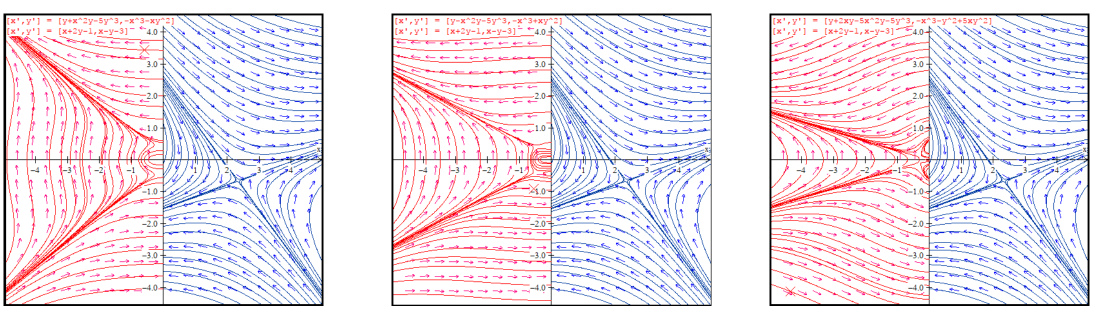
(2) $F = F_4$, $a = -5$:

$$\dot{X} = \begin{cases} (y - x^2y - 5y^3, -x^3 + xy^2) & \text{if } x < 0, \\ (x + 2y - 1, x - y - 3) & \text{if } x > 0. \end{cases} \tag{2.16}$$

(3) $F = F_5$, $a = -5$, $b = -5$:

$$\dot{X} = \begin{cases} (y + 2xt - 5x^2y - 5y^3, -x^3 - y^2 + 5xy^2) & \text{if } x < 0, \\ (x + 2y - 1, x - y - 3) & \text{if } x > 0. \end{cases} \tag{2.17}$$

From part (4) of Theorem (1.14), systems (2.15)–(2.17) may have at most two limit cycles (see Fig. 5).



(a) Flow of system (2.15) **(b)** Flow of system (2.16) **(c)** Flow of system (2.17)

Figure 5: Nilpotent center-linear saddle system (1.14) when $\mu \neq 0$, $k < -\frac{\delta^2}{\mu^2}$

Now, consider the piecewise differential systems placed in three zones separated by two straight lines and formed by a nilpotent center and two Hamiltonian saddles:

$$(\dot{x},\dot{y}) = \begin{cases} (F_y(x,y), -F_x(x,y)) & \text{if } x<-1, \\ \left(H_y^{(1)}(x,y), -H_x^{(1)}(x,y)\right) & \text{if } -1<x<1, \\ \left(H_y^{(2)}(x,y), -H_x^{(2)}(x,y)\right) & \text{if } x>1, \end{cases} \tag{2.18}$$

where

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}$$

with $F = F_i$ for $i = 1,2,3,4,5$ and

$$H^{(j)} = -\frac{1}{2}\alpha_j x^2 - \frac{1}{2}\delta_j y^2 - \beta_j xy - \gamma_j x + \mu_j y \ \text{ for } \ j=1,2.$$

Theorem 2.2. *Let $k = a$ or b. System (2.18) has at most two limit cycles.*

Proof. Suppose that there is a periodic solution of system (2.18) that passes through the points $(-1,y_1)$, $(-1,y_2)$, $(1,y_3)$ and $(1,y_4)$ with $y_1<y_2$ and $y_4<y_3$.

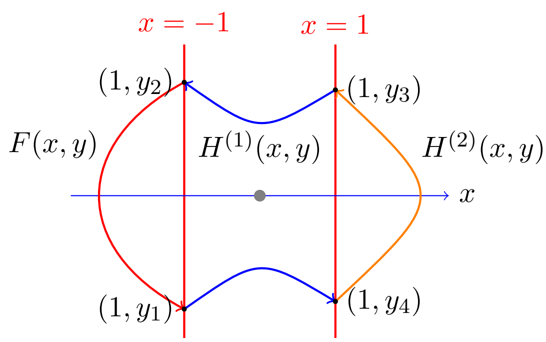


Figure 6: Closed orbit of the system (2.18) with center-saddle-saddle

Note that the solutions of Hamiltonian systems lie along the level curves of the Hamiltonian, then we have

$$F(-1,y_1) = F(-1,y_2), \tag{2.19}$$
$$H^{(1)}(-1,y_2) = H^{(1)}(1,y_3), \tag{2.20}$$
$$H^{(1)}(-1,y_1) = H^{(1)}(1,y_4), \tag{2.21}$$
$$H^{(2)}(1,y_3) = H^{(2)}(1,y_4). \tag{2.22}$$

From equation (2.19) we get

$$\frac{1}{4}(y_1-y_2)(y_1+y_2)(2(a-1)-b(y_1^2-y_2^2)) = 0. \tag{2.23}$$

Since $y_1 \neq y_2$, equation (2.23) gives

$$y_1 = -y_2 \ \text{ or } \ b(y_1^2+y_2^2) = -2(a-1).$$

Note that $b>0$ and $a \geq 0$, and $b(y_1^2+y_2^2) = -2(a-1)$ is not possible.

