Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2024, 1–13

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REGULARITY OF SOLUTIONS TO $\vec{p}$-LAPLACIAN PROBLEM WITH A LOWER ORDER TERM AND A HARDY POTENTIAL

Abstract. In this paper, we study the existence and regularity results for an anisotropic elliptic problem involving a lower order term and a Hardy potential. Interestingly, our study reveals that the use of the Hardy inequality is dispensable due to the inclusion of the lower order term, which dominates the Hardy term. This inclusion not only improves the regularity of solutions but also eliminates the need to impose constraints on the coefficient of the Hardy term.

2020 Mathematics Subject Classification. 35J60, 35B45, 35D30, 35B65.

Key words and phrases. Anisotropic problems, lower order terms, Hardy potential, L^m data, fixed point theorem.

1 Overview and necessary preliminaries

Anisotropic equations play a crucial role in a wide range of mathematical models. One prominent example is their application in the study of fluid dynamics, where they capture the behavior of fluids with varying conductivities in different directions (see [2]). Moreover, these equations find significance in the field of biology, particularly in modeling the spread of epidemic diseases in heterogeneous environments, as explored by Bendahmane, Langlais, and Saad in their work (refer to [3]). These instances highlight the versatility and importance of anisotropic equations in various scientific disciplines.

This paper focuses on the study of an anisotropic elliptic problem described by the following equations

$$\begin{cases} -\Delta_{\vec{p}} u + \nu |u|^{s-2} u = \mu \dfrac{|u|^{\overline{p}-1}}{|x|^{\overline{p}}} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$ (with $N > 2$) having a smooth boundary $\partial\Omega$. The vector $\vec{p} = (p_1, ..., p_N) \in \mathbb{R}^N$ satisfies the following conditions:

$$1 < p^- = \min_{1\le i\le N} \{p_i\} \le p_i \le p^+ = \max_{1\le i\le N} \{p_i\}, \quad 1 < \overline{p} < N, \tag{1.2}$$

here, $\overline{p}$ represents the harmonic mean of p_i and is defined as

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}\,.$$

The anisotropic Laplace operator $\Delta_{\vec{p}} u$ is given by

$$\Delta_{\vec{p}} u = \sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right], \quad \text{where } \partial_i u = \frac{\partial u}{\partial x_i}, \quad \forall\, i = 1, \ldots, N.$$

This study assumes $\nu > 0$, $\mu > 0$, and f belonging to $L^m(\Omega)$ with $1 < m < \frac{N}{\overline{p}}$. Additionally, the condition

$$s > \overline{p}^{\,*} \tag{1.3}$$

is satisfied, where $\overline{p}^{\,*} = \frac{N\overline{p}}{N-\overline{p}}$.

The natural functional framework for problem (1.1) is the anisotropic Sobolev spaces $W^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$, which are defined as follows:

$$W^{1,\vec{p}}(\Omega) = \Big\{ u \in W^{1,1}(\Omega) : \ \ \partial_i u \in L^{p_i}(\Omega), \ \ \forall\, i = 1, \ldots, N \Big\}$$

and

$$W_0^{1,\vec{p}}(\Omega) = \Big\{ u \in W_0^{1,1}(\Omega) : \ \ \partial_i u \in L^{p_i}(\Omega), \ \ \forall\, i = 1, \ldots, N \Big\}.$$

The space $W_0^{1,\vec{p}}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}.$$

Equipped with this norm, $W_0^{1,\vec{p}}(\Omega)$ is a separable and reflexive Banach space.

The theory of such spaces was developed in [7, 13–15]. In particular, it has been proved in [15] that if $\overline{p} < N$, then the following continuous embedding holds:

$$W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall\, r \in [1, \overline{p}^{\,*}].$$

Moreover, this embedding is compact if $r < \overline{p}^*$. A Sobolev-type inequality is also demonstrated, showing the existence of a positive constant C, which depends only on $|\Omega|$, such that

$$\|u\|^{p^+}_{L^{\overline{p}^*}(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_i u\|^{p_i}_{L^{p_i}(\Omega)}, \quad \forall\, u \in W_0^{1,\vec{p}}(\Omega). \tag{1.4}$$

Furthermore, for each $i = 1, \ldots, N$, there exists a constant $C_i > 0$ (see [12, Lemma 1.1]) such that

$$\|u\|_{L^{p_i}(\Omega)} \leq C_i \|\partial_i u\|_{L^{p_i}(\Omega)}, \quad \forall\, u \in W_0^{1,\vec{p}}(\Omega). \tag{1.5}$$

Problem (1.1) with $\mu = 0$, in the isotropic case (i.e., $p_i = p$ for all i), has been extensively studied in the literature. For further details and references, we recommend consulting the work [4].

In a previous work [5], the author demonstrated the existence of a weak solution for the boundary value problem defined by the equations

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $f \in L^m(\Omega)$. The obtained weak solution belongs to the space $W_0^{1,\vec{p}}(\Omega)$. The paper extensively discussed various cases by considering different values of m.

