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MULTIPLE SOLUTIONS FOR A CLASS OF $(p(x), q(x))$-BIHARMONIC OPERATOR WITH HARDY POTENTIAL AND NONLOCAL SOURCE TERM

Abstract. We study a class of nonhomogeneous $(p(x), q(x))$-biharmonic problems which is seldom studied because the nonlinearity has nonstandard growth and contains a nonlocal term and a Hardy potential. Based on variational methods, especially the abstract critical point result of Bonanno–Candito-D'Aguí [*Adv. Nonlinear Stud.* **14** (2014), no. 4, 915–939] and a recent three critical points theorem of Bonanno–Marano [*Appl. Anal.* **89** (2010), 1–10], we prove the existence of at least one non-zero critical point and the existence of at least three distinct critical points Our results generalize and extend several existing results.

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1 Introduction

This paper deals on a biharmonic problem involving $(p(x), q(x))$ exponents

$$\begin{cases} \Delta^2_{p(x)}u + \Delta^2_{q(x)}u + \theta(x)\,\dfrac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u)\Big(\displaystyle\int_\Omega F(x,u)\,dx\Big)^r & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with boundary of class C^1, p and q are in a class of functions denoted by

$$C_+(\overline{\Omega}) := \big\{t \in C(\overline{\Omega}) : \ \ t(x) > 1 \ \text{ for all } \ x \in \overline{\Omega}\big\},$$

θ is a real function in $L^\infty(\Omega)$ with $\operatorname*{ess\,inf}_{x\in\overline{\Omega}} \theta(x) > 0$, r is a positive constant, s is a constant such that $1 < s < \frac{N}{2}$, $\lambda > 0$ is a real parameter and $f : \Omega \to \mathbb{R}$ is a Carathéodory function satisfying the following growth condition:

(H_1) $\qquad a_1|t|^{\alpha(x)-1} \le f(x,t) \le a_2|t|^{\beta(x)-1}$ for all $x \in \overline{\Omega}$,

where $\alpha(x), \beta(x) \in C_+(\overline{\Omega})$ such that

$$\sup_{x\in\Omega} \alpha(x) = \alpha^+ \le \inf_{x\in\Omega} \beta(x) = \beta^-.$$

Throughout this paper, we assume that

(H_2) $\qquad 1 < s < q^- \le q(x) \le \max\big(q^+, (r+1)\beta^+\big) < p^- \le p(x) \le p^+ < \dfrac{N}{2}.$

Here,

$$\Delta^2_{\gamma(x)}u := \Delta\big(|\Delta u|^{\gamma(x)-2}\Delta u\big), \ \ \forall\, \gamma \in \{p, q\},$$

is the so-called $\gamma(x)$-biharmonic operator which is not homogeneous, and thus some techniques which can be applied when $\gamma(x)$ is a positive constant such as the Lagrange Multiplier Theorem, will fail in this new situation.

Recently, the investigation of differential equations and variational problems with variable exponent has become a new and interesting topic. The study of various mathematical problems with variable exponent has been received considerable attention in recent years. These problems arise in various fields, including electrorheological fluids, image processing, and elastic mechanics, making it an intriguing area for research [6,24,26]. In this direction, an increased interest among the researchers has been observed to extend the study of problem (1.1). Our problem (1.1) is a particular case of the following problems:

$$\begin{cases} \Delta^2_{p(x)}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and

$$\begin{cases} -g(u)\Delta_{p(x)}u = \lambda f(x,u)\Big(\displaystyle\int_\Omega F(x,u)\,dx\Big)^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.2) was first studied in 2020 by El Khalil et al. in [10] in the case where $f(x,u) = \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}$, λ is a positive real number and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary $\partial\Omega$. The authors established the existence of at least one non-decreasing sequence of nonnegative eigenvalues to problem (1.2) by using the Hardy–Rellich inequality for $p(x) < \frac{N}{2}$. When $f(x,u) = \lambda|u|^{p(x)-2}u$ in problem (1.2), Ayoujil–EI Amross [2] used the Ljusternik–Schnirelmann critical point theorem and found that there are multiple eigenvalues to this problem. If $f(x,u) = \lambda V(x)|u|^{q(x)-2}u$ in problem (1.2), $1 < q(x) < p(x) < \frac{N}{2} < s(x)$, $V(x) \in L^{s(x)}$, there are multiple eigenvalues of this problem in the neighbourhood of the origin (see [14]). We refer reader to [1] and the references therein.

