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EXISTENCE AND MULTIPLICITY RESULTS FOR p-HAMILTONIAN SYSTEMS

Abstract. In this paper, we give some new criteria that guarantee the existence of at least one weak solution and two weak solutions for a p-Hamiltonian boundary value problem generated by impulsive effects. To ensure the existence of these solutions, we use variational methods and critical point theory as our main tools.

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1 Introduction

In this research, we prove the existence of at least one weak solution and two weak solutions to the following second-order impulsive p-Hamiltonian system

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \mu\nabla G(t,u), \ \text{ a.e. } t \in J, \\ \triangle(|u_i'(t_j)|^{p-2}u_i'(t_j)) = I_{ij}(u_i(t_j)), \ \ i = 1,2,\dots,N, \ \ j = 1,2,\dots,m, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.1}$$

Here, we assume that

- $N \geq 1$, $m \geq 2$, $p > 1$, $T > 0$ and $\lambda > 0$;
- the function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in $[0,T]$ and is C^1 in $\mathbb{R}^N$;
- $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $G(\,\cdot\,,x)$ is continuous on $[0,T]$ for all $x \in \mathbb{R}^N$, and $G(t,\,\cdot\,)$ is C^1 on $\mathbb{R}^N$ for almost every $t \in [0,T]$;
- $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $J = [0,T] \setminus \{t_1, t_2, \dots, t_m\}$, $u(t) = (u_1(t), \dots, u_N(t))$ and $\triangle(u_i'(t_j)) = u_i'(t_j^+) - u_i'(t_j^-)$ such that $u_i'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u_i'(t)$;
- the functions $I_{ij} : \mathbb{R} \to \mathbb{R}$ ($i = 1,2,\dots,N$ and $j = 1,2,\dots,m$) satisfy $|I_{ij}(s)| \leq L_{ij}|s|^{p-1}$ for every $s \in \mathbb{R}$;
- $A(t) = (a_{ij}(t))_{N\times N}$ is an $N \times N$ continuous symmetric matrix and there is a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x, x) \geq \underline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0,T]$.

The study of the multiplicity of the solutions of Hamiltonian systems, as particular cases of dynamical systems, is mathematically important and interesting from a practical point of view. This is because these systems constitute a natural framework for the mathematical models of many natural phenomena in fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, etc. Inspired by the monographs [27] and [32], the existence and multiplicity of weak solutions for Hamiltonian systems have been investigated by many authors using variational methods (see, e.g., [13,14,16,18,20,28,30,39,43,46] and the references therein).

In recent years, critical points theorems were widely used to solve differential equations (see [3, 7, 10–12, 19, 25] and references therein).

In contrast to Hamiltonian systems, for the general case $p > 1$, the study of the existence and multiplicity of periodic solutions is recent (see [21, 40]). In [40], Xu and Tang proved the existence of periodic solutions for the problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \nabla F(t,u), \ \text{ a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{1.2}$$

by minimax methods in the critical point theory. In [26], Ma and Zhang obtained some results on the existence and multiplicity of non-trivial periodic solutions for system (1.2). These results generalize the corresponding results in [34]. In [21], two existence results have been established by the least action principle and the Mountain-pass lemma for ordinary p-Laplacian systems with nonlinear boundary conditions.

In [25], based on two general three critical points theorems due respectively to Ricceri (see [33]) and Averna–Bonanno (see [4]), the authors proved the existence of three solutions for the p-Hamiltonian system

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \mu\nabla G(t,u), \ \text{ a.e. } t \in J, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

In this article, we use three theorems of Bonanno to prove the existence of one weak solution and two weak solutions for problem (1.1).

2 Preliminaries

For a given non-empty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions:

$$\beta(r_1, r_2) := \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup\limits_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$
$$\rho_2(r_1, r_2) = \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, and

$$\rho(r) = \sup_{v\in\Phi^{-1}(r,\infty)} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{\Phi(v) - r}$$

for all $r \in \mathbb{R}$.

The following critical point theorems due to Bonanno will be used to prove our mail results.

Theorem 2.1 ([6, Theorem 5.1]). *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in (\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)})$, there is $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([6, Theorem 5.5]). *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there is $r \in \mathbb{R}$, with $\inf\limits_X \Phi < r < \sup\limits_X \Phi$, such that*

$$\rho(r) > 0,$$

and for each $\lambda > \frac{1}{\rho(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in (\frac{1}{\rho(r)}, +\infty)$, there is $u_{0,\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r, +\infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.3 ([5, Theorem 3.2]). *Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup\limits_{u\in\Phi^{-1}(r,+\infty)} \Psi(u) < +\infty$ and assume that for each $\lambda \in \big(0, \frac{r}{\sup_{u\in\Phi^{-1}(r,+\infty)} \Psi(u)}\big)$, the functional $J_\lambda = \Phi - \lambda\Phi$ satisfies the* (PS)*-condition and is unbounded from below. Then for each $\lambda \in \big(0, \frac{r}{\sup_{u\in\Phi^{-1}(r,+\infty)} \Psi(u)}\big)$, the functional J_λ admits two distinct critical points.*