Moreover, the same author conducted a study in [6] concerning the following problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right] + g(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where f is a given function belonging to some Lebesgue space, and g is a nonlinear term that exhibits natural growth with respect to the gradient, while satisfying the sign condition $g(x, \sigma, \xi)\sigma \geq 0$. The author proved the existence of a weak solution u in $W_0^{1,\vec{p}}(\Omega)$ when $f \in L^1(\Omega)$ and certain conditions on g are met. In addition to these contributions, the author has obtained several other noteworthy results, which can be explored in detail within the referenced publication. To learn more about anisotropic problems including issues such as degeneracy or singularity, we recommend that the reader consult references such as [8, 9, 16].

On the other hand, the authors in [1] investigated the following problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + b|u|^{r-2}u = a\,\dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $a > 0$ and $b > 0$. The authors derived the following results:

(R1) If $r > 2^*$ and f belongs to $L^m(\Omega)$ with $\frac{r}{r-1} \leq m < \frac{N}{2}\,\frac{r-2}{r-1}$, then there exists a weak solution u in $W_0^{1,2}(\Omega) \cap L^{m(r-1)}(\Omega)$ under certain conditions on the matrix M.

(R2) If $r > \frac{2^*}{2}$ and f belongs to $L^m(\Omega)$ with $\frac{N(r+1)}{Nr+1} \leq m < \frac{2N}{N+2}$, then there exists a weak solution u in $W_0^{1,2}(\Omega) \cap L^{rm^*}(\Omega)$ under certain conditions on the matrix M.

(R3) If $r > 2^*$ and f belongs to $L^m(\Omega)$ with $1 < m < \frac{r}{r-1}$, then there exists a distributional solution u in $W_0^{1,q}(\Omega) \cap L^{m(r-1)}(\Omega)$ with $q = 2m\,\frac{r-1}{r}$ under certain conditions on the matrix M.

These results highlight different scenarios based on the value of r and the range of m. The existence of weak or distributional solutions is proved in suitable function spaces, incorporating conditions on the matrix M.

In this paper, we draw upon the findings from the aforementioned works (1.6), (1.7), and (1.8) to establish a connection and combine their results to form the problem described in (1.1). Specifically,

(i) When considering the case of $\nu = \mu = 0$, problem (1.1) corresponds to the one presented in (1.6).

(ii) If $\mu = 0$ and $g(x, u, \nabla u) = \nu|u|^{s-2}u$, problems (1.1) and (1.7) are equivalent.

(iii) In the isotropic case where $p_i = 2$ for all i, problem (1.1) is identical to (1.8) with $M \equiv 1$.

By acknowledging these connections, we integrate and build upon the previous works, utilizing their insights and results to address the problem presented in (1.1).

Regarding subsequences, we will need the following useful topological trick of uniqueness.

Lemma 1.1 ([11, Lemma 1.1]). *Let X be a topological space and (x_n) be a sequence in X with the property that there exists $x \in X$ such that for any subsequence of (x_n), it is possible to extract a further subsequence that converges to x. Then the entire sequence (x_n) converges to x.*

2 Existence and regularity theorems

We begin this section by providing the definition of weak solutions for problem (1.1).

The first result deals with the existence of finite energy solutions in $L^{m(s-1)}(\Omega)$ for a given f, under the condition $s > \overline{p}^*$.

Theorem 2.1. *Let us assume that (1.2) and (1.3) hold true. Furthermore, suppose that $f \in L^m(\Omega)$, where*

$$\frac{s}{s-1} \leq m < \frac{N(s-\overline{p})}{\overline{p}(s-1)}. \tag{2.1}$$

Then there exists a weak solution $u \in W_0^{1,\vec{p}}(\Omega) \cap L^{m(s-1)}(\Omega)$ to problem (1.1). That is,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \varphi \, dx + \nu \int_\Omega |u|^{s-2}u\varphi \, dx = \mu \int_\Omega \frac{|u|^{\overline{p}-1}}{|x|^{\overline{p}}}\varphi \, dx + \int_\Omega f\varphi \, dx, \tag{2.2}$$

for every $\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.1. It should be noted that the weak formulation (2.2) makes sense. Indeed, since $u, \varphi \in W_0^{1,\vec{p}}(\Omega)$, we have $|\partial_i u|^{p_i-1}|\partial_i \varphi| \in L^1(\Omega)$. Furthermore, employing Hölder's inequality and the fact that $u \in L^{m(s-1)}(\Omega)$, we obtain

$$\int_\Omega |u|^{s-1} \, dx \leq |\Omega|^{1-\frac{1}{m}} \left(\int_\Omega |u|^{m(s-1)} \, dx \right)^{\frac{1}{m}} < \infty.$$

Additionally, considering conditions (1.3) and (2.1), we can derive

$$\int_\Omega \frac{|u|^{\overline{p}-1}}{|x|^{\overline{p}}} \, dx \leq \left(\int_\Omega |u|^{m(s-1)} \, dx \right)^{\frac{\overline{p}-1}{m(s-1)}} \left(\int_\Omega \frac{1}{|x|^{\frac{\overline{p}m(s-1)}{m(s-1)-\overline{p}+1}}} \, dx \right)^{\frac{m(s-1)-\overline{p}+1}{m(s-1)}} < \infty.$$

The second result addresses the case where the summability of f leads to the existence of infinite energy solutions $u \in W_0^{1,\vec{q}}(\Omega)$, with $\vec{q} = (q_1, \ldots, q_N)$ and $1 < q_i < p_i$ for every $i = 1, \ldots, N$.