Problem (1.3) is called a bi-nonlocal problem due to the presence of the terms $g(u)$ and $[\int_\Omega F(x,u)\,dx]^r$, which means that the first equation in (1.3) is no longer a point identity. This phenomenon raises some mathematical difficulties which make the study of such problems particularly interesting. In [7], Corrêa–Costa studied problem (1.3) with $\lambda = 1$ and

$$Q_1 u^{\gamma(x)-1} \le f(x,u) \le Q_2 u^{q(x)-1} \ \text{ and } \ A_0 + Au^{\alpha(x)} \le g(u) \le B_0 + Bu^{\beta(x)},$$

where A_0, A, B_0, B, Q_1, Q_2 are positive constants and $\alpha(x), \beta(x), \gamma(x), q(x) \in C_+(\overline{\Omega})$ satisfy some suitable conditions. By using Krasnoselskii's genus, they proved the existence of infinitely many solutions for (1.3). We refer the reader to the papers [15–17] and the references therein.

From here it is natural to ask whether problems (1.2) and (1.3) have multiple solutions when they contain two operators, namely, the biharmonic operator, a Hardy potential operator and a nonlocal source term?

In 2023, Khaleghi and Razani in [20] considered the following $(p(x), q(x))$-biharmonic elliptic equation with singular term:

$$\begin{cases} \Delta^2_{p(x)} u + \Delta^2_{q(x)} u + \theta(x)\,\dfrac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with boundary of class $C^1, p, q \in C_+(\overline{\Omega})$ and satisfy

$$\max\left\{2, \frac{N}{2}\right\} < q^- \le q(x) \le q^+ < p^- \le p(x) \le p^+ < +\infty.$$

By using variational methods and critical point results, the authors established the existence of multiple solutions via the standard and restrictive (AR) condition for function f, due to Ambrosetti and Rabinowitz [23] (see also [1, 3, 21]).

Motivated by the above papers, we consider combining problems (1.3) and (1.4) to study our problem (1.1). Obviously, the combination of two nonhomogeneous operators, a Hardy potential operator and a nonlocal source term will undoubtedly bring more difficulties. Contrary to [20], we will use the case of $p^+ < \frac{N}{2}$ and suppose that the function f changes its sign and does not satisfy the additional condition of Ambrosetti–-Rabinowitz. In particular, to overcome these difficulties, we have to develop some subtle techniques. We shall prove the existence of at least one non-zero critical point via [5, Theorem 3.1] and the existence of at least three distinct critical points which represent the weak solutions of system (1.1) via the recent three critical points theorem of Bonanno-Marano [4].

To the best of our knowledge, there are no results concerning the existence of at least three distinct weak solutions for the problem defined by problem (1.1) via the recent three critical points theorem of Bonanno–Marano [4] with nonlocal source term. In this context, the results of our paper can be seen as a generalization of the above results. There is no doubt that our new approach employed in this article could be applied to study the other elliptic equations and systems involving the variable-order fractional $(p_1(x,\cdot), p_2(x,\cdot))$-Laplacian.

The study is organized as follows. In Section 2, we introduce our primary tools and review some fundamental information that will be needed later. In Section 3, it can be shown that there is a weak solution to problem (1.1). In Section 4, it is confirmed that there are several weak solutions to problem (1.1).

2 Background setting and results

In the whole paper, denote

$$\gamma^- := \inf_{x\in\Omega} \gamma(x) \ \text{ and } \ \gamma^+ := \sup_{x\in\Omega} \gamma(x)$$

for $\gamma \in \{p, q, \beta, \alpha\}$. We denote the variable exponent Lebesgue space [11] by

$$L^{p(x)}(\Omega) = \left\{\Omega \to \mathbb{R} : \ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}$$

with the Luxemburg norm

$$|u|_{p(x)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx \le 1\Big\}.$$

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \tag{2.1}$$

holds true.

In what follows and for $\gamma \in C_+(\overline{\Omega})$, put

$$[\zeta]^\gamma := \max\big\{\zeta^{\gamma^-}, \zeta^{\gamma^+}\big\}, \quad [\zeta]_\gamma := \min\big\{\zeta^{\gamma^-}, \zeta^{\gamma^+}\big\}.$$

So, we have:

(i) $[\zeta]^{\frac{1}{\gamma}} = \max\big\{\zeta^{\frac{1}{\gamma^-}}, \zeta^{\frac{1}{\gamma^+}}\big\}$,

(ii) $[\zeta]_{\frac{1}{\gamma}} = \min\big\{\zeta^{\frac{1}{\gamma^-}}, \zeta^{\frac{1}{\gamma^+}}\big\}$,

(iii) $[\zeta]_{\frac{1}{\gamma}} = a \Longleftrightarrow \zeta = [a]^\gamma$, $[\zeta]^{\frac{1}{\gamma}} = a \Longleftrightarrow \zeta = [a]_\gamma$,

(iv) $[\zeta]_\gamma[\alpha]_\gamma \le [\zeta\alpha]_\gamma \le [\zeta\alpha]^\gamma \le [\zeta]^\gamma[\alpha]^\gamma$.

Now, let us recall the following proposition [18, Proposition 2.7].