Here, we recall some basic concepts that will be used in what follows. Let

$$W_T^{1,p} = \Big\{u : [0, T] \to \mathbb{R}^N : \ u \text{ is absolutely continuous}, \ u(0) = u(T), \ u' \in L^p([0, T], \mathbb{R}^N)\Big\},$$

be endowed with the norm

$$\|u\| = \Big(\int_0^T |u'(t)|^p + \big(A(t)|u(t)|^{p-2}u(t), u(t)\big)\, dt\Big)^{\frac{1}{p}}.$$

Observe that

$$\big(A(t)|x|^{p-2}x,x\big)=|x|^{p-2}\sum_{i,j=1}^{N}a_{ij}(t)x_ix_j\le|x|^{p-2}\sum_{i,j=1}^{N}|a_{ij}(t)|\,|x_i|\,|x_j|\le\Big(\sum_{i,j=1}^{N}\|a_{ij}(t)\|_\infty\Big)|x|^p.$$

Then there exists a constant $\overline{\lambda}\le\sum\limits_{i,j=1}^{N}\|a_{ij}(t)\|_\infty$ such that $(A(t)|x|^{p-2}x,x)\le\overline{\lambda}|x|^p$ for all $x\in\mathbb{R}^N$. So,

$$\min\{1,\underline{\lambda}\}|||u|||^p\le\|u\|^p\le\max\{1,\overline{\lambda}\}|||u|||^p, \tag{2.1}$$

where

$$|||u|||=\Big(\int_0^T|u(t)|^p\,dt+\int_0^T|u'(t)|^p\,dt\Big)^{\frac1p}$$

is the usual norm of $W_T^{1,p}$. Let

$$k_0=\sup_{u\in W_T^{1,p}\setminus\{0\}}\frac{\|u\|_\infty}{\|u\|},\quad \|u\|_\infty=\sup_{t\in[0,T]}|u(t)|, \tag{2.2}$$

where $|\cdot|$ is the usual norm of $\mathbb{R}^N$. Since $W_T^{1,p}\hookrightarrow C^0$ is compact, one has $k_0<+\infty$ and for each $u\in W_T^{1,p}$, there exists $\xi\in[0,T]$ such that $|u(\xi)|=\min_{t\in[0,T]}|u(t)|$. Hence, by Hölder's inequality, one has

$$\begin{aligned}|u(t)|&=\Big|\int_\xi^t u'(s)\,ds+u(\xi)\Big|\le\int_0^T|u'(s)|\,ds+\frac1T\int_0^T|u(\xi)|\,ds\\&\le\int_0^T|u'(s)|\,ds+\frac1T\int_0^T|u(s)|\,ds\le T^{\frac1q}\Big(\int_0^T|u'(s)|^p\,ds\Big)^{\frac1p}+T^{-\frac1p}\Big(\int_0^T|u(s)|^p\,ds\Big)^{\frac1p}\\&\le\max\{T^{\frac1q},T^{-\frac1p}\}\Big(\Big(\int_0^T|u'(s)|^p\,ds\Big)^{\frac1p}+\Big(\int_0^T|u(s)|^p\,ds\Big)^{\frac1p}\Big)\\&\le\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}\Big(\int_0^T|u'(s)|^p\,ds+\int_0^T|u(s)|^p\,ds\Big)^{\frac1p}=\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}|||u|||\end{aligned}$$

for each $t\in[0,T]$ and $q=\frac{p}{p-1}$. So, by (2.1) and the above expression, we obtain

$$\|u\|_\infty\le\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}|||u|||\le\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}\big(\min\{1,\underline{\lambda}\}\big)^{-\frac1p}\|u\|.$$

From this and (2.2) it follows that

$$k_0\le k=\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}\big(\min\{1,\underline{\lambda}\}\big)^{-\frac1p}.$$

For all $v\in W_T^{1,p}$ we have

$$\begin{aligned}-\int_0^T\big(|u'(t)|^{p-2}u'(t)\big)'v(t)\,dt+\int_0^T\big(A(t)|u(t)|^{p-2}u(t),v(t)\big)\,dt\\-\lambda\int_0^T\big(\nabla F(t,u(t)),v(t)\big)\,dt-\mu\int_0^T\big(\nabla G(t,u(t)),v(t)\big)\,dt=0,\end{aligned}$$

according to the condition of problem (1.1),

$$\int_0^T \Big[\big(|u'(t)|^{p-2}u'(t), v'(t) \big) + \big(A(t)|u(t)|^{p-2}u(t), v(t) \big) \Big]\, dt$$
$$+ \sum_{j=1}^{p} \sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) - \lambda \int_0^T \big(\nabla F(t, u(t)), v(t) \big)\, dt - \mu \int_0^T \big(\nabla G(t, u(t)), v(t) \big)\, dt = 0 \quad (2.3)$$

for all $v \in W_T^{1,p}$. As usual, a weak solution to problem (1.1) is any $u \in W_T^{1,p}$ that satisfies in (2.3).