Theorem 2.2. *Let hypotheses (1.2) and (1.3) hold, and $f \in L^m(\Omega)$ such that*

$$1 < m < \frac{s}{s-1}, \tag{2.3}$$

and let $\overline{p}$ and m satisfy one of the following assumptions:

$$1 + \frac{N-1}{N+1} < \overline{p} < N \quad \text{and} \quad 1 < m < \frac{s}{s-1}, \tag{2.4}$$

$$1 < \overline{p} < 1 + \frac{N-1}{N+1} \quad \text{and} \quad \frac{s}{(s-1)\overline{p}} < m < \frac{s}{s-1}. \tag{2.5}$$

Then there exists a weak solution $u \in W_0^{1,\vec{q}}(\Omega) \cap L^{m(s-1)}$ *to problem* (1.1), *where* $q_i = \frac{m(s-1)}{s} p_i$, $\forall\, i = 1, \dots, N$. *That is,*

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi \, dx + \nu \int_{\Omega} |u|^{s-2} u \varphi \, dx = \mu \int_{\Omega} \frac{|u|^{\overline{p}-1}}{|x|^{\overline{p}}} \varphi \, dx + \int_{\Omega} f \varphi \, dx$$

for every $\varphi \in C_0^1(\Omega)$.

Remark 2.2. The assumption (1.3) in Theorem 2.1 guarantees that

$$\Big[\frac{s}{s-1}, \frac{N(s-\overline{p})}{\overline{p}(s-1)}\Big) \neq \varnothing.$$

The hypothesis (2.3) implies that $q_i \le p_i$ for all $i = 1, \dots, N$. By the assumptions (2.4) and (1.3), we have $q_i > 1$ and $\frac{q_i}{p_i - 1} > 1$ for every $i = 1, \dots, N$.

Remark 2.3. In the isotropic case, i.e., when $p_i = 2$, the results of Theorem 2.1 and Theorem 2.2 coincide with regularity results for elliptic equation problems involving Hardy potential (see Theorem 2.1 and Theorem 3.1 in [1]).

3 Proof of the main results

In this paper, we will use the truncation function $T_k(s) = \min\{k, \max\{-k, s\}\}$ for $k > 0$ and $s \in \mathbb{R}$.

Let us consider the approximation problems defined as follows:

$$\begin{cases} -\Delta_{\vec{p}} u_n + \nu |u_n|^{s-2} u_n = \mu \dfrac{|T_n(u_n)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} + f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f_n = T_n(f)$.

Lemma 3.1. *For every* $n \in \mathbb{N}^*$, *problem* (3.1) *has a weak solution* $u_n \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $n \in \mathbb{N}^*$ be fixed. We define a map S as

$$\begin{aligned} S:\ L^{\overline{p}}(\Omega) &\longrightarrow L^{\overline{p}}(\Omega), \\ v &\longmapsto S(v) = w, \end{aligned}$$

where w is the unique solution of the following problem:

$$\begin{cases} -\Delta_{\vec{p}} w + \nu |w|^{s-2} w = \mu \dfrac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} + f_n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The map S is well defined because the existence of a unique weak solution $w \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ for problem (3.2) is guaranteed in the work [6].

We multiply both sides of the first equality of (3.2) by a test function φ and integrate over Ω, then apply Green's formula. Choosing $\varphi = w$ and using the fact that

$$|f_n| \le n \ \text{ and } \ \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} \le n^{\overline{p}},$$

we obtain

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i w|^{p_i} \, dx + \nu \int_{\Omega} |w|^s \, dx \le (\mu n^{\overline{p}} + n) \int_{\Omega} |w| \, dx.$$

Taking out the non-negative term on the left-hand side and applying Hölder's inequality, we can further estimate the right-hand side as follows:

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i w|^{p_i} \, dx \leq (\mu n^{\overline{p}} + n)|\Omega|^{\frac{1}{(\overline{p}^*)'}} \left(\int_{\Omega} |w|^{\overline{p}^*} \, dx \right)^{\frac{1}{\overline{p}^*}}.$$

Or, in terms of the norms, we have

$$\sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}(\Omega)}^{p_i} \leq (\mu n^{\overline{p}} + n)|\Omega|^{\frac{1}{(\overline{p}^*)'}} \|w\|_{L^{\overline{p}^*}(\Omega)}. \tag{3.3}$$

From inequality (1.4), there exists a positive constant C such that

$$\|w\|_{L^{\overline{p}^*}(\Omega)}^{p^+} \leq C(\mu n^{\overline{p}} + n)|\Omega|^{\frac{1}{(\overline{p}^*)'}} \|w\|_{L^{\overline{p}^*}(\Omega)}.$$

This implies that

$$\|w\|_{L^{\overline{p}^*}(\Omega)} \leq C_n.$$

Since $\overline{p} < \overline{p}^*$, we have

$$\|w\|_{L^{\overline{p}}(\Omega)} \leq C_n \tag{3.4}$$

for some constant C_n independent of v and w. Thus we have shown that the ball $\mathcal{B}$ in $L^{\overline{p}}(\Omega)$ of radius C_n is invariant under the map S.