Proposition 2.1. *For every $u \in L^{p(x)}(\Omega)$, one has*

$$\big[\,|u|_{p(x)}\,\big]_p \le \int_\Omega |u(x)|^{p(x)}\,dx \le \big[\,|u|_{p(x)}\,\big]^p.$$

Proposition 2.2 ([12])**.** *If $p, q \in C_+(\overline{\Omega})$ and $q(x) \le p(x)$ a.e. on Ω, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and there exists a constant c_q such that*

$$|u|_{q(x)} \le c_q|u|_{p(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ for $k = 1, 2$ is defined as

$$W^{k,p(x)}(\Omega) := \Big\{u \in L^{p(x)}(\Omega) : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \le k\Big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha}{}_1 x_1 \cdots \partial^{\alpha}{}_N x_N}$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index with $|\alpha| = \sum\limits_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)}$$

is a reflexive separable Banach space. Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with the norm $\|u\|_{1,p(x)} = |Du|_{p(x)}$. In what follows, let

$$X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$$

be equipped with the norm

$$\|u\| := \inf\Big\{\mu > 0 \int_\Omega \Big|\frac{\Delta u}{\mu}\Big|^{p(x)} dx \le 1\Big\}.$$

The modular on X is the mapping $\rho_{p(x)} : X \to \mathbb{R}$ defined by $\rho_{p(x)}(u) = \int\limits_\Omega |\Delta u|^{p(x)}\,dx$. This mapping satisfies the same properties as in Proposition 2.3. More precisely, we have the following result.

Proposition 2.3. *For all $u \in L^{p(x)}(\Omega)$, we have*

1. $\|u\| < 1$ *(resp.* $= 1, > 1) \Longleftrightarrow \rho_{p(x)}(u) < 1$ *(resp.* $= 1, > 1)$.
2. $[\,\|u\|\,]_p \leq \rho_{p(x)}(u) \leq [\,\|u\|\,]^p$.

Proposition 2.4 ([9])**.** *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$, and $1 \leq p(x)q(x) \leq \infty$ a.e. $x \in \Omega$. Let $w \in L^{q(x)}(\Omega)$, $w \neq 0$. Then*

$$\left[\,|w|_{p(x)q(x)}\,\right]_p \leq \left|\,|w|^{p(x)}\,\right|_{q(x)} \leq \left[\,|w|_{p(x)q(x)}\,\right]^p.$$

We recall that the critical Sobolev exponent is defined as follows:

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-2p(x)}, & p(x) < \dfrac{N}{2}, \\ +\infty, & p(x) \geq \dfrac{N}{2}. \end{cases}$$

As a consequence of Proposition 2.2, if $q(x) \leq p(x)$ a.e on Ω, we have

$$W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega) \text{ and } W^{2,p(x)}(\Omega) \hookrightarrow W^{2,q(x)}(\Omega).$$

In particular, one has

$$X \hookrightarrow W_0^{1,p^-}(\Omega) \cap W^{2,p^-}(\Omega).$$

Lemma 2.1 ([13])**.** *Let G be a measurable subset in $\mathbb{R}^N$ and $0 < \operatorname{meas}(G) < +\infty$. If $f : G \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and*

$$|f(x,u)| \leq a(x) + b|u|^{\frac{p_1(x)}{p_2(x)}} \quad a.e. \ (x,u) \in G \times \mathbb{R},$$

where $p_1(x), p_2(x) \in C_+(\overline{\Omega})$, $0 < a(x) \in L^{p_2(x)}(G)$, $b > 0$, then the Nemytskii operator defined by $N_f(u)(x) = f(x,u(x))$ maps $L_{p_1(x)}(G)$ into $L_{p_2(x)}(G)$, and it is continuous and bounded.

In order to formulate our existence result, we need the following preliminary definitions and theorems.

Definition 2.1. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $d \in \mathbb{R}$. The functional $I := \Phi - \Psi$ is said to verify the Palais–Smale condition cut of upper at d (in short $(PS)^{[d]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \in X$ such that

- $I(u_n)$ is bounded,
- $\lim\limits_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
- $\Phi(u_n) < d$ for each $n \in \mathbb{N}$,

has a convergent subsequence.

If $d = \infty$, the functional $I := \Phi - \Psi$ fulfill the Palais–Smale condition.

Our main existence result follows from the following theorem.

Theorem 2.1 ([5, (Theorem 3.1])**.** *Let X be a real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{x\in X} \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist a positive constant $d \in \mathbb{R}$ and $\overline{x} \in X$ with $0 < \Phi(\overline{x}) < d$ such that

$$\frac{\sup\limits_{x\in\Phi^{-1}(]-\infty,d])} \Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$$

and for each

$$\lambda \in \Lambda := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup\limits_{x \in \Phi^{-1}(]-\infty, d])} \Psi(x)}\right[,$$

$I_\lambda = \Phi - \lambda\Psi$ *fulfills the* $(PS)^{[d]}$*-condition. Then for every* $\lambda \in \Lambda$*, there is* $x_\lambda \in \Phi^{-1}(]0, d])$ *such that* $I_\lambda(x_\lambda) \le I_\lambda(x)$ *for all* $x \in \Phi^{-1}(]0, d])$ *and* $I'_\lambda(u_\lambda) = 0$.