3 Main results

For two given non-negative constants θ_i for $i = 1, 2$ and a given positive constant d with $\theta_i^p \neq (\frac{1-s}{1+s})\overline{\lambda} T k^p d^p$, put

$$a_d(\theta_i) := \frac{\int_0^T \max_{|u|<\theta_i} \big[F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t)) \big]\, dt - \int_0^T F(t, d)\, dt}{\theta_i^p - (\frac{1-s}{1+s})\overline{\lambda} T k^p d^p},$$

$$\mu_1 := \frac{(1-s)\theta_1^p - (1+s)\overline{\lambda} T k^p d^p - \lambda p k^p \int_0^T \max_{|u|<\theta_1} F(t, u)\, dt + \lambda p k^p \int_0^T F(t, d)\, dt}{p k^p \int_0^T \max_{|u|<\theta_1} G(t, u)\, dt},$$

$$\mu_2 := \frac{(1-s)\theta_2^p - (1+s)\overline{\lambda} T k^p d^p - \lambda p k^p \int_0^T \max_{|u|<\theta_2} F(t, u)\, dt + \lambda p k^p \int_0^T F(t, d)\, dt}{p k^p \int_0^T \max_{|u|<\theta_2} G(t, u)\, dt}$$

and

$$s := k^p \sum_{j=1}^{m} \sum_{i=1}^{N} L_{ij} < 1.$$

Now, we present an application of Theorem 2.1 that we will used to obtain one nontrivial weak solution.

Theorem 3.1. *Assume that there exist three nonnegative constants θ_1, θ_2, and d with*

$$\theta_1^p < \Big(\frac{1+s}{1-s} \Big) \overline{\lambda} T k^p d^p < \theta_2^p \quad (3.1)$$

such that

(A$_1$) $\int_0^T F(t, d)\, dt \geq 0$ *for every* $t \in [0, T]$;

(A$_2$) $a_d(\theta_2) < a_d(\theta_1)$.

Moreover, $\lambda \in \frac{(1-s)}{pk^p} \big(\frac{1}{a_d(\theta_1)}, \frac{1}{a_d(\theta_2)} \big)$ *and potential* $G(t, x)$ *for all* $(t, x) \in [0, T] \times (0, +\infty)$*, is nonnegative. Then for every* $\mu \in (\mu_1, \mu_2)$*, problem* (1.1) *admits at least one nontrivial weak solution* $u_1 \in W_T^{1,p}$.

Proof. Let $X = W_T^{1,p}$ be endowed with $\|\cdot\|$. We introduce the functionals ϕ, $\psi : X \to \mathbb{R}$ for each u in X as follows:

$$\phi(u) = \frac{1}{p} \|u\|^p + \sum_{j=1}^{m} \sum_{i=1}^{N} \int_0^{u_i(t_j)} I_{ij}(t)\, dt$$

and

$$\psi(u) = \int_0^T F(t, u(t))\, dt + \frac{\mu}{\lambda} \int_0^T G(t, u(t))\, dt,$$

and put $J_\lambda(u) := \phi(u) - \lambda\psi(u)$. Let us prove that the functionals ϕ and ψ satisfy the conditions. It is well known that ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\psi'(u)(v) = \int_0^T \big(\nabla F(t, u(t)), v(t)\big)\, dt + \frac{\mu}{\lambda} \int_0^T \big(\nabla G(t, u(t)), v(t)\big)\, dt$$

for every $v \in X$ as well as being sequentially weakly upper semicontinuous. Furthermore, $\psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that ψ' is strongly continuous on X. To this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$; then $\{u_n\}$ converges uniformly to u on T as $n \to \infty$ (see [44]). Since ∇F, ∇G are continuous functions in $\mathbb{R}$ for every $t \in T$,

$$\nabla F(t, u_n) + \frac{\mu}{\lambda} \nabla G(t, u_n) \to \nabla F(t, u) + \frac{\mu}{\lambda} \nabla G(t, u)$$

as $n \to \infty$. Hence $\psi'(u_n) \to \psi'(u)$ as $n \to \infty$. Thus we have proved that ψ' is strongly continuous on X, which implies that ψ' is a compact operator by Proposition 26.2 of [44]. Furthermore, $\phi' : X \to X^*$ admits a continuous inverse, where

$$\phi'(u)(v) = \int_0^T \Big[|u'(t)|^{p-2} u'(t) v'(t) + A(t) |u(t)|^{p-2} u(t) v(t) \Big]\, dt$$

for every $v \in X$. Clearly, the weak solutions of problem (1.1) are exactly the solutions of the equation $J'_\lambda(u) = 0$. Now, put

$$r_1 := \frac{(1-s)}{p} \left(\frac{\theta_1}{k}\right)^p, \quad r_2 := \frac{(1-s)}{p} \left(\frac{\theta_2}{k}\right)^p \text{ and } w(t) := d.$$

It is easy to verify that $w \in X$ and

$$\frac{(1-s)\underline{\lambda} T}{p}\, d^p \le \phi(w) \le \frac{(1-s)\overline{\lambda} T}{p}\, d^p.$$