Now, we will prove the continuity of the map S. Let $v \in L^{\overline{p}(\Omega)}$ and let (v_k) be a sequence of functions converged to v in $L^{\overline{p}}(\Omega)$. We denote $w_k = S(v_k)$ and $w = S(v)$. To prove that $w_k \to w$ in $L^{\overline{p}}(\Omega)$, it suffices to demonstrate that $w_k \to w$ in $W_0^{1,\vec{p}}(\Omega)$ because $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{\overline{p}}(\Omega)$. According to Lemma 1.1, to verify that $w_k \to w$ in $W_0^{1,\vec{p}}(\Omega)$, it is sufficient to show that for any subsequence of (w_k), it is possible to extract a further subsequence that converges to w.

Let $(w_{\sigma(k)})$ be a subsequence of (w_k). Firstly, since $v_{\sigma(k)} \to v$ in $L^{\overline{p}}(\Omega)$ as $\sigma(k) \to \infty$, we can extract a subsequence $(v_{\sigma_1(k)})$ of $(v_{\sigma(k)})$ such that

$$v_{\sigma_1(k)} \xrightarrow{\sigma_1(k) \to \infty} v \quad \text{a.e. in } \Omega. \tag{3.5}$$

Secondly, for every integer $\sigma_1(k)$, one has

$$\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} \leq n^{\overline{p}}. \tag{3.6}$$

Then from (3.5) and (3.6) we can apply the dominated convergence theorem to conclude that

$$\left\| \frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} \right\|_{L^{\alpha}(\Omega)} \xrightarrow{\sigma_1(k) \to \infty} 0 \,, \quad \forall \alpha \geq 1. \tag{3.7}$$

Thirdly, we have $w_{\sigma_1(k)}$ and w satisfying the equation

$$-\Delta_{\vec{p}} w_{\sigma_1(k)} + \Delta_{\vec{p}} w + \nu \Big[|w_{\sigma_1(k)}|^{s-2} w_{\sigma_1(k)} - |w|^{s-2} w \Big] = \mu \left[\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} \right],$$

thus

$$-\sum_{i=1}^{N} \partial_i \Big[|\partial_i w_{\sigma_1(k)}|^{p_i-2} \partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2} \partial_i w \Big] \, dx + \nu \Big[|w_{\sigma_1(k)}|^{s-2} w_{\sigma_1(k)} - |w|^{s-2} w \Big]$$
$$= \mu \left[\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} \right]. \tag{3.8}$$

Choosing $w_{\sigma_1(k)} - w$ as a test function in (3.8), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \Big[|\partial_i w_{\sigma_1(k)}|^{p_i-2}\partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2}\partial_i w\Big]\big(\partial_i w_{\sigma_1(k)} - \partial_i w\big)\,dx$$
$$+ \nu \int_{\Omega} \Big[|w_{\sigma_1(k)}|^{s-2}w_{\sigma_1(k)} - |w|^{s-2}w\Big](w_{\sigma_1(k)} - w)\,dx$$
$$= \mu \int_{\Omega} \left[\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}}\right](w_{\sigma_1(k)} - w)\,dx.$$

Since

$$\nu \int_{\Omega} \Big[|w_{\sigma_1(k)}|^{s-2}w_{\sigma_1(k)} - |w|^{s-2}w\Big](w_{\sigma_1(k)} - w)\,dx \geq 0,$$

and by Hölder's inequality, we have

$$\sum_{i=1}^{N} \int_{\Omega} \Big[|\partial_i w_{\sigma_1(k)}|^{p_i-2}\partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2}\partial_i w\Big]\big(\partial_i w_{\sigma_1(k)} - \partial_i w\big)\,dx$$
$$\leq \mu \left\|\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}}\right\|_{L^{p_i'}(\Omega)} \|w_{\sigma_1(k)} - w\|_{L^{p_i}(\Omega)}.$$

By (1.5), there exists $C > 0$ such that

$$\sum_{i=1}^{N} \int_{\Omega} \Big[|\partial_i w_{\sigma_1(k)}|^{p_i-2}\partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2}\partial_i w\Big]\big(\partial_i w_{\sigma_1(k)} - \partial_i w\big)\,dx$$
$$\leq \mu\, C \left\|\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}}\right\|_{L^{p_i'}(\Omega)} \big\|\partial_i(w_{\sigma_1(k)} - w)\big\|_{L^{p_i}(\Omega)}.$$

Using (3.3) and (3.4), we deduce that

$$\sum_{i=1}^{N} \int_{\Omega} \Big[|\partial_i w_{\sigma_1(k)}|^{p_i-2}\partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2}\partial_i w\Big]\big(\partial_i w_{\sigma_1(k)} - \partial_i w\big)\,dx$$
$$\leq C_n \left\|\frac{|T_n(v_{\sigma_1(k)})|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} - \frac{|T_n(v)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}}\right\|_{L^{p_i'}(\Omega)}.$$

Consequently, from (3.7), we obtain

$$\lim_{\sigma_1(k)\to\infty} \sum_{i=1}^{N} \int_{\Omega} \Big[|\partial_i w_{\sigma_1(k)}|^{p_i-2}\partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2}\partial_i w\Big]\partial_i(w_{\sigma_1(k)} - w)\,dx = 0.$$

Finally, following the same line of reasoning as in Lemma 2.4 of [10], we can extract a subsequence $(w_{\sigma_2(k)})$ from $(w_{\sigma_1(k)})$ such that $(w_{\sigma_2(k)})$ converges to w in $W_0^{1,\vec{p}}(\Omega)$. This establishes the continuity of S.