The multiplicity result is due to the following

Theorem 2.2 ([4])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Suppose that there exist $d > 0$ and $\overline{x} \in X$, with $d < \Phi(\overline{x})$, such that

(i) $\dfrac{\sup\limits_{\Phi(x)<d} \Psi(x)}{d} < \dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) *for each* $\lambda \in \Lambda_d := \left]\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{d}{\sup\limits_{\Phi(x)\le d} \Psi(x)}\right[$, $I_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for any $\lambda \in \Lambda_d$, $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

In what follows, let

$$\delta(x) = \sup\left\{\delta > 0 : \ B(x, \delta) \subseteq \Omega\right\}$$

and let

$$R := \sup_{x \in \Omega} \delta(x).$$

It is clear that there exists $x^0 = (x_1^0, \ldots, x_N^0) \in \Omega$ such that $B(x^0, R) \subseteq \Omega$.

3 Existence result

In what follows, we recall the Hardy–Rellich inequality [8].

Lemma 3.1. *For $1 < s < N/2$ and $u \in W_0^{1,s}(\Omega) \cap W^{2,s}(\Omega)$, we have*

$$\int\limits_\Omega \frac{|u(x)|^s}{|x|^{2s}}\, dx \le \frac{1}{\mathcal{H}_s} \int\limits_\Omega |\Delta u(x)|^s\, dx,$$

where

$$\mathcal{H}_s := \Big(\frac{N(s-1)(N-2s)}{s^2}\Big)^s.$$

Note that a weak solution of problem (1.1) is defined as follows.

Definition 3.1. $u \in X$ is a weak solution of Problem (1.1) if $u = \Delta u = 0$ on $\partial\Omega$ and

$$\int\limits_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\, dx + \int\limits_\Omega |\Delta u|^{q(x)-2}\Delta u \Delta v\, dx + \int\limits_\Omega \theta(x)\, \frac{|u|^{s-2}}{|x|^{2s}} uv\, dx - \lambda\Big(\int\limits_\Omega F(x, u)\, dx\Big)^r \int\limits_\Omega f(x, u) v\, dx = 0$$

for every $v \in X$.

Let $\Phi : X \to \mathbb{R}$ be a functional defined by

$$\Phi(u) = \int\limits_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx + \int\limits_\Omega \frac{1}{q(x)} |\Delta u|^{q(x)}\, dx + \frac{1}{s}\int\limits_\Omega \theta(x) \frac{|u(x)|^s}{|x|^{2s}}\, dx.$$

We mention that Φ is coercive. In fact, due to Proposition 2.3, for $u \in X$ with $\|u\| \geq 1$, one has

$$\Phi(u) \geq \frac{1}{p^+}\int\limits_\Omega |\Delta u|^{p(x)}\, dx \geq \frac{1}{p^+} \|u\|^{p^-}.$$

Due to conditions (H_1) and (H_2) and Proposition 2.2, Φ is well defined and continuously Gâteaux differentiable (for more details, see [22]), moreover,

$$\langle \Phi'(u), v\rangle = \int\limits_\Omega \left(|\Delta u|^{p(x)-2}\Delta u \Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v + \theta(x)\frac{|u(x)|^{s-2}uv}{|x|^{2s}}\right) dx$$

for $u, v \in X$.

Let

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Psi(u) = \frac{1}{r+1}\left(\int\limits_\Omega F(x,u)\, dx\right)^{r+1}.$$

Note that Ψ is well defined and

$$\langle \Psi'(u), v\rangle = \left(\int\limits_\Omega F(x,u)\, dx\right)^r \int\limits_\Omega f(x,u)v\, dx$$

for all $u, v \in X$. Moreover, $\Psi'(u)$ is compact. In fact, the compact embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$, $1 < \beta(x) < p^*(x)$, implies the compactness of $\Psi'(u)$. Indeed, let $(u_k)_k \subset X$ be a sequence such that $u_k \rightharpoonup u$. Thus there is a subsequence, still denoted by $(u_k)_k$, such that $u_k \to u$, strongly in $L^{\beta(x)}(\Omega)$. We claim that the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ is continuous, since $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (f). Thus $N_f(u_k) \to N_f(u)$ in $L^{\frac{\beta(x)}{\beta(x)-1}}(\Omega)$. In view of Hölder's inequality mentioned in (2.1) and the compact embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$, $1 < \beta(x) < p^*(x)$, for all $v \in X$, one has

$$\left|\Psi'(u_k)(v) - \Psi'(u)(v)\right| = \left|\left(\int\limits_\Omega F(x,u_k)\, dx\right)^r \int\limits_\Omega f(x,u_k)v\, dx - \left(\int\limits_\Omega F(x,u)\, dx\right)^r \int\limits_\Omega f(x,u)v\, dx\right|.$$