In particular, from (3.1) we conclude that

$$r_1 < \phi(w) < r_2.$$

On the other hand, for all $u \in X$, we have

$$\phi^{-1}(-\infty, r_2) = \big\{u \in X :\ \phi(u) < r_2\big\} = \big\{u \in X :\ |u| < c_2\big\},$$

from which it follows that

$$\begin{aligned} \sup_{u \in \phi^{-1}(-\infty, r_2)} \psi(u) &= \sup_{u \in \phi^{-1}(-\infty, r_2)} \left[\int_0^T \Big(F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t)) \Big)\, dt \right] \\ &\le \int_0^T \max_{|u(t)| < \theta_2} \Big[F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t)) \Big]\, dt. \end{aligned}$$

Arguing as before, we obtain

$$\begin{aligned} \sup_{u \in \phi^{-1}(-\infty, r_1)} \psi(u) &= \sup_{u \in \phi^{-1}(-\infty, r_1)} \left[\int_0^T \Big(F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t)) \Big)\, dt \right] \\ &\le \int_0^T \max_{|u(t)| < \theta_1} \Big[F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t)) \Big]\, dt. \end{aligned}$$

Since $w(t) > 0$ for each $t \in T$, assumption (A_1) ensures that

$$\psi(w) \geq \int_0^T F(t,d)\,dt.$$

Then, due to the fact that $G \geq 0$, we get

$$\int_0^T \max_{|u|<\theta_2} \Big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\Big]\,dt \geq \int_0^T F(t,d)\,dt,$$

and thus $a_d(\theta_2) \geq 0$. At this point, we have

$$\begin{aligned}
\beta(r_1,r_2) &\leq \frac{\sup\limits_{u\in\phi^{-1}(-\infty,r_2)} \psi(u) - \psi(w)}{r_2 - \phi(w)} \\
&\leq \frac{\int_0^T \max\limits_{|u|<\theta_2} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt - \int_0^T F(t,d)\,dt}{\frac{(1-s)}{p}\,(\frac{\theta_2}{k})^p - \frac{(1+s)\overline{\lambda}T}{p}\,d^p} \\
&= \frac{pk^p}{(1-s)}\,\frac{\int_0^T \max\limits_{|u|<\theta_2} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt - \int_0^T F(t,d)\,dt}{\theta_2^p - (\frac{1+s}{1-s})\overline{\lambda}Tk^p d^p} \\
&= \frac{pk^p}{(1-s)}\,a_d(\theta_2).
\end{aligned}$$

Since $a_d(\theta_2) \geq 0$, hypothesis (A_2) implies that

$$\int_0^T \max_{|u|<\theta_1} \Big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\Big]\,dt < \int_0^T F(t,d)\,dt.$$

So,

$$\begin{aligned}
\rho_2(r_1,r_2) &\geq \frac{\psi(w) - \sup\limits_{u\in\phi^{-1}(-\infty,r_1)} \psi(u)}{\phi(w) - r_1} \\
&\geq \frac{\int_0^T F(t,d)\,dt - \int_0^T \max\limits_{|u|<\theta_1} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt}{\frac{(1+s)\overline{\lambda}T}{p}\,d^p - \frac{(1-s)}{p}\,(\frac{\theta_1}{k})^p} \\
&= \frac{pk^p}{(1-s)}\,\frac{\int_0^T F(t,d)\,dt - \int_0^T \max\limits_{|u|<\theta_1} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt}{(\frac{1+s}{1-s})\overline{\lambda}Tk^p d^p - \theta_1^p} \\
&= \frac{pk^p}{(1-s)}\,a_d(\theta_1).
\end{aligned}$$

Hence, from assumption (A_2), $\beta(r_1,r_2) < \rho_2(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in \frac{(1-s)}{pk^p}\,(\frac{1}{a_d(\theta_1)}, \frac{1}{a_d(\theta_2)})$, the functional J_λ admits at least one critical point u_1 such that

$$r_1 < \phi(u_1) < r_2. \qquad \square$$

Theorem 3.2. *Assume that there exist two constants θ and $\overline{d}$ with*

$$\Big(\frac{1+s}{1-s}\Big)\overline{\lambda}Tk^p\overline{d}^{\,p} < \theta^p$$

such that

(A$_3$) $\int_0^T F(t,\overline{d})\,dt \geq 0$ *for every* $t \in [0,T]$;

(A$_4$) $\lim_{|x|\to 0} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = \lim_{|x|\to +\infty} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly, for almost every* $t \in [0,T]$.

(A$_5$) *There exist the constants* $c > 0$ *and* $1 \leq q < p$ *such that*

$$|\nabla F(t,x)| \leq c(1+|x|^{q-1})$$

for all $x \in \mathbb{R}^N$ *and almost every* $t \in [0,T]$.