From equation (2.20) and using $y_1 = -y_2$, we get

$$\frac{\delta_1}{2}(y_1^2-y_3^2) + \beta_1(y_3-y_1) - \mu_1(y_3-y_1) + 2\gamma_1 = 0. \tag{2.24}$$

Similarly, equation (2.21) gives

$$\frac{\delta_1}{2}\,(y_1^2 - y_4^2) + \beta_1(y_1 + y_4) + \mu_1(y_1 - y_4) + 2\gamma_1 = 0. \tag{2.25}$$

Further, from equation (2.22) and using $y_3 \neq y_4$, we find that

$$\delta_2(y_3 + y_4) = -2(\beta_2 - \mu_2). \tag{2.26}$$

Assume $\delta_1 = 0$. Then equations (2.24) and (2.25) become

$$(\beta_1 - \mu_1)(y_3 - y_1) = -2\gamma_1, \tag{2.27}$$

$$(\beta_1 + \mu_1)y_1 + (\beta_1 - \mu_1)y_4 = -2\gamma_1, \tag{2.28}$$

respectively.

If $\beta_1 - \mu_1 \neq 0$, then from (2.27) and (2.28) we get

$$y_3 = -2\frac{\gamma_1}{\beta_1 - \mu_1} + y_1, \quad y_4 = -2\,\frac{\gamma_1}{\beta_1 - \mu_1} - \frac{\beta_1 + \mu_1}{\beta_1 - \mu_1}\,y_1,$$

and $y_1 = -y_2$ is a parameter. In this case, we have a period annulus around the origin for system (2.18).

If $\beta_1 = \mu_1 \neq 0$, then from (2.27) we have $\gamma_1 = 0$ and $y_1 = 0$, which is a contradiction, since $y_1 \neq 0$. Hence, either $\beta_1 \neq \mu_1$ or $\beta_1 = \mu_1 = 0$, and we have a period annulus in this case.

Next, assume that $\delta_1 \neq 0$. From subtraction of equations (2.24) and (2.25) we obtain

$$\frac{1}{2}\,(y_3 - y_4)\big(\delta_1(y_3 + y_4) + 2(\beta_1 - \mu_1)\big) = 2(\beta_1 + \mu_1)y_1. \tag{2.29}$$

Multiplying equation (2.29) by δ_2 and using equation (2.26), we get

$$(y_3 - y_4)\big(- \delta_1(\beta_2 - \mu_2) + \delta_2(\beta_1 - \mu_1)\big) = 2\delta_2(\beta_1 + \mu_1)y_1. \tag{2.30}$$

If $\delta_2 = 0$, then $\beta_2 = \mu_2$. From equation (2.24), we can obtain y_3 in terms of y_1, and from (2.25), we can find y_4 in terms of y_1. Note that $y_1 = -y_2$ is a parameter. Hence, in this case, system (2.18) has a period annulus and at most one limit cycle formed by saddle separatrices.

Now, assume that $\delta_1\delta_2 \neq 0$. Let $\beta_i - \mu_i = l_i$, $i = 1, 2$, and $\Delta = \delta_2 l_1 - \delta_1 l_2$. Then y_1, y_2, y_3 and y_4 are related by the following equations:

$$y_2 = -y_1,$$

$$y_4 = -y_3 - 2\,\frac{l_2}{\delta_2}\,,$$

$$y_3 = \frac{\delta_2^2 l_1 y_1 - \Delta l_2}{\delta_2\Delta}\,, \tag{2.31}$$

$$\frac{\delta_1}{2}\,(y_1^2 - y_3^2) + l_1(y_3 - y_1) + 2\gamma_1 = 0. \tag{2.32}$$

Note that $\Delta = 0$ implies $l_1 = 0$. Hence, $l_2 = 0$ and from (2.26), $\delta_2 = 0$, which is a contradiction.

From equations (2.31) and (2.32), we get a quadratic equation for y_1. It has at most two positive real roots and hence system (2.18) will have at most two limit cycles. □

3 Limit cycles of piecewise smooth integrable systems

In this section, we discuss the limit cycles of piecewise differential systems placed in two zones separated by one straight line and formed by an integrable degenerate center and Hamiltonian saddle, and systems placed in three zones separated by two straight lines formed by one integrable degenerate center and two Hamiltonian saddles.

The differential systems

$$(\dot{x},\dot{y}) = \big(y(x^2-y^2)-2x^4y,\, x(x^2+y^2)-2x^3y^2\big), \tag{3.1}$$
$$(\dot{x},\dot{y}) = \big(-y(3x^2+y^2),\, x(x^2-y^2)\big) \tag{3.2}$$

are integrable systems with degenerate centers at $(0,0)$. The integrals of systems (3.1) and (3.2) are given by

$$I_1(x,y) = \ln(x^2+y^2-1) - \frac{1}{2}\ln(x^4+y^4) - \tan^{-1}\Big(\frac{x^2}{y^2}\Big),$$
$$I_2(x,y) = \frac{1}{2}\ln(x^2+y^2) - \frac{x^2}{x^2+y^2},$$

respectively.