The continuity of $F(x,u)$ with respect to u ensures that

$$F(x,u_k) \to F(x,u) \text{ for almost every } x.$$

Moreover, there exists $C > 0$ such that

$$|F(x,u_k)| \leq C|u_k|^{\beta(x)}.$$

Applying the dominated Convergence theorem, we can conclude that

$$\int\limits_\Omega F(x,u_k)\, dx \longrightarrow \int\limits_\Omega F(x,u)\, dx \text{ as } k \to +\infty.$$

From condition (H_1), it follows that the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ is continuous, as $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function that satisfies (H_1). Consequently, we conclude that $N_f(u_k) \to N_f(u)$ in $L^{\frac{\beta(x)}{\beta(x)-1}}(\Omega)$.

Using Hölder's inequality, for any $v \in X$, we obtain

$$\begin{aligned}\Big| \int_\Omega f(x,u_k)v\,dx - \int_\Omega f(x,u)v\,dx \Big| &\le \int_\Omega \big|(f(x,u_k)-f(x,u))v\big|\,dx \\ &\le 2\big\|N_f(u_k)-N_f(u)\big\|_{\frac{\beta(x)}{\beta(x)-1}}\|v\|_{\beta(x)} \\ &\le 2c_\beta\big\|N_f(u_k)-N_f(u)\big\|_{\frac{\beta(x)}{\beta(x)-1}}\|v\|,\end{aligned}$$

where c_β is the embedding constant of the embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$, $1 < \beta(x) < p^*(x)$. Thus $\Psi'(u_k) \to \Psi'(u)$ in X^*, i.e. Ψ' is completely continuous, thus Ψ' is compact.

Moreover, we have

Proposition 3.1. *The operator $\Phi' : X \to X^*$ is coercive and uniformly monotone and admits a continuous inverse in X^*.*

Proof. For the coercivity, it is obvious that for any $u \in X$ with $\|u\| \ge 1$, we have

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|} \ge \|u\|^{p^- - 1},$$

which assures the coercivity of Φ'.

For the rest of the proof and using the assertion on the function θ, one has

$$\int_\Omega \frac{\theta(x)}{|x|^{2s}}\big(|u|^{s-2}u - |v|^{s-2}v\big)(u-v)\,dx \ge \frac{\operatorname*{ess\,inf}_{x\in\overline{\Omega}}\theta(x)}{(\mathrm{diam}(\Omega))^{2s}}\int_\Omega \big(|u|^{s-2}u - |v|^{s-2}v\big)(u-v)\,dx.$$

Now, let $U_\gamma = \{x \in \Omega : \ \gamma(x) \ge 2\}$ and $V_\gamma = \{x \in \Omega : \ 1 < \gamma(x) < 2\}$. Using the elementary inequality [25], for $\gamma > 1$, there exists a positive constant C_γ such that if $\gamma \ge 2$, then

$$\Big\langle |x|^{\gamma-2}x - |y|^{\gamma-2}y, x-y \Big\rangle \ge C_\gamma |x-y|^\gamma \ \text{ for } \ \gamma \ge 2,$$

and if $1 < \gamma < 2$, then

$$\Big\langle |x|^{\gamma-2}x - |y|^{\gamma-2}y, x-y \Big\rangle \ge C_\gamma \frac{|x-y|^2}{(|x|+|y|)^{2-\gamma}} \ \text{ for } \ 1 < \gamma < 2,$$

where $\langle \cdot\,, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$. For $\gamma \ge 2$, due to

$$\begin{aligned}\Big\langle \Phi'(u) - \Phi'(v), u - v \Big\rangle &= \int_\Omega \Big(|\Delta u|^{p(x)-2}\Delta u - |\Delta v|^{p(x)-2}\Delta v\Big)(\Delta u - \Delta v)\,dx \\ &+ \int_\Omega \Big(|\Delta u|^{q(x)-2}\Delta u - |\Delta v|^{q(x)-2}\Delta v\Big)(\Delta u - \Delta v)\,dx + \int_\Omega \frac{\theta(x)}{|x|^{2s}}\big(|u|^{s-2}u - |v|^{s-2}v\big)(u-v)\,dx,\end{aligned}$$

and taking into account the above three inequalities, by Proposition 2.3, for any $u, v \in X$, one has

$$\Big\langle \Phi'(u) - \Phi'(v), u - v \Big\rangle \ge C_p\big[\,\|u-v\|\,\big]_p.$$

Similarly, if $1 < \gamma < 2$, then

$$(\Phi'(u) - \Phi'(v))(u-v) \ge \int_\Omega \frac{C_p|\Delta u - \Delta v|^2}{(|\Delta u| + |\Delta v|)^{2-p(x)}}\,dx > 0.$$

Thus we have that Φ' is strictly monotone in X. Furthermore, one has

Lemma 3.2. *The operator Φ' is a mapping of (S_+)-type, i.e. if $u_n \rightharpoonup u$ in X, and*

$$\overline{\lim_{n\to\infty}} \big\langle \Phi'(u_n) - \Phi'(u), u_n - u \big\rangle \leq 0,$$

then $u_n \to u$ in X.