(A$_6$) *For any* $i \in \{1,2,\dots,N\}$ *and* $j \in \{1,2,\dots,m\}$, *there exist the constants* $a_{ij} > 0$, $b_{ij} > 0$ *and* $\gamma_{ij} \in [0,1]$ *such that*

$$I_{ij}(y) \geq -a_{ij} - b_{ij}y^{\gamma_{ij}} \ \ (y \geq 0) \ \ \text{and} \ \ I_{ij}(y) \leq a_{ij} + b_{ij}(-y)^{\gamma_{ij}} \ \ (y \leq 0).$$

Let $\lambda > \lambda_3$, *where*

$$\lambda_3 := \frac{(1-s)}{pk^p} \frac{(\frac{1+s}{1-s})\overline{\lambda}Tk^p\overline{d}^p - \theta^p}{\int_0^T F(t,\overline{d})\,dt - \int_0^T \max_{|u|<\theta}\big(F(t,u) + \frac{\mu}{\lambda}G(t,u)\big)\,dt},$$

whose potential $G(t,x)$ *for all* $(t,x) \in [0,T]\times(0,+\infty)$ *is nonnegative. Then for every* $\mu \in (0,\mu_3)$, *where*

$$\mu_3 := \frac{(1-s)\theta^p - (1+s)\overline{\lambda}Tk^pd^p - \lambda pk^p\int_0^T \max_{|u|<\theta}F(t,u)\,dt + \lambda pk^p\int_0^T F(t,d)\,dt}{pk^p\int_0^T \max_{|u|<\theta}G(t,u)\,dt},$$

problem (1.1) *admits at least one nontrivial weak solution* $u_3 \in W_T^{1,p}$.

Proof. Since the critical points of the functional $J := \phi - \lambda\psi$ on X are exactly the weak solutions of problem (1.1), our aim is to apply Theorem 2.1 to ϕ and ψ. It is well-known that ϕ is a continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, ψ is continuously Gateaux differentiable and sequentially weakly continuous. Owning to the assumption (A$_6$), we have

$$\int_0^z I_{ij}(t)\,dt \geq -a_{ij}z - \frac{b_{ij}}{\gamma_{ij}+1}\,z^{\gamma_{ij}+1} = -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}\,|z|^{\gamma_{ij}+1} \ \ (z \geq 0)$$

and

$$\int_z^0 I_{ij}(t)\,dt \leq -a_{ij}z - \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij}+1}\,z^{\gamma_{ij}+1} = a_{ij}|z| + \frac{b_{ij}}{\gamma_{ij}+1}\,|z|^{\gamma_{ij}+1} \ \ (z < 0).$$

Therefore, for every $i \in \{1,2,\dots,N\}$, $j \in \{1,2,\dots,m\}$ and $z \in \mathbb{R}$,

$$\int_0^z I_{ij}(t)\,dt \geq -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}\,|z|^{\gamma_{ij}+1}. \tag{3.2}$$

Thanks to (A$_4$), fixing $0 < \varepsilon < \frac{\min\{1,\lambda\}}{\mu}$ small enough, we can find a constant $C_\varepsilon > 0$ such that

$$|G(t,x)| \leq C_\varepsilon + \frac{\varepsilon}{p}\,|x|^p \tag{3.3}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. Also, taking (A_5) into account, we get

$$|F(t,x)| \le c|x| + \frac{c}{q}\,|x|^q \tag{3.4}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. Now, by (3.2), (3.3) and (3.4), for all $u \in X$ and $\lambda \in \mathbb{R}^+$, we obtain

$$\begin{aligned}
\phi(u) - \lambda\psi(u) &= \frac{1}{p}\,\|u\|^p - \lambda\int_0^T F(t,u(t))\,dt - \mu\int_0^T G(t,u(t))\,dt + \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(t)\,dt \\
&\ge \frac{1}{p}\,\|u\|^p - \lambda\int_0^T \Big(c|u(t)| + \frac{c}{q}\,|u(t)|^q\Big)\,dt - \mu\int_0^T \Big(C_\varepsilon + \frac{\varepsilon}{p}\,|u(t)|^p\Big)\,dt \\
&\quad - \sum_{j=1}^{m}\sum_{i=1}^{N} a_{ij}|u(t_j)| - \sum_{j=1}^{m}\sum_{i=1}^{N}\frac{b_{ij}}{\gamma_{ij}+1}\,|u(t_j)|^{\gamma_{ij}+1} \\
&\ge \frac{1}{p}\left(1 - \frac{\mu\varepsilon}{\min\{1,\underline{\lambda}\}}\right)\|u\|^p - \frac{1}{q}\left(\min\{1,\underline{\lambda}\}\right)^{-\frac{q}{p}}\lambda c\|u\|^q - \left(\min\{1,\underline{\lambda}\}\right)^{-\frac{1}{p}} T^{\frac{1}{q}}\lambda c\|u\| \\
&\quad - \mu C_\varepsilon T - \sum_{j=1}^{m}\sum_{i=1}^{N} a_{ij}|u(t_j)| - \sum_{j=1}^{m}\sum_{i=1}^{N}\frac{b_{ij}}{\gamma_{ij}+1}\,|u(t_j)|^{\gamma_{ij}+1}.
\end{aligned}$$