Consider the system

$$(\dot{x},\dot{y}) = \begin{cases} \big(I_{1y}(x,y), -I_{1x}(x,y)\big) & \text{if } x<0, \\ \big(H_y(x,y), -H_x(x,y)\big) & \text{if } x>0. \end{cases} \tag{3.3}$$

Theorem 3.1. *Let $x_0 = -\frac{\beta\mu+\delta\gamma}{\alpha\delta-\beta^2} > 0$. System (3.3) has at most one limit cycle which consists of saddle separatrices of the right subsystem if and only if $\mu = 0$.*

Proof. Suppose that there is a periodic solution of system (3.3) passing through $(0,y_1)$ and $(0,y_2)$ with $y_1<y_2$. Then we have $H(0,y_1)=H(0,y_2)$ and $I_1(0,y_1)=I_1(0,y_2)$. This implies that

$$\ln(y_1^2-1)-\ln(y_1^2) = \ln(y_2^2-1)-\ln(y_2^2), \quad \delta(y_1+y_2) = 2\mu.$$

Assume that $\delta \neq 0$ and $k = \frac{2\mu}{\delta}$. Then

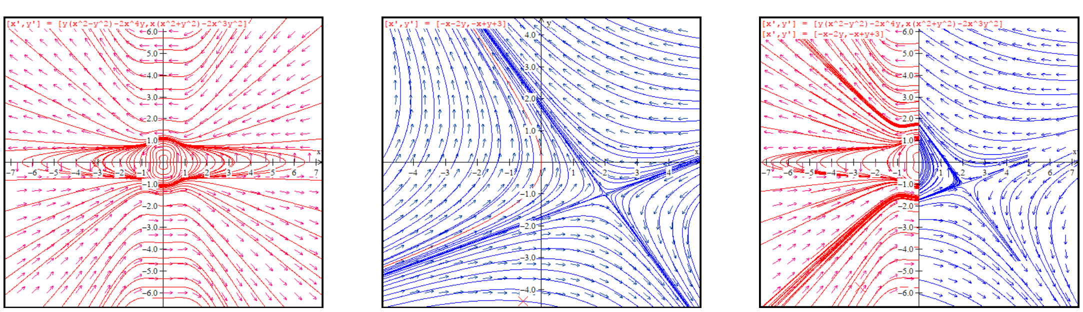
$$(y_1^2-1)(y_2^2) = (y_2^2-1)(y_1^2), \quad y_1 = k-y_2.$$

Therefore, we get $y_1 = -y_2 = k-y_2$. This gives $k=\mu=0$. Hence, system (3.3) has a period annulus consisting of periodic orbits passing through $(0,y)$ for all $y>1$ if and only if $\mu=0$. □

Example 3.1. Consider system (3.3) with $\beta=1$, $\delta=2$, $\mu=0$, $\alpha=-1$ and $\gamma=3$:

$$\dot{X} = \begin{cases} (y(x^2-y^2)-2x^4y,\, x(x^2+y^2)-2x^3y^2) & \text{if } x<0, \\ (-x-2y, -x+y+3) & \text{if } x>0. \end{cases} \tag{3.4}$$

From Theorem 3.1, system (3.4) has at most one limit cycle consisting of saddle separatrices.



(a) Flow of system (3.1) **(b)** Linear saddle **(c)** Flow of system (3.4)

Figure 7: Integrable center-linear saddle system (3.4)

Now, consider the piecewise differential systems formed by a degenerate center and Hamiltonian saddle

$$(\dot{x}, \dot{y}) = \begin{cases} \big(I_{2y}(x,y), -I_{2x}(x,y)\big) & \text{if } x < 0, \\ \big(H_y(x,y), -H_x(x,y)\big) & \text{if } x > 0. \end{cases} \tag{3.5}$$

Theorem 3.2. *Let $x_0 = -\frac{\beta\mu+\delta\gamma}{\alpha\delta-\beta^2} > 0$. System (3.5) has at most one limit cycle which consists of saddle separatrices of the right subsystem if and only if $\mu = 0$.*

Proof. Let A and B be the constants given by equation (1.7). Assume that there is a periodic solution of system (3.5) passing through the points $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Then $H(0, y_1) = H(0, y_2)$ and $I_2(0, y_1) = I_2(0, y_2)$.