Ultimately, by the Sobolev embedding, it is easy to prove that S is compact on $L^{\overline{p}}(\Omega)$. Therefore, by Schauder's fixed point theorem, there exists u_n in $W_0^{1,\vec{p}}(\Omega) \cap L^{\infty}(\Omega)$, for every fixed n, such that $S(u_n) = u_n$. □

In the remainder of this section, we will use the symbol C to represent various constants that depend solely on the characteristics of p_i, ν, s, μ, $|\Omega|$ and $\|f\|_{L^m(\Omega)}$.

3.1 Proof of Theorem 2.1

Throughout the ensuing discussion, let $u_n \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ represent a solution to problem (3.1). In the following lemma, we provide $L^{m(s-1)}$-estimates for the finite energy solutions u_n of problem (3.1).

Lemma 3.2. *Under the assumptions of Theorem 2.1, there exists a positive constant C, independent of n, such that*

$$\|u_n\|_{L^{m(s-1)}(\Omega)} \le C, \tag{3.9}$$

$$\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \le C. \tag{3.10}$$

Proof. Let us use $\varphi(u_n) = |u_n|^\lambda u_n$ with $\lambda := (m-1)(s-1) - 1 \ge 0$ (since $m \ge \frac{s}{s-1}$) as a test function in (3.1). Using the fact that $T_n(u_n) \le |u_n|$, we have

$$(\lambda+1)\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} |u_n|^\lambda \, dx + \nu \int_\Omega |u_n|^{\lambda+s} \, dx \le \mu \int_\Omega \frac{|u_n|^{\lambda+\overline{p}}}{|x|^{\overline{p}} + \frac{1}{n}} \, dx + \int_\Omega |f_n||u_n|^{\lambda+1} \, dx. \tag{3.11}$$

If we omit the operator term from the left-hand side, we find

$$\nu \int_\Omega |u_n|^{\lambda+s} \, dx \le \mu \int_\Omega \frac{|u_n|^{\lambda+\overline{p}}}{|x|^{\overline{p}} + \frac{1}{n}} \, dx + \int_\Omega |f_n| \, |u_n|^{\lambda+1} \, dx. \tag{3.12}$$

Applying the Hölder inequality with exponent m on the right-hand side of (3.12) and taking into consideration $(\lambda+1)m' = \lambda + s$, we can deduce

$$\int_\Omega |f_n| \, |u_n|^{\lambda+1} \, dx \le \Big(\int_\Omega |f|^m \, dx\Big)^{\frac{1}{m}} \Big(\int_\Omega |u_n|^{(\lambda+1)m'} \, dx\Big)^{\frac{1}{m'}} = C\Big(\int_\Omega |u_n|^{\lambda+s} \, dx\Big)^{\frac{1}{m'}}. \tag{3.13}$$

Recalling that $s > \overline{p}$ (from (1.3)), we have $\lambda + \overline{p} < \lambda + s$. Applying Hölder's inequality with indices $(\frac{\lambda+s}{\lambda+\overline{p}}, \frac{\lambda+s}{s-\overline{p}})$, we obtain

$$\int_\Omega \frac{|u_n|^{\lambda+\overline{p}}}{|x|^{\overline{p}} + \frac{1}{n}} \, dx \le \Big(\int_\Omega |u_n|^{\lambda+s} \, dx\Big)^{\frac{\lambda+\overline{p}}{\lambda+s}} \Big(\int_\Omega \frac{1}{|x|^{\overline{p}\,\frac{\lambda+s}{s-\overline{p}}}} \, dx\Big)^{\frac{s-\overline{p}}{\lambda+s}}.$$

The assumption $m < \frac{N(s-\overline{p})}{\overline{p}(s-1)}$ implies $\overline{p}\,\frac{\lambda+s}{s-\overline{p}} < N$. Consequently, from the above inequality, it follows that

$$\int_\Omega \frac{|u_n|^{\lambda+\overline{p}}}{|x|^{\overline{p}} + \frac{1}{n}} \, dx \le C\Big(\int_\Omega |u_n|^{\lambda+s} \, dx\Big)^{\frac{\lambda+\overline{p}}{\lambda+s}}. \tag{3.14}$$

Combining (3.12)–(3.14) with the fact that $\lambda + s = m(s-1)$, we arrive at

$$\int_\Omega |u_n|^{m(s-1)} \, dx \le C\Big(\int_\Omega |u_n|^{m(s-1)} \, dx\Big)^{\frac{1}{m'}} + C\Big(\int_\Omega |u_n|^{m(s-1)} \, dx\Big)^{\frac{\lambda+\overline{p}}{\lambda+s}}.$$

As $\frac{1}{m'} < 1$ and $\frac{\lambda+\overline{p}}{\lambda+s} < 1$, we get

$$\int_\Omega |u_n|^{m(s-1)} \, dx \le C. \tag{3.15}$$

Therefore, from (3.15) it follows (3.9).