Proof. Let $u_n \rightharpoonup u$ in X and $\overline{\lim\limits_{n\to\infty}} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$. The strict monotonicity of Φ' implies that

$$0 \leq \underline{\lim_{n\to\infty}} (\Phi'(u_n) - \Phi'(u))(u_n - u) \leq \overline{\lim_{n\to\infty}} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0,$$

therefore,

$$\lim_{n\to\infty} (\Phi'(u_n) - \Phi'(u))(u_n - u) = 0.$$

Thus

$$\overline{\lim_{n\to\infty}} \big\langle J'(u_n) - J'(u), u_n - u \big\rangle \leq 0,$$

where $J' : X \to X^*$ is defined as

$$J(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx,$$

$$J'(u)(v) = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx.$$

Noticing that $J'(u)$ is a mapping of (S_+)-type, we get $u_n \to u$ in X. Thus the operator Φ' is a mapping of (S_+)-type. □

Lemma 3.3. *The operator Φ' is a homeomorphism.*

Proof. As the proof is similar to the work of Kefi et al. [19], we omit it here. □

Remark 3.1. Under assumption $(H1)$, one has

$$\frac{1}{p^+} [\,\|u\|\,]_p \leq \Phi(u) \leq K\big([\,\|u\|\,]^p + \|u\|^s\big),$$

where

$$K = \max\Big\{\frac{2}{s}, \frac{2|\theta|_\infty}{s\mathcal{H}_s}\Big\}.$$

Proof. Due to the assertion $1 < s < q^+ < p^- \leq p^+ < \frac{N}{2}$ and Proposition 2.3, one has

$$\frac{1}{p^+}[\,\|u\|\,]_p \leq \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx$$
$$\leq \Phi(u) \leq \frac{1}{s} \int_\Omega |\Delta u|^{p(x)}\, dx + \frac{1}{s} \int_\Omega |\Delta u|^{q(x)}\, dx + \frac{1}{s} \int_\Omega \theta(x) \frac{|u(x)|^s}{|x|^{2s}}\, dx.$$

By using Hardy's inequality, we deduce that

$$\frac{1}{p^+} [\,\|u\|\,]_p \leq \Phi(u) \leq K\big([\,\|u\|\,]^p + \|u\|^s\big),$$

where $K = \max\{\frac{2}{s}, \frac{2|\theta|_\infty}{s\mathcal{H}_s}\}$, and then the proof is completed. □

Remark 3.2. If $I'_\lambda(u) = 0$, we have

$$\int_\Omega \Big(|\Delta u|^{p(x)-2}\Delta u \Delta v + |\Delta u|^{q(x)-2}\Delta u \Delta v + \theta(x)\,\frac{|u|^{s-2}uv}{|x|^{2s}} \Big)\,dx - \lambda \Big(\int_\Omega F(x,u)\,dx \Big)^r \int_\Omega f(x,u)v\,dx = 0$$

for every $u, v \in X$, then the critical points of I_λ are the weak solutions of Problem (1.1).

Lemma 3.4. *I_λ fulfills the Palais–Smale condition for every $\lambda > 0$.*

Proof. Let $\{u_n\} \subseteq X$ be a Palais–Smale sequence, so one has

$$\sup_n I_\lambda(u_n) < +\infty \quad \text{and} \quad \|I'_\lambda(u_n)\|_{X^*} \to 0. \tag{3.1}$$

Let us show that $\{u_n\} \subseteq X$ contains a convergent subsequence. By Hölder's inequality and Proposition 2.4, we have

$$\begin{aligned}
\langle \Psi'(u), u\rangle &= \Big(\int_\Omega F(x,u)\,dx \Big)^r \int_\Omega f(x,u)u\,dx, \\
&\le \alpha_2 \Big(\alpha_2 \int_\Omega \frac{1}{\beta(x)} |u|^{\beta(x)}\,dx \Big)^r \int_\Omega |u|^{\beta(x)-1}u\,dx, \\
&\le \frac{\alpha_2^{r+1}}{(\beta^-)^r} \Big(\int_\Omega |u|^{\beta(x)}\,dx \Big)^{r+1}, \\
&\le \frac{\alpha_2^{r+1}}{(\beta^-)^r} \Big(\max(|u|^{\beta^+}_{\beta(x)}, |u|^{\beta^-}_{\beta(x)}) \Big)^{r+1}, \\
&\le \frac{\alpha_2^{r+1}}{(\beta^-)^r} \max\Big(c_\beta^{(r+1)\beta^+} \|u\|^{(r+1)\beta^+}, c_\beta^{(r+1)\beta^-} \|u\|^{(r+1)\beta^-} \Big).
\end{aligned}$$