Hence,

$$\ln(y_1^2) = \ln(y_2^2) \Longrightarrow y_1 = -y_2, \quad \delta(y_1 + y_2) = 2\mu \Longrightarrow \mu = 0.$$

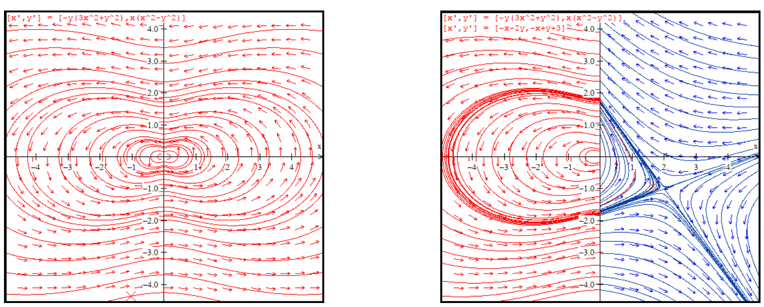
Therefore, for each $|y| < \min\{A, B\}$, there is a periodic orbit passing through the points $(0, \pm y)$. Thus, there is a period annulus inside a limit cycle formed by saddle separatrices of system (3.5) if and only if $\mu = 0$. $\square$

Example 3.2. Consider the center-saddle system (3.5) with $\beta = 1$, $\delta = 2$, $\mu = 0$, $\alpha = -1$ and $\gamma = 3$:

$$\dot{X} = \begin{cases} \big(-y(3x^2+y^2), x(x^2-y^2)\big) & \text{if } x < 0, \\ (-x-2y, -x+y+3) & \text{if } x > 0. \end{cases} \tag{3.6}$$

In this case, the center is integrable and degenerate and the saddle on the right side is linear Hamiltonian with saddle point (x_0, y_0) such that $x_0 > 0$.

From Theorem 3.2, system (3.6) may have at most one limit cycle consisting of separatrices of the saddle.



(a) Flow of system (3.2) **(b)** Center-saddle (3.6)

Figure 8: Integrable degenerate center-linear saddle

Next, consider the piecewise differential system placed in three zones formed by one degenerate center (3.1) and two Hamiltonian saddles:

$$(\dot{x}, \dot{y}) = \begin{cases} \big(I_{1y}(x,y), -I_{1x}(x,y)\big) & \text{if } x < -1, \\ \big(H_y^{(1)}(x,y), -H_x^{(1)}(x,y)\big) & \text{if } -1 < x < 1, \\ \big(H_y^{(2)}(x,y), -H_x^{(2)}(x,y)\big) & \text{if } x > 1 \end{cases} \tag{3.7}$$

Theorem 3.3. *Consider system (3.7).*

(1) *If $\delta_1\delta_2 \neq 0$, then system (3.7) has at most one periodic solution.*

(2) *If $\delta_1 = 0$, $\delta_2(\mu_1 - \beta_1) \neq 0$, then system (3.7) has at most one limit cycle.*

(3) *If $\delta_1 \neq 0$ and $\delta_2 = 0$, then system (3.7) has at most one limit cycle.*

(4) *If $\delta_2 = \delta_1 = \mu_2 - \beta_2 = 0$, then system (3.7) has a period annulus.*

Proof. Suppose that there is a periodic solution of system (3.7) which passes through the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ with $y_1 < y_2$ and $y_4 < y_3$. Since the solutions of the integrable systems lie along level curves of the first integrals, we have

$$I_1(-1, y_1) = I_1(-1, y_2), \tag{3.8}$$
$$H^{(1)}(-1, y_2) = H^{(1)}(1, y_3), \tag{3.9}$$
$$H^{(1)}(-1, y_1) = H^{(1)}(1, y_4), \tag{3.10}$$
$$H^{(2)}(1, y_3) = H^{(2)}(1, y_4). \tag{3.11}$$

From equation (3.8), we get

$$2\ln\left(\frac{y_1}{y_2}\right) - \frac{1}{2}\ln\left(\frac{{y_1}^4 + 1}{{y_2}^4 + 1}\right) - \arctan({y_1}^{-2}) + \arctan({y_2}^{-2}) = 0. \tag{3.12}$$

The real roots of equation (3.12) satisfy $y_1^2 = y_2^2$. But $y_1 \neq y_2$. Hence, $y_1 = -y_2$.

From equations (3.9) and (3.10) we have

$$\frac{\delta_1}{2}(y_2^2 - y_3^2) + \beta_1(y_2 + y_3) + \mu_1(y_2 - y_3) + 2\gamma_1 = 0. \tag{3.13}$$

Similarly, from equation (3.10) we have

$$\frac{\delta_1}{2}(y_1^2 - y_4^2) + \beta_1(y_1 + y_4) + \mu_1(y_1 - y_4) + 2\gamma_1 = 0. \tag{3.14}$$

Also, from equation (3.11) we find that

$$y_3 = y_4 - 2\nu_2. \tag{3.15}$$

Subtracting equation (3.14) from equation (3.13), in view of (3.15), we get

$$y_1 = -y_2 = -\frac{(y_4 - \nu_2)(\nu_1 - \nu_2)}{l_1}, \tag{3.16}$$

where

$$l_i = \frac{\mu_i + \beta_i}{\delta_i}, \quad \nu_i = \frac{\mu_i - \beta_i}{\delta_i}, \quad k_i = \frac{\gamma_i}{\delta_i} \text{ for } i = 1, 2.$$