Since $p > q$ and ε is small enough,

$$\lim_{\|u\|\to+\infty}\big[\phi(u) - \lambda\psi(u)\big] = +\infty, \tag{3.5}$$

which means that the functional J_λ is coercive. Let $r := \frac{(1-s)}{p}\,(\frac{\theta}{k})^p$ and $\overline{w}(x) = \overline{d}$. We obtain

$$\rho(r) \ge \frac{pk^p}{(1-s)}\,\frac{\int_0^T F(t,\overline{d})\,dt - \int_0^T \max_{|u|<\theta}\big(F(t,u) + \frac{\mu}{\lambda}\,G(t,u)\big)\,dt}{(\frac{1+s}{1-s})\overline{\lambda}Tk^p\overline{d}^{\,p} - \theta^p}.$$

So, from our assumption it follows that $\rho(r) > 0$. Hence, from Theorem 2.2 for each $\lambda > \lambda_3$, the functional J_λ admits at least one local minimum u_3 such that

$$\phi(u_3) > r,$$

and the conclusion is achieved. □

Now, we present an application of Theorem 2.2 which will be used to obtain two nontrivial weak solutions.

Theorem 3.3. *Suppose F and G satisfy the assumptions (A_i) for $i = 4,5,6$ and there are $M > 0$ and $\sigma > p$ such that*

(A_7) $0 < \sigma F(t,x) \le \langle \nabla F(t,x), x\rangle$ *for all $x \in \mathbb{R}^N$ with $|x| \ge M$ and a.e. $t \in [0,T]$.*

Let $\lambda \in (0,\lambda_4)$, where

$$\lambda_4 := \frac{(1-s)}{pk^p}\,\frac{\theta^p}{\int_0^T \max_{|u|<\theta}\big(F(t,u) + \frac{\mu}{\lambda}\,G(t,u)\big)\,dt},$$

whose potential $G(t,x)$ for all $(t,x) \in [0,T]\times(0,+\infty)$ is non-negative. Then for every $\mu \in (0,\mu_4)$, where

$$\mu_4 := \frac{(1-s)\theta^p - \lambda pk^p \int_0^T \max_{|u|<\theta} F(t,u)\,dt}{pk^p \int_0^T \max_{|u|<\theta} G(t,u)\,dt},$$

problem (1.1) admits two distinct critical points.

Proof. We prove this theorem by using the same reasoning as in the proof of Theorem 2.3. First, we show that J_λ satisfies the (PS)-condition. Suppose that $\{u_n\}_{n=1}^\infty$ is a (PS)-sequence of J_λ, that is, there exists $C>0$ such that

$$J_\lambda(u_n)\to C,\quad J'_\lambda(u_n)\to 0 \text{ as } n\to\infty.$$

Assume that $\|u_n\|\to+\infty$. Then (3.5) contradicts $J_\lambda(u_n)\to C$; hence $\{u_n\}_{n=1}^\infty$ is bounded in $W_T^{1,p}$. We may assume that there exists $u_0\in W_T^{1,p}$ satisfying $u_n\to u_0$ weakly in $W_T^{1,p}$, $u_n\to u_0$ in $L^p[0,T]$, $u_n(t)\to u_0(t)$ for almost every $t\in[0,T]$. Observe that

$$\begin{aligned}J'_\lambda(u_n)(u_n-u_0)=\int_0^T\Big[\big(|u_n'(t)|^{p-2}u_n'(t),u_n'(t)-u_0'(t)\big)+\big(A(t)|u_n(t)|^{p-2}u_n(t),u_n(t)-u_0(t)\big)\Big]\,dt\\ -\lambda\int_0^T\big(\nabla F(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt-\mu\int_0^T\big(\nabla G(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt\\ +\sum_{j=1}^m\sum_{i=1}^N I_{ij}\big((u_n)_i(t_j)\big)\big((u_n)_i(t_j)-(u_0)_i(t_j)\big).\end{aligned}$$

We already know that

$$J'_\lambda(u_n)(u_n-u_0)\to 0 \text{ as } n\to\infty.$$

By (A_4). given $\varepsilon>0$, we can find a constant $C_\varepsilon>0$ such that

$$|\nabla G(t,x)|\le C_\varepsilon+\varepsilon|x|^{p-1}$$

for every $x\in\mathbb{R}^N$ and almost every $t\in[0,T]$. So,

$$\int_0^T\big(\nabla G(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt\to 0 \text{ as } n\to\infty.$$

Moreover, by (A_5),

$$\int_0^T\big(\nabla F(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt\to 0 \text{ as } n\to\infty.$$

Also,

$$\sum_{j=1}^m\sum_{i=1}^N I_{ij}\big((u_n)_i(t_j)\big)\big((u_n)_i(t_j)-(u_0)_i(t_j)\big)\to 0 \text{ as } n\to\infty.$$

Therefore,

$$\int_0^T\Big[\big(|u_n'(t)|^{p-2}u_n'(t),u_n'(t)-u_0'(t)\big)+\big(A(t)|u_n(t)|^{p-2}u_n(t),u_n(t)-u_0(t)\big)\Big]\,dt\to 0 \text{ as } n\to\infty.$$