Using (3.11) and (3.15), we can conclude that

$$\int_\Omega |\partial_i u_n|^{p_i} |u_n|^\lambda \, dx \le C, \quad \forall\, i = 1, \dots, N.$$

Since $\lambda \geq 0$, it follows that

$$\int_{\{|u_n|\geq 1\}} |\partial_i u_n|^{p_i}\, dx \leq C, \quad \forall\, i = 1, \dots, N. \tag{3.16}$$

On the other hand, using $T_1(u_n)$ as a test function in (3.1) and dropping the positive lower order term, we obtain

$$\sum_{i=1}^{N} \int_{\{|u_n|<1\}} |\partial_i T_1(u_n)|^{p_i}\, dx \leq \mu \int_{\Omega} \frac{|T_n(u_n)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}}\, dx + C. \tag{3.17}$$

Using Hölder's inequality on the right-hand side of (3.17), from (3.15) one gets

$$\begin{aligned}\int_{\Omega} \frac{|T_n(u_n)|^{\overline{p}-1}}{|x|^{\overline{p}}}\, dx &\leq \Big(\int_{\Omega} |u_n|^{m(s-1)}\, dx \Big)^{\frac{\overline{p}-1}{m(s-1)}} \Big(\int_{\Omega} \frac{1}{|x|^{\frac{\overline{p}m(s-1)}{m(s-1)-\overline{p}+1}}}\, dx \Big)^{1-\frac{\overline{p}-1}{m(s-1)}} \\ &\leq C \Big(\int_{\Omega} \frac{1}{|x|^{\frac{\overline{p}m(s-1)}{m(s-1)-\overline{p}+1}}}\, dx \Big)^{1-\frac{\overline{p}-1}{m(s-1)}}. \end{aligned} \tag{3.18}$$

Observe now that since $s > \overline{p}^*$, we have

$$m \geq \frac{s}{s-1} > \frac{N(\overline{p}-1)}{(N-\overline{p})(s-1)},$$

which implies

$$\frac{\overline{p}m(s-1)}{m(s-1)-\overline{p}+1} < N.$$

Consequently, using (3.18), we derive

$$\text{the sequence } \frac{|T_n(u_n)|^{\overline{p}-1}}{|x|^{\overline{p}} + \frac{1}{n}} \text{ is bounded in } L^1(\Omega). \tag{3.19}$$

Combining (3.17) and (3.19), we can deduce that

$$\int_{\{|u_n|<1\}} |\partial_i T_1(u_n)|^{p_i}\, dx \leq C, \quad \forall\, i = 1, \dots, N.$$

Taking into account both this result and (3.16), we obtain the final outcome (3.10). □

From Lemma 3.2, there exists a subsequence of (u_n) (still denoted by (u_n)) and a function $u \in W_0^{1,\vec{p}}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,\vec{p}}(\Omega) \text{ and a.e. in } \Omega. \tag{3.20}$$

Now, adapting the approach of the proof of Theorem 2.3 in [5], we can show that there exists a subsequence (still denoted (u_n)) such that for all $i = 1, \dots, N$,

$$\partial_i u_n \to \partial_i u \text{ strongly in } L^{r_i}(\Omega) \text{ and a.e. in } \Omega, \ \forall\, r_i < p_i. \tag{3.21}$$

By (3.20), (3.21) and applying the Lebesgue dominated convergence theorem, for every $i = 1, \dots, N$ we obtain the following result:

$$|\partial_i u_n|^{p_i-2}\partial_i u_n \to |\partial_i u|^{p_i-2}\partial_i u \text{ strongly in } L^{p_i'}(\Omega). \tag{3.22}$$

Let φ be any function in $W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Since the sequence (u_n) is bounded in $L^{m(s-1)}(\Omega)$, it follows that

$$\lim_{n\to\infty} \int_{\Omega} |u_n|^{s-2}u_n\varphi\, dx = \int_{\Omega} |u|^{s-2}u\varphi\, dx. \tag{3.23}$$

On the other hand, by (3.19), (3.20) and the Lebesgue's theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{|T_n(u_n)|^{\overline{p}-1}}{|x|^{\overline{p}}+\frac{1}{n}}\varphi\,dx=\int_\Omega \frac{|u|^{\overline{p}-1}}{|x|^{\overline{p}}}\varphi\,dx,\quad \forall\,\varphi\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega). \tag{3.24}$$