Now, eliminating y_1 from equations (3.13), (3.14) and (3.15), we obtain

$$Py_4^2 + Qy_4 + R = 0, \tag{3.17}$$

where

$$P = \nu_2^2 - l_1^2, \quad Q = -2(\nu_1 + \nu_2)P, \quad R = \nu_2^4 + \nu_1^2\nu_2^2 + P\nu_1\nu_2 + (4k_1 - 2\nu_2^2)l_1^2.$$

Equation (3.17) has at most one positive or negative root which gives the value of y_4 and the values of y_1, y_2, y_3, satisfying equations (3.8)–(3.11), can be determined from equations (3.15) and (3.16). Hence, if $\delta_1\delta_2(\mu_1 + \beta_1) \neq 0$ and $\delta_1 \neq 0$, then system (3.7) has at most one periodic solution.

Now, consider the case $\delta_1 = 0$. From equations (3.13) and (3.14) we get

$$-(\beta_1 + \mu_1)y_1 + (\beta_1 - \mu_1)y_3 + 2\gamma_1 = 0, \tag{3.18}$$
$$(\beta_1 + \mu_1)y_1 + (\beta_1 - \mu_1)y_4 + 2\gamma_1 = 0. \tag{3.19}$$

Further, if $\beta_1 - \mu_1 \neq 0$, then from equations (3.18) and (3.19) we get

$$y_3 = \frac{-2\gamma_1}{\beta_1 - \mu_1} + \frac{\beta_1 + \mu_1}{\beta_1 - \mu_1}y_1, \quad y_4 = \frac{-2\gamma_1}{\beta_1 - \mu_1} - \frac{\beta_1 + \mu_1}{\beta_1 - \mu_1}y_1.$$

Substituting y_3 and y_4 in (3.12), we get only one value for y_1, when $\delta_2 \neq 0$, and hence only one periodic solution.

If $\delta_1 = 0$, $\delta_2 = 0$ and $\mu_2 - \beta_2 = 0$, then y_1 and y_2 are determined in terms of parameters y_3 and y_4. Hence, in this case, we get a period annulus. Similarly, if $\delta_2 = 0$ and $\delta_1 \neq 0$, then the system has at most one periodic solution. □

Now, consider the piecewise differential system in three zones formed by a degenerate center (3.2) and two Hamiltonian saddles:

$$(\dot{x}, \dot{y}) = \begin{cases} \big(I_{2y}(x,y), -I_{2x}(x,y)\big) & \text{if } x < -1, \\ \big(H_y^{(1)}(x,y), -H_x^{(1)}(x,y)\big) & \text{if } -1 < x < 1, \\ \big(H_y^{(2)}(x,y), -H_x^{(2)}(x,y)\big) & \text{if } x > 1. \end{cases} \tag{3.20}$$

Theorem 3.4. *Consider system (3.20).*

(1) *If $\delta_1\delta_2 \neq 0$, then the system (3.20) has at most one periodic solution.*

(2) *If $\delta_1 = 0$, $\delta_2(\mu_1 - \beta_1) \neq 0$, then system (3.20) has one limit cycle.*

(3) *If $\delta_1 \neq 0$ and $\delta_2 = 0$, then system (3.20) has at most one limit cycle.*

(4) *If $\delta_2 = \delta_1 = \mu_2 - \beta_2 = 0$, then system (3.20) has a period annulus.*

Proof. Suppose that there is a periodic solution of system (3.20) which passes through the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ with $y_1 < y_2$ and $y_4 < y_3$. Since the solutions of the integrable systems lie along level curves of the first integrals, we have

$$I_2(-1, y_1) = I_2(-1, y_2), \tag{3.21}$$
$$H^{(1)}(-1, y_2) = H^{(1)}(1, y_3), \tag{3.22}$$
$$H^{(1)}(-1, y_1) = H^{(1)}(1, y_4), \tag{3.23}$$
$$H^{(2)}(1, y_3) = H^{(2)}(1, y_4). \tag{3.24}$$

From equation (3.21) we get

$$\ln\left(\frac{y_1^2+1}{y_2^2+1}\right) - \frac{2}{y_1^2+1} + \frac{2}{y_2^2+1} = 0.$$

The real roots of the equation satisfy $y_1^2 = y_2^2$. But $y_1 \neq y_2$, so $y_1 = -y_2$. The rest of the proof is similar to that of Theorem 3.3. □

4 Limit cycles of piecewise system separated by rays

In this section, we discuss the piecewise differential system separated by rays and formed by the linear integrable system with saddle and center. We convert systems into polar form and study a number of limit cycles for a piecewise smooth differential system of type center-saddle separated by rays $\theta = k$ and center-saddle-saddle type separated by two rays $\theta_i = k_i$ for $i = 1, 2$.