This, together with the weak convergence of $u_n\to u_0$ in $W_T^{1,p}$, implies that

$$u_n\to u_0 \text{ in } W_T^{1,p} \text{ as } n\to\infty.$$

Hence J_λ satisfies the (PS)-condition. Finally, we prove that J_λ is unbounded from below. Owning to the assumption (A_7), we can find $\delta>0$ such that for every $M>0$, one has

$$|F(t,x)|>M|x|^\sigma \text{ for } 0<|x|\le\delta \text{ and almost every } t\in[0,T].$$

We choose a nonzero nonnegative function $v \in C_0^\infty([0,T])$ and take $\varepsilon > 0$ small enough. Then we obtain

$$\begin{aligned} J(\varepsilon v) &= \frac{1}{p}\|\varepsilon v\|^p - \lambda\int_0^T F(t,\varepsilon v(t))\,dt - \mu\int_0^T G(t,\varepsilon v(t))\,dt + \sum_{j=1}^m\sum_{i=1}^N \int_0^{\varepsilon v_i(t)(t_j)} I_{ij}(t)\,dt \\ &\le \frac{\varepsilon^p}{p}\|v\|^p - \lambda M\varepsilon^\sigma\int_0^T |v(t)|^\sigma\,dt - \sum_{j=1}^m\sum_{i=1}^N a_{ij}|u(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1} \\ &< \frac{\varepsilon^p}{p}\|v\|^p - \lambda M\varepsilon^\sigma\int_0^T |v(t)|^\sigma\,dt - \sum_{j=1}^m\sum_{i=1}^N a_{ij}|\varepsilon v_i(t)(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|\varepsilon v_i(t)(t_j)|^{\gamma_{ij}+1}. \end{aligned}$$

Since $\sigma > p$, this condition guarantees that J_λ is unbounded from below. Now, we have

$$\frac{\sup\limits_{u\in\phi^{-1}(r,+\infty)}\psi(u)}{r} \le \frac{\int_0^T \max\limits_{|u|<\theta}\left(F(t,u)+\frac{\mu}{\lambda}G(t,u)\right)dt}{\frac{(1-s)}{p}\left(\frac{\theta}{k}\right)^p} = \frac{pk^p}{(1-s)}\,\frac{\int_0^T \max\limits_{|u|<\theta}\left(F(t,u)+\frac{\mu}{\lambda}G(t,u)\right)dt}{\theta^p}.$$

Finally, for each $\lambda \in \Big(0, \frac{r}{\sup_{u\in\phi^{-1}(r,+\infty)}\psi(u)}\Big)$, problem (1.1) admits two distinct critical points. □

4 Applications

In this section, we point out some consequences and applications of the results previously obtained.

Theorem 4.1. *Assume that there exist two positive constants θ and d with*

$$\Big(\frac{1+s}{1-s}\Big)\overline{\lambda}Tk^pd^p < \theta^p$$

such that assumption (A_1) *in Theorem 3.1 holds. Furthermore, suppose that*

$$(\mathrm{A}_8)\quad \frac{\int_0^T \max\limits_{|v|<\theta} F(t,v)\,dt}{\theta^p} < \frac{\int_0^T F(t,d)\,dt}{(\frac{1+s}{1-s})\overline{\lambda}Tk^pd^p}.$$

Then for each

$$\lambda \in \frac{(1-s)}{pk^p}\left(\frac{(\frac{1+s}{1-s})\overline{\lambda}Tk^pd^p}{\int_0^T F(t,d)\,dt}, \frac{\theta^p}{\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt}\right),$$

the problem

$$\begin{cases} -\big(|u'|^{p-2}u'\big)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u), & a.e.\ t\in J, \\ \triangle\big(|u_i'(t_j)|^{p-2}u_i'(t_j)\big) = I_{ij}(u_i(t_j)), & i=1,2,\dots,N,\quad j=1,2,\dots,m, \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases}$$

admits at least one nontrivial weak solution.

Proof. The conclusion follows from Theorem 3.2, by taking $\theta_1 = 0$, $\theta_2 = \theta$ and $\mu = 0$. Indeed, owing to assumption (A_8), one has

$$a_\eta(\theta) = \frac{\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt - \int_0^T F(t,d)\,dt}{\theta^p - (\frac{1+s}{1-s})\overline{\lambda}Tk^pd^p} < \frac{\Big(1-\frac{(\frac{1+s}{1-s})\overline{\lambda}Tk^pd^p}{\theta^p}\Big)\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt}{\theta^p - (\frac{1+s}{1-s})\overline{\lambda}Tk^pd^p} = \frac{1}{\theta^p}\int_0^T \max_{|v|<\theta}F(t,v)\,dt.$$

On the other hand,

$$a_\eta(0) = \frac{\int\limits_0^T F(t,d)\,dt}{(\frac{1+s}{1-s})\overline{\lambda} T k^p d^p}\,.$$

Hence, in view of (A_8), Theorem 3.2 ensures the conclusion. □

Now, we suppose that $\nabla F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function. We note the following lemma, which is useful to obtain the results on the existence of nonnegative solutions.