Using the convergence results (3.22)–(3.24) and $f_n\to f$ in $L^1(\Omega)$, we can then take the limit as $n\to+\infty$ in the identities

$$\sum_{i=1}^N\int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n\partial_i\varphi\,dx+\nu\int_\Omega |u_n|^{s-2}u_n\varphi\,dx=\mu\int_\Omega \frac{|T_n(u_n)|^{\overline{p}-1}}{|x|^{\overline{p}}+\frac{1}{n}}\,dx+\int_\Omega f_n\varphi\,dx, \tag{3.25}$$

for all $\varphi\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$. This yields

$$\sum_{i=1}^N\int_\Omega |\partial_i u|^{p_i-2}\partial_i u\partial_i\varphi\,dx+\nu\int_\Omega |u|^{s-2}u\varphi\,dx=\mu\int_\Omega \frac{|u|^{\overline{p}-1}}{|x|^{\overline{p}}}\,\varphi\,dx+\int_\Omega f\varphi\,dx.$$

So, the proof of Theorem 2.1 is complete.

3.2 Proof of Theorem 2.2

Lemma 3.3. *Suppose that the hypotheses of Theorem 2.2 are satisfied and there exists a positive constant C, independent of n, such that*

$$\|u_n\|_{W_0^{1,\vec{q}}(\Omega)\cap L^{m(s-1)}(\Omega)}\le C. \tag{3.26}$$

Proof. Let $\varepsilon>0$. We consider the function

$$\varphi(u_n)=\frac{u_n}{(|u_n|+\varepsilon)^\rho}$$

as a test function in equation (3.1), where $0<\rho:=1-(m-1)(s-1)<1$ (since $1<m<\frac{s}{s-1}$). Using the fact that

$$\partial_i\varphi(u_n)=\frac{(1-\rho)|u_n|+\varepsilon}{(|u_n|+\varepsilon)^{\rho+1}}\,\partial_i u_n,$$

we obtain the following inequality:

$$\sum_{i=1}^N\int_\Omega \frac{(1-\rho)|u_n|+\varepsilon}{(|u_n|+\varepsilon)^{\rho+1}}\,|\partial_i u_n|^{p_i}\,dx+\nu\int_\Omega \frac{|u_n|^s}{(|u_n|+\varepsilon)^\rho}\,dx\le\mu\int_\Omega \frac{|u_n|^{\overline{p}-\rho}}{|x|^{\overline{p}}+\frac{1}{n}}\,dx+\int_\Omega f_n|u_n|^{1-\rho}\,dx. \tag{3.27}$$

Since $1-\rho>0$, we have

$$\frac{(1-\rho)|u_n|+\varepsilon}{(|u_n|+\varepsilon)^{\rho+1}}\,|\partial_i u_n|^{p_i}\ge(1-\rho)\,\frac{|\partial_i u_n|^{p_i}}{(|u_n|+\varepsilon)^\rho}\,,\quad\forall\,i=1,\dots,N.$$

The previous estimate and inequality (3.27), yields

$$(1-\rho)\sum_{i=1}^N\int_\Omega \frac{|\partial_i u_n|^{p_i}}{(|u_n|+\varepsilon)^\rho}\,dx+\nu\int_\Omega \frac{|u_n|^s}{(|u_n|+\varepsilon)^\rho}\,dx\le\mu\int_\Omega \frac{|u_n|^{\overline{p}-\rho}}{|x|^{\overline{p}}+\frac{1}{n}}\,dx+\int_\Omega f_n|u_n|^{1-\rho}\,dx. \tag{3.28}$$

Omitting the operator term on the left-hand side of (3.28) and, subsequently, taking the limit as ε tends to zero, we have

$$\nu\int_\Omega |u_n|^{s-\rho}\,dx\le\mu\int_\Omega \frac{|u_n|^{\overline{p}-\rho}}{|x|^{\overline{p}}+\frac{1}{n}}\,dx+\int_\Omega f_n|u_n|^{1-\rho}\,dx,$$

which is the same as (3.12) with $\rho = -\lambda$. Starting from this inequality and working as in the proof of Lemma 3.2 (see inequality (3.9)), we can use this fact and (3.28) to obtain

$$\int_\Omega \frac{|\partial_i u_n|^{p_i}}{(|u_n| + \varepsilon)^\rho} \, dx \leq C, \;\; \forall\, i = 1, \dots, N.$$

From the previous estimate and by applying Hölder's inequality with exponents $\frac{p_i}{q_i}$ and $(\frac{p_i}{q_i})'$, for any $i = 1, \dots, N$ we get

$$\begin{aligned} \int_\Omega |\partial_i u_n|^{q_i} \, dx &= \int_\Omega \frac{|\partial_i u_n|^{q_i}}{(|u_n| + \varepsilon)^{\rho \frac{q_i}{p_i}}} (|u_n| + \varepsilon)^{\rho \frac{q_i}{p_i}} \, dx \\ &\leq C \Bigg(\int_\Omega \frac{|\partial_i u_n|^{p_i}}{(|u_n| + \varepsilon)^\rho} \, dx \Bigg)^{\frac{q_i}{p_i}} \Bigg(\int_\Omega (|u_n| + \varepsilon)^{\rho \frac{q_i}{p_i - q_i}} \, dx \Bigg)^{\frac{p_i - q_i}{p_i}} \\ &\leq C \Bigg(\int_\Omega (|u_n| + \varepsilon)^{\rho \frac{q_i}{p_i - q_i}} \, dx \Bigg)^{\frac{p_i - q_i}{p_i}}. \end{aligned} \tag{3.29}$$