A linear integrable system with a saddle point at (α, β), $\alpha > 0$, and oriented in anticlockwise direction is given by

$$(\dot{x}, \dot{y}) = \big(a(x-\alpha) - (y-\beta), -(x-\alpha) + a(y-\beta)\big), \quad -1 < a < 0. \tag{4.1}$$

Its integral is given by

$$a \ln\left(\frac{y - x - \beta + \alpha}{y + x - \alpha - \beta}\right) + \ln\big((y-\beta)^2 - (x-\alpha)^2\big) = \ln c^2.$$

Therefore, the solution passing through $(\rho, 0)$ is

$$a\ln\left(\frac{y-x+\alpha-\beta}{y+x-\alpha-\beta}\right)+\ln\left((y-\beta)^2-(x-\alpha)^2\right)=a\ln\left(\frac{-\rho+\alpha-\beta}{\rho-\alpha-\beta}\right)+\ln\left(\beta^2-(\rho-\alpha)^2\right).$$

This solution intersects the y-axis (i.e., $\theta = \frac{\pm\pi}{2}$) at two points $(0, y_1)$ and $(0, y_2)$ which will satisfy

$$a\ln\left(\frac{y_i+\alpha-\beta}{y_i-\alpha-\beta}\right)+\ln\left((y_i-\beta)^2-\alpha^2\right)=a\ln\left(\frac{-\rho+\alpha-\beta}{\rho-\alpha-\beta}\right)+\ln\left(\beta^2-(\rho-\alpha)^2\right) \tag{4.2}$$

for $i = 1, 2$. Here, note that $y_1 \neq -y_2$.

Similarly, for any fix $0 < \theta = \phi < \frac{\pi}{2}$, the solution given by (4.2) intersects two rays $\theta = \phi$ and $\theta = -\phi$, which are not symmetric about the x-axis.

The saddle separatrices, invariant eigenspaces of system (4.1), are

$$\frac{y-\beta}{x-\alpha} = \pm 1.$$

The saddle separatrices intersect the rays $\theta = \pm\phi$ at the points that are symmetric to the x-axis.

Polar form of the general planar system $\dot{x} = F(x, y), \quad \dot{y} = G(x, y)$ is given by

$$\frac{dr}{d\theta} = r\,\frac{F(r\cos\theta, r\sin\theta)\cos\theta + G(r\cos\theta, r\sin\theta)\sin\theta}{-F(r\cos\theta, r\sin\theta)\sin\theta + G(r\cos\theta, r\sin\theta)\cos\theta}\,.$$

Hence, polar form of the integrable systems (3.1) and (3.2) are given, respectively, by

$$\frac{dr}{d\theta} = -2r(r^2-1)\,\frac{\cos^3\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\,, \tag{4.3}$$

$$\frac{dr}{d\theta} = 4r\,\frac{\cos^3\theta\sin\theta}{\cos^4\theta-4\cos^2\theta\sin^2\theta-\sin^4\theta}\,. \tag{4.4}$$

The integrals of equations (4.3) and (4.4) are

$$\begin{aligned}\frac{r^2-1}{r^2} &= \frac{1}{2\cos^4(\theta)-2\cos^2(\theta)+1}\,e^{-2\tan^{-1}(2\cos^2(\theta)-1)},\\ r^4 &= \frac{1}{4\cos^4(\theta)-2\cos^2(\theta)-1}\,e^{\frac{\sqrt{5}}{10}\tanh^{-1}(\frac{\sqrt{5}}{10}(8\cos^2(\theta)-2))},\end{aligned}$$

respectively.

Let $n \in \mathbb{N}$. The generalized trigonometric functions $x(\theta) = Cs(\theta)$, $y(\theta) = Sn(\theta)$ satisfy the properties mentioned in Proposition 4.1 [1].

Proposition 4.1.

(1) $Cs^{2n}(\theta) + nSn^2(\theta) = 1.$

(2) *$Cs(\theta)$ and $Sn(\theta)$ are T-period functions with $T = 2\sqrt{\frac{\pi}{n}}\,\frac{\Gamma(\frac{1}{2n})}{\Gamma(\frac{n+1}{2n})}$, where Γ denotes the Gamma function.*

(2) $\displaystyle\int_0^T Sn^p(\theta)Cs^q(\theta)\,d\theta = 0$ *if p or q is odd.*

(3) $\displaystyle\int_0^T Sn^p(\theta)Cs^q(\theta)\,d\theta = \frac{2}{\sqrt{n^{p+1}}}\,\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2n})}{\Gamma(\frac{p+1}{2}+\frac{q+1}{2n})}$ *if both p and q are even.*

Consider the system $(\dot{x}, \dot{y}) = (-y, x^{2n-1})$ with a nilpotent center at the origin. If we substitute $x = RCs(\theta)$, $y = RSn(\theta)$, then in these coordinates the system becomes

$$\dot{R} = \frac{x^{2n-1}(-y) + y(x^{2n-1})}{R^{2n-1}}, \quad \dot{\theta} = \frac{x(x^{2n-1}) - ny(-y)}{R^{n+1}},$$

which is equivalent to $(\dot{R}, \dot{\theta}) = (0, 1)$.