So, for n and $\|u_n\|$ large enough, from Proposition 2.3, one has

$$\begin{aligned}
\big\langle I'_\lambda(u_n), u_n \big\rangle &= \big\langle \Phi'_\lambda(u_n), u_n \big\rangle - \lambda \big\langle \Psi'_\lambda(u_n), u_n \big\rangle \\
&\ge \|u_n\|^{p-} - \lambda\,\frac{\alpha_2^{r+1}}{(\beta^-)^r} \max\Big(c_\beta^{(r+1)\beta^+} \|u_n\|^{(r+1)\beta^+}, c_\beta^{(r+1)\beta^-} \|u_n\|^{(r+1)\beta^-} \Big).
\end{aligned}$$

Moreover, using (3.1), we have

$$\|u_n\|^{p-} \le \lambda\,\frac{\alpha_2^{r+1}}{(\beta^-)^r} \max\Big(c_\beta^{(r+1)\beta^+} \|u_n\|^{(r+1)\beta^+}, c_\beta^{(r+1)\beta^-} \|u_n\|^{(r+1)\beta^-} \Big),$$

since $(r+1)\beta^+ < p^-$. Then $\{u_n\}$ is bounded, and passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$. Since $\Psi'(u)$ is compact, by Lemma 3.2, $u_n \to u$ (strongly) in X and so, I_λ fulfills the Palais–Smale condition. ☐

Our existence result is the following

Theorem 3.1. *Suppose that there exist $d, \delta > 0$ such that*

$$K\bigg(\Big[\frac{2\delta N}{R^2 - (\frac{R}{2})^2} \Big]^p + \Big(\frac{2\delta N}{R^2 - (\frac{R}{2})^2} \Big)^s \bigg) m\Big(R^N - \Big(\frac{R}{2}\Big)^N \Big) < d, \tag{3.2}$$

where $m := \frac{\pi^{N/2}}{N/2\Gamma(N/2)}$ *is the measure of a unit ball in* $\mathbb{R}^N$ *and* Γ *is the Gamma function. So, for any* $\lambda \in]A_\delta, B_d[$, *with*

$$A_\delta := \frac{K\big([\frac{2\delta N}{R^2-(\frac{R}{2})^2}]^p + (\frac{2\delta N}{R^2-(\frac{R}{2})^2})^s\big)m(R^N - (\frac{R}{2})^N)}{\frac{a_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\,([\delta]_\alpha m(\frac{R}{2})^N)^{r+1}}\,, \tag{3.3}$$

and

$$B_d := \frac{d}{\frac{([c_\beta]^\beta)^{r+1}(p^+)^{\frac{r+1}{p^-}}}{(r+1)(\beta^-)^{r+1}}\,([[d]^{\frac{1}{p}}]^\beta)^{r+1}}\,, \tag{3.4}$$

problem (1.1) *has at least one non-trivial weak solution.*

Proof. We try to prove our existence result using Theorem 2.1. For this purpose, We have to show that all conditions of Theorem 2.1 are fulfilled.

For a given $\lambda > 0$, we mention that by Lemma 3.4 the functional I_λ satisfies the $(PS)^{[d]}$ condition. Let $d > 0$ and $\delta > 0$ be as in (3.2) and let $w \in X$ be defined by

$$w(x) := \begin{cases} 0, & x \in \Omega \setminus B(x^0, R), \\ \delta, & x \in B\Big(x^0, \dfrac{R}{2}\Big), \\ \dfrac{\delta}{R^2 - (\frac{R}{2})^2}\Big(R^2 - \displaystyle\sum_{i=1}^N (x_i - x_i^0)^2\Big), & x \in B(x^0, R) \setminus B\Big(x^0, \dfrac{R}{2}\Big), \end{cases} \tag{3.5}$$

where $x = (x_1, \ldots, x_N) \in \Omega$. Then

$$\sum_{i=1}^N \frac{\partial^2 w}{\partial x_i^2}(x) = \begin{cases} 0, & x \in (\Omega \setminus B(x^0, R)) \cup B\Big(x^0, \dfrac{R}{2}\Big) \\ -\dfrac{2\delta N}{R^2 - (\frac{R}{2})^2}\,, & x \in B(x_0, R) \setminus B\Big(x^0, \dfrac{R}{2}\Big). \end{cases}$$