Now we take $q_i = \theta p_i$ with $\theta \in [0, 1)$ such that

$$\frac{\rho q_i}{p_i - q_i} = \frac{\rho \theta}{1 - \theta} = m(s - 1), \;\; \forall\, i = 1, \dots, N, \tag{3.30}$$

the previous equality is equivalent to

$$\theta = \frac{m(s-1)}{s} < 1, \;\; \text{and} \;\; q_i = \frac{m(s-1)}{s} p_i, \;\; \forall\, i = 1, \dots, N.$$

Therefore, using (3.29), (3.30) and the boundedness of the sequence (u_n) in $L^{m(s-1)}(\Omega)$, we can write

$$\int_\Omega |\partial_i u_n|^{q_i} \, dx \leq C \Bigg(\int_\Omega (|u_n| + \varepsilon)^{m(s-1)} \, dx \Bigg)^{1-\theta} \leq C, \;\; \forall\, i = 1, \dots, N. \tag{3.31}$$

If the parameters $\overline{p}$ and m satisfy assumption (2.4) or (2.5) of Theorem 2.2, we can conclude that $q_i > 1$ for every $i = 1, \dots, N$. Therefore, using (3.31), we obtain estimate (3.26). $\square$

In order to prove this theorem, we modify the proof of Theorem 2.1. It is sufficient to replace only (3.22) by the following

$$|\partial_i u_n|^{p_i - 2} \partial_i u_n \to |\partial_i u|^{p_i - 2} \partial_i u \;\; \text{strongly in} \;\; L^{\frac{q_i}{p_i - 1}}(\Omega), \tag{3.32}$$

for every $\frac{q_i}{p_i - 1} > 1$, $\forall\, i = 1, \dots, N$. Thus, by (3.32), (3.23) and (3.24), we can pass to the limit as $n \to +\infty$ in (3.25). Consequently, we have that the limit function u is a weak solution of (1.1) possessing the regularity stated in Theorem 2.2.

Acknowledgments

The authors would like to express their gratitude to the referees for their valuable comments and insightful suggestions.

References

[1] A. Adimurthi, L. Boccardo, G. R. Cirmi and L. Orsina, The regularizing effect of lower order terms in elliptic problems involving Hardy potential. *Adv. Nonlinear Stud.* **17** (2017), no. 2, 311–317.

[2] J. Bear, *Dynamics of Fluids in Porous Media*. Elsevier, New York, NY, 1972.

[3] M. Bendahmane, M. Langlais and M. Saad, On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease. *Nonlinear Anal.* **54** (2003), no. 4, 617–636.

[4] G. R. Cirmi, Regularity of the solutions to nonlinear elliptic equations with a lower-order term. *Nonlinear Anal.* **25** (1995), no. 6, 569–580.

[5] A. Di Castro, Existence and regularity results for anisotropic elliptic problems. *Adv. Nonlinear Stud.* **9** (2009), no. 2, 367–393.

[6] A. Di Castro, Anisotropic elliptic problems with natural growth terms. *Manuscripta Math.* **135** (2011), no. 3-4, 521–543.

[7] I. Fragalà, F. Gazzola and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **21** (2004), no. 5, 715–734.

[8] H. Khelifi, Application of the Stampacchia lemma to anisotropic degenerate elliptic equations. *J. Innov. Appl. Math. Comput. Sci.* **3** (2023), no. 1, 75–82.

[9] H. Khelifi, Existence and regularity of solutions of nonlinear anisotropic elliptic problem with Hardy potential. *Asymptotic Analysis* **137** (2024), 291–303.

[10] H. Khelifi and F. Mokhtari, Nonlinear degenerate anisotropic elliptic equations with variable exponents and L^1 data. *J. Partial Differ. Equ.* **33** (2020), no. 1, 1–16.

[11] H. Le Dret, *Équations aux Dérivées Partielles Elliptiques Non Linéaires.* (French) *[Nonlinear Elliptic Partial Differential Equations]* Mathématiques & Applications (Berlin) [Mathematics & Applications], 72. Springer, Heidelberg, 2013.

[12] F. Mokhtari, Anisotropic parabolic problems with measures data. *Differ. Equ. Appl.* **2** (2010), no. 1, 123–150.

[13] J. Rákosník, Some remarks to anisotropic Sobolev spaces, I. *Beiträge Anal.* no. 13 (1979), 55–68.

[14] J. Rákosník, Some remarks to anisotropic Sobolev spaces, II. *Beiträge Anal.* no. 15 (1980), 127–140 (1981).

[15] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi. (Italian) *Ricerche Mat.* **18** (1969), 3–24.

[16] M. A. Zouatini, H. Khelifi and F. Mokhtari, Anisotropic degenerate elliptic problem with a singular nonlinearity. *Adv. Oper. Theory* **8** (2023), no. 1, Paper no. 13, 24 pp.

(Received 07.07.2023; revised 16.09.2024; accepted 14.12.2023)

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