Let $-1 < a < 0$ and $0 < \phi \le \frac{\pi}{2}$. Consider the piecewise smooth planar differential system separated by the rays $\theta = \phi$ and $\theta = -\phi$ passing through the origin:

$$(\dot{x}, \dot{y}) = (a(x-\alpha) - (y-\beta), -(x-\alpha) + a(y-\beta)) \text{ if } -\phi < \theta < \phi, \tag{4.5}$$

$$(\dot{R}, \dot{\theta}) = (0, 0) \text{ if } \phi < \theta < 2\pi - \phi, \tag{4.6}$$

where (r, θ) are the polar coordinates and (R, θ) are the generalized polar coordinates.

Theorem 4.1. *The piecewise smooth differential system formed by (4.5) and (4.6) has exactly one limit cycle, which consists of saddle separatrices of (4.5) and solution of (4.6) passing through the point of intersection of saddle separatrix and the ray $\theta = \phi$.*

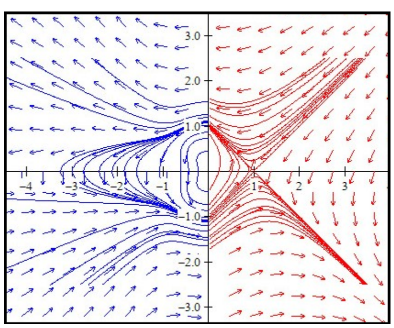
Proof. The only saddle separatrix of system (4.5) is symmetric about the x-axis. Also, all solutions of system (4.6) are symmetric and system (4.6) has a global center. Hence, the proof is complete. □

Let $0 < \phi \le \frac{\pi}{2}$ and $-1 < a < 0$. Consider a piecewise smooth planar differential system separated by the rays $\theta = -\phi$ and $\theta = \phi$ passing through the origin:

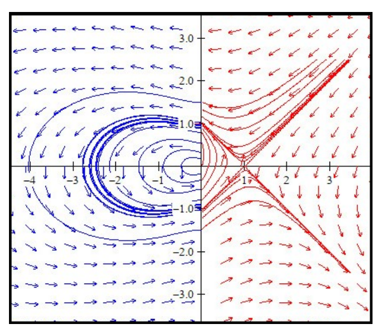
$$(\dot{x}, \dot{y}) = \begin{cases} \big(a(x-\alpha) - (y-\beta), -(x-\alpha) + a(y-\beta)\big) & \text{if } -\phi < \theta < \phi, \\ \big(y(x^2 - y^2) - 2x^4 y, x(x^2 + y^2) - 2x^3 y^2\big) & \text{if } \phi < \theta < 2\pi - \phi. \end{cases} \tag{4.7}$$

Theorem 4.2. *The piecewise smooth system (4.7) has one limit cycle if $|\alpha \pm \beta| \le 1$, otherwise there is no limit cycle.*

Proof. The proof is similar to that of Theorem 4.1. □



(a) System (4.7) when $\theta = \frac{\pi}{2}$



(b) System (4.8) when $\theta = \frac{\pi}{2}$

Figure 9: Integrable piecewise system separated by rays

Let $0 < \phi \le \frac{\pi}{2}$ and $-1 < a < 0$. Consider the piecewise smooth planar differential system separated by the rays $\theta = -\phi$ and $\theta = \phi$ passing through the origin:

$$(\dot{x}, \dot{y}) = \begin{cases} \big(a(x-\alpha) - (y-\beta), -(x-\alpha) + a(y-\beta)\big) & \text{if } -\phi < \theta < \phi, \\ \big(-y(3x^2 + y^2), x(x^2 - y^2)\big) & \text{if } \phi < \theta < 2\pi - \phi. \end{cases} \tag{4.8}$$

Theorem 4.3. *The piecewise smooth system (4.8) has exactly one limit cycle.*

Proof. The proof is similar to that of Theorem 4.1. □

Acknowledgement

The authors would like to express sincere gratitude to the reviewers for their valuable suggestions and comments.

References

[1] M. J. Álvarez and A. Gasull, Generating limit cycles from a nilpotent critical point via normal forms. *J. Math. Anal. Appl.* **318** (2006), no. 1, 271–287.

[2] A. A. Andronow and C. E. Chaikin, *Theory of Oscillations*. English Language Edition Edited Under the Direction of Solomon Lefschetz. Princeton University Press, Princeton, NJ, 1949.

[3] D. P. Atherton, Nonlinear control systems. In *R. C. Dorf (Ed.), Systems, Controls, Embedded Systems, Energy, and Machines*, pp. 11–72, *The Electrical Engineering Handbook Series, CRC Press, Taylor* & *Francis Group, Boca Raton, London, New York*, 2006.

[4] M. Bernardo, Ch. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems: Theory and Applications*. Applied Mathematics and Sciences, Vol. 163. Springer Science & Business Media, London, 2008.