Applying Remark 3.1, one has

$$\begin{aligned} &\frac{1}{p^+}\left[\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\right]_p m\Big(R^N - \Big(\frac{R}{2}\Big)^N\Big) \\ &\qquad < \Phi(w) \le K\bigg(\bigg[\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\bigg]^p + \bigg(\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\bigg)^s\bigg)m\Big(R^N - \Big(\frac{R}{2}\Big)^N\Big), \end{aligned}$$

so, $\Phi(w) < d$. On the other hand,

$$\begin{aligned} \Psi(w) &\ge \frac{1}{r+1}\bigg(\int_\Omega F(x, w)\,dx\bigg)^{r+1} \ge \frac{a_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\bigg(\int_{B(x^0, \frac{R}{2})} |\delta|^{\alpha(x)}\,dx\bigg)^{r+1} \\ &\ge \frac{a_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\Big([\delta]_\alpha m\Big(\frac{R}{2}\Big)^N\Big)^{r+1}. \end{aligned}$$

Thus we deduce that

$$\frac{\Psi(w)}{\Phi(w)} > \frac{\frac{a_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\,([\delta]_\alpha m(\frac{R}{2})^N)^{r+1}}{K\big([\frac{2\delta N}{R^2-(\frac{R}{2})^2}]^p + (\frac{2\delta N}{R^2-(\frac{R}{2})^2})^s\big)m(R^N - (\frac{R}{2})^N)}\,.$$

Using Remark 2.3, for any $u \in \Phi^{-1}((-\infty, d])$, we have

$$\frac{1}{p^+}\,\big[\,\|u\|\,\big]_p \le \Phi(u) \le d.$$

Hence, from Proposition 2.4 and Remark 3.1, we deduce

$$\begin{aligned}\Psi(u) &\leq \frac{1}{r+1}\left(\int_\Omega F(x,u)\,dx\right)^{r+1}\\ &\leq \frac{1}{(r+1)(\beta^-)^{r+1}}\left([\,|u|_{\beta(x)}\,]^{\beta}\right)^{r+1}\\ &\leq \frac{1}{(r+1)(\beta^-)^{r+1}}\left([c_\beta\|u\|]^{\beta}\right)^{r+1}. \end{aligned} \tag{3.6}$$

Therefore,

$$\sup_{\Phi(u)\leq d}\Psi(u) \leq \frac{([c_\beta]^{\beta})^{r+1}(p^+)^{\frac{r+1}{p^-}}}{(r+1)(\beta^-)^{r+1}}\left([\,[d]^{\frac{1}{p}}\,]^{\beta}\right)^{r+1}.$$

As a result, the criteria of Theorem 2.1 are confirmed. So, for any

$$\lambda \in]A_\delta, B_d[\subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{d}{\sup\limits_{u\in\Phi^{-1}(]-\infty,d])}\Psi(u)}\right[,$$

I_λ admits at least one non-zero critical point, which is a weak solution of the problem. □

4 Multiplicity

Theorem 4.1. *For any $\lambda \in]A_\delta, B_d[$, where A_δ and B_d are those from Theorem 3.1 defined by (3.3) and (3.4), problem (1.1) admits at least three weak solutions.*

Proof. Note that Φ and Ψ fulfill the regularity assumptions of Theorem 2.2. Let us verify conditions (i) and (ii) of this Theorem. For this purpose, let

$$\frac{1}{p^+}\left[\frac{2\delta N}{R^2-(\frac{R}{2})^2}\right]_p m\Big(R^N - \Big(\frac{R}{2}\Big)^N\Big) = d$$

and let $w \in X$ be defined by (**??**). So, applying Remark 3.1, one has

$$\begin{aligned}\Phi(w) &= \int_\Omega \frac{1}{p(x)}\,|\Delta w|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}\,|\Delta w|^{q(x)}\,dx + \frac{1}{s}\int_\Omega \theta(x)\,\frac{|w(x)|^s}{|x|^{2s}}\,dx\\ &> \frac{1}{p^+}\left[\frac{2\delta N}{R^2-(\frac{R}{2})^2}\right]_p m\Big(R^N - \Big(\frac{R}{2}\Big)^N\Big) = d.\end{aligned}$$

Therefore, assumption (i) of Theorem 2.2 holds. Let us show that I_λ is coercive for any $\lambda > 0$.

From (**??**), one has

$$\Psi(u) \leq \frac{([c_\beta]^{\beta})^{r+1}}{(r+1)(\beta^-)^{r+1}}\left([\,\|u\|\,]^{\beta}\right)^{r+1}$$

besides, from Remark 3.1, $\frac{1}{p^+}[\,\|u\|\,]_p \leq \Phi(u)$. So,

$$I_\lambda(u) \geq \frac{1}{p^+}\left[\,\|u\|\,\right]_p - \frac{([c_\beta]^{\beta})^{r+1}}{(r+1)(\beta^-)^{r+1}}\left([\,\|u\|\,]^{\beta}\right)^{r+1},$$

and using $(r+1)\beta^+ < p^-$, we deduce that I_λ is coercive and, consequently, condition (ii) is fulfilled, which assures that all hypotheses of Theorem 4.1 are satisfied. Then, for any $\lambda \in]A_\delta, B_d[$, I_λ has at least three distinct critical points which represent the weak solutions of Problem (1.1). □

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