Lemma. *Let $\nabla F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a nonnegative function. Suppose that $u \in X$ is a weak solution of problem (1.1). Then u is nonnegative.*

Proof. Put $u^- = -\min\{u, 0\}$. Then $u^- \in X$. Taking into account that u is a weak solution and choosing $v = u^-$, we obtain

$$\begin{aligned}
0 &\le \lambda \int\limits_0^T \big(\nabla F(t,u(t)), u^-(t)\big)\,dt + \mu \int\limits_0^T \big(\nabla G(t,u(t)), u^-(t)\big)\,dt \\
&= \int\limits_0^T \Big[\big(|u'(t)|^{p-2}u'(t), (u^-)'(t)\big) + \big(A(t)|u(t)|^{p-2}u(t), u^-(t)\big)\Big]\,dt + \sum_{j=1}^m \sum_{i=1}^N I_{ij}(u_i(t_j))u_i^-(t_j) \\
&= -\|u^-\|^p - \sum_{j=1}^m \sum_{i=1}^N I_{ij}(u_i(t_j))u_i^-(t_j).
\end{aligned}$$

That is, $u^- = 0$ a.e. in $[0,T]$. Hence our claim is proved. □

Now, we point out a result when the nonlinear term has separable variables. To be precise, let $m : [0,T] \to \mathbb{R}$ be a function such that $m \in L^1([0,T])$, $m(t) \ge 0$ a.e. $t \in [0,T]$, $m \ne 0$, and let $\nabla H : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative and continuous function. Consider the following problem:

$$\begin{cases}
-\big(|u'|^{p-2}u'\big)' + A(t)|u|^{p-2}u = \lambda m(t)\nabla H(u(t)), \;\; \text{a.e. } t \in J, \\
\triangle\big(|u_i'(t_j)|^{p-2}u_i'(t_j)\big) = I_{ij}(u_i(t_j)), \;\; i = 1,2,\dots,N, \;\; j = 1,2,\dots,m, \\
u(0) - u(T) = u'(0) - u'(T) = 0.
\end{cases} \tag{4.1}$$

Theorem 4.2. *Assume that (A_5) and (A_6) hold and there exist $\sigma > p$ and $M > 0$ such that*

$$0 < \sigma H(s) \le s\nabla H(s) \tag{4.2}$$

for all $s \in \mathbb{R}^N$ with $|s| \ge M$. Then for each $\lambda \in (0, \lambda^*)$, where

$$\lambda^* := \frac{(1-s)}{pk^p\|m\|_{L^1([0,T])}} \max_{\theta > 0} \frac{\theta^p}{H(\theta)}\,,$$

problem (4.1) has at least two nonnegative and non-zero weak solutions.

Corollary. *Let $\nabla F : \mathbb{R}^N \to \mathbb{R}$ be nonnegative and continuous function and assume (4.2) holds. Then for each that $\lambda \in (0, \lambda^{**})$, where*

$$\lambda^{**} := \frac{(1-s')}{pk^p} \max_{\theta > 0} \frac{\theta^p}{H(\theta)} \quad \text{and} \quad s' := k^p \sum_{j=1}^m I_j < 1,$$

the problem

$$\begin{cases}
-\big(|u'|^{p-2}u'\big)' + |u|^{p-2}u = \lambda \nabla H(u(t)), \;\; \text{a.e. } t \in J, \\
\triangle\big(|u'(t_j)|^{p-2}u'(t_j)\big) = I_j(u(t_j)), \;\; j = 1,2,\dots,m, \\
u(0) - u(T) = u'(0) - u'(T) = 0
\end{cases}$$

has two nonnegative and non-zero classical solutions.

Proof. This is a consequence of Theorem 4.2 with $\mu = 0$, $A(t) = I$, where I is the identity matrix of order $p \times p$, and $m(t) = 1$ for all $t \in [0, T]$. □

Example 4.1. Consider $p = 4$ and the function $\nabla H(t) = 5t^4 + 1$ satisfying (4.2). We observe that $\max_{\theta>0} \frac{\theta^4}{H(\theta)} = \frac{\sqrt[4]{27}}{4}$, and for each $\lambda \in (0, 0.066)$,

$$\begin{cases} -\big(|u'|^2u'\big)' + |u|^2u = \lambda\nabla H(u(t)), \;\; \text{a.e. } t \in (0,1), \\ \triangle\big(|u'(t)|^2u'(t)\big) = I(u(t)), \\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases}$$

admits at least two non-zero and nonnegative solutions.

Example 4.2. Consider $p = 3$ and the function

$$h(t) = \begin{cases} \frac{3}{2}\sqrt{t} + 5t^4, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

We observe that it is enough to pick, for instance, $\mu = 4$ and that (4.2) holds. Moreover, $\max_{\theta>0} \frac{\theta^3}{H(\theta)} = \frac{2\sqrt[7]{54}}{7}$, and for each $\lambda \in (0, 0.08)$,

$$\begin{cases} -\big(|u'|u'\big)' + |u|u = \lambda\nabla H(u(t)), \;\; \text{a.e. } t \in (0,1), \\ \triangle\big(|u'(t)|u'(t)\big) = I(u(t)), \\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases}$$

admits at least two non-zero and nonnegative solutions.

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