[5] J. L. Cardoso, J. Llibre, D. D. Novaes and D. J. Tonon, Simultaneous occurrence of sliding and crossing limit cycles in piecewise linear planar vector fields. *Dyn. Syst.* **35** (2020), no. 3, 490–514.

[6] D. de Carvalho Braga and L. F. Mello, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane. *Nonlinear Dynam.* **73** (2013), no. 3, 1283–1288.

[7] F. S. Dias, J. Llibre and C. Valls, Polynomial Hamiltonian systems of degree 3 with symmetric nilpotent centers. *Math. Comput. Simulation* **144** (2018), 60–77.

[8] E. Freire, E. Ponce, F. Rodrigo and F. Torres, Bifurcation sets of continuous piecewise linear systems with two zones. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8** (1998), no. 11, 2073–2097.

[9] E. Freire, E. Ponce and F. Torres, A general mechanism to generate three limit cycles in planar Filippov systems with two zones. *Nonlinear Dynam.* **78** (2014), no. 1, 251–263.

[10] M. A. Henson and D. E. Seborg, *Nonlinear Process Control*. Prentice Hall PTR Upper Saddle River, New Jersey, 1997.

[11] J. Llibre and E. Ponce, Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms* **19** (2012), no. 3, 325–335.

[12] J. Llibre and Y. Tang, Limit cycles of discontinuous piecewise quadratic and cubic polynomial perturbations of a linear center. *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019), no. 4, 1769–1784.

[13] J. Llibre and A. E. Teruel, *Introduction to the Qualitative Theory of Differential Systems. Planar, Symmetric and Continuous Piecewise Linear Systems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser/Springer, Basel, 2014.

[14] J. Llibre and C. Valls, Limit cycles of planar piecewise differential systems with linear Hamiltonian saddles. *Symmetry* **13** (2021), no. 7, 1128, 10 pp.

[15] J. Llibre and C. Valls, Limit cycles of piecewise differential systems with linear Hamiltonian saddles and linear centres. *Dyn. Syst.* **37** (2022), no. 2, 262–279.

[16] R. Lum and L. O. Chua, Global properties of continuous piecewise linear vector fields. Part I: Simplest case in $\mathbb{R}^2$. *Int. J. Circuit Theory Appl.* **19** (1991), no. 3, 251–307.

[17] R. Lum and L. O. Chua, Global properties of continuous piecewise linear vector fields. Part II: Simplest symmetric case in $\mathbb{R}^2$. *Int. J. Circuit Theory Appl.* **20** (1992), no. 1, 9–46.

[18] K. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability*. Academic Press, 2014.

[19] A. A. Pechenkin, Understanding of the history of the Belousov–Zhabotinsky reaction. *Studia Philosophica IV* **40** (2004), 106–130.

[20] N. Phatangare, S. Kendre and K. Masalkar, Bifurcations of limit cycles in piecewise smooth Hamiltonian system with boundary perturbation. *Differ. Equ. Appl.* **14** (2022), no. 4, 499–524.

[21] N. Phatangare, K. Masalkar and S. Kendre, Limit cycles of piecewise smooth differential systems of the type nonlinear centre and saddle. *J. Difference Equ. Appl.* **30** (2024), no. 12, 1900–1922.

[22] N. Phatangare, K. Masalkar and S. Kendre, Limit cycles in piecewise smooth differential systems of focus-focus and saddle-saddle dynamics. *Comput. Methods Differ. Equ.* **13** (2025), no. 2, 395–419.

[23] M. H. Poincaré, Sur L'intégration algébrique des équations différentielles du premier ordre et du premier degré. *Rend. Circ. Matem.* **5** (1891), 161–191.

[24] D. J. W. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems.* World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 70. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[25] M. A. Teixeira, Perturbation Theory for Non-smooth Systems. In: *Gaeta, G. (eds) Perturbation Theory. Encyclopedia of Complexity and Systems Science Series*, pp. 503–517, Springer, New York, NY, 2009.

[26] B. Van der Pol, On "relaxation-oscillations". *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science (7)* **2** (2009), no. 11, 978–992.

[27] Ba. Van der Pol, Theory of the amplitude of frfeE. forced triode vibrations. *Radio Review* **1** (1920), 701–710, 754–762.

[28] A. M. Zhabotinsky, BPeriodical oxidation of malonic acid in solution (a study of the Belousov reaction kinetics). *Biofizika* **9** (1964), 306–311.

(Received 20.07.2024; revised 16.03.2025; accepted 06.05.2025)

Authors' addresses:

Nanasaheb Phatangare
Department of Mathematics, Fergusson College, Pune – 411004, India (M.S.)
E-mail: `nmphatangare@gmail.com`

Krishnat Masalkar
Department of Mathematics, Abasaheb Garware College, Pune – 411004, India (M.S.)
E-mail: `krishnatmasalkar@gmail.com`

Subhash Kendre
Department of Mathematics, Savitribai Phule Pune University, Pune – 411007, India (M.S.)
E-mails: `sdkendre@yahoo.com`, `drsubhash2010@gmail.com`