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Svetlin G. Georgiev, Khaled Zennir, Keltoum Bouhali, Aissa Boukarou

EXISTENCE OF CLASSICAL SOLUTIONS FOR FLEXIBLE SATELLITE SYSTEMS IN VISCOELASTICITY

Abstract. In this paper, we investigate a satellite system in viscoelasticity with a double flexible panel and external disturbances. The existence of at least one classical solution is proved under a minimal assumption on the relaxation function. Besides, we established the existence of at least two non-negative classical solutions with dynamic boundary conditions. To analyze the results obtained in different ways, various methods are used, including new iterative approaches with certain topological properties.

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1 Introduction

Thinking about mathematical models of the spacecraft as a field of modern technology began in the late of the last century. However, the large-scale implementation of these systems was hampered by the insufficient level of development of computer technology. Only in the mid-1950s this field of knowledge began rapidly growth, due to the rapid pace of development of the computer industry. One of the main tasks solved by means of a satellite system is the search and analysis of connections of geographically distributed objects with the help of components inside them, which help them to accomplish all the tasks allotted to them, being in totally different orbits and helping us in various ways on the Earth. Recently, there has been an increasingly noticeable trend towards the transition from quantitative data presentation formats to qualitative ones, when spatial relationships between objects are defined explicitly. Most database management system developers offer tools for working with spatial data which need theoretical studies for systems with the presence of deferred mathematical terms. It is well known fact that a satellite system stays in orbit because of the balance between gravitation pull and centrifugal force. This is from the quantitative point of view, where the angular velocity is determined by the force balance equation, which balances the gravitational and centrifugal forces. In this direction, let us consider a mathematical model of a flexible satellite system based on an internal nonlinear disturbance.

For $x \in [0, l]$, $l > 0$ represents the position and $t \geq 0$ is the time, let $y = y(x, t)$, $z = z(x, t)$ be the transverse displacements of the left and right panels of a flexible satellite system in viscoelasticity introduced by the following problem:

$$\begin{aligned} &\rho A y_{tt} + EI y_{xxxx} - EI \int_0^t \zeta(t-s) y_{xxxx}(s, x)\, ds = f_1(x, t), \quad x \in \Big[0, \frac{l}{2}\Big], \\ &\rho A z_{tt} + EI z_{xxxx} - EI \int_0^t \zeta(t-s) z_{xxxx}(s, x)\, ds = f_2(x, t), \quad x \in \Big[\frac{l}{2}, l\Big], \end{aligned} \tag{1.1}$$

subject to the dynamic boundary conditions

$$\begin{aligned} &y_x\Big(x = \frac{l}{2}, t\Big) = z_x\Big(x = \frac{l}{2}, t\Big) = 0, \\ &y_{xx}(x = 0, t) = z_{xx}(x = l, t) = 0, \\ &y_{xxx}(x = 0, t) = z_{xxx}(x = l, t) = 0, \\ &y\Big(x = \frac{l}{2}, t\Big) = z\Big(x = \frac{l}{2}, t\Big) = w\Big(x = \frac{l}{2}, t\Big), \\ &m w_{tt}\Big(x = \frac{l}{2}, t\Big) = \eta(t) + EI y_{xxx}\Big(x = \frac{l}{2}, t\Big) - EI \int_0^t \zeta(t-s) y_{xxx}\Big(x = \frac{l}{2}, s\Big)\, ds \\ &\qquad - EI z_{xxx}\Big(x = \frac{l}{2}, t\Big) + EI \int_0^t \zeta(t-s) z_{xxx}\Big(x = \frac{l}{2}, s\Big)\, ds + d\Big(x = \frac{l}{2}, t\Big), \quad t \geq 0, \end{aligned} \tag{1.2}$$

and the initial conditions

$$\begin{aligned} &y(x, t = 0) = y_0(x), \quad y_t(x, t = 0) = y_1(x), \quad x \in \Big[0, \frac{l}{2}\Big], \\ &z(x, t = 0) = z_0(x), \quad z_t(x, t = 0) = z_1(x), \quad x \in \Big[\frac{l}{2}, l\Big]. \end{aligned} \tag{1.3}$$

Here ρ, A, EI, m are the non-negative constants.

(H1) Let $\mathcal{B}$ be a fixed non-negative constant. The sources f_1, f_2 and the external disturbance d are of $\mathcal{C}([0, \infty) \times [0, l])$, where

$$|f_1|, |f_2|, |d| \leq \mathcal{B} \text{ on } [0, \infty) \times [0, l],$$

and the relaxation function $\zeta \in \mathcal{C}([0,\infty))$ is defined to satisfy

$$\int\limits_0^t |\zeta(t-s)|\, ds \leq \mathcal{B}, \quad t \in [0,\infty).$$

The functionals

$$y_0, y_1, z_0, z_1 \in \mathcal{C}([0,l])$$

and

$$|y_0|, |y_1|, |z_0|, |z_1| \leq \mathcal{B} \;\text{ on }\; [0,l].$$

Here, η is unknown and will be defined later.

Each equation of (1.1) is a viscoelastic Euler–Bernoulli beam, while the coupled system is connected to a central body (see [2,15]). As in [6], we apply a control force at the center body of the system given by a function $\eta(t)$. It is shown that using the Galerkin approximation method, the well-posedness is guaranteed, and then the stability for solution is obtained. The results in [11] extend those in [7] from a qualitative point of view, where the well-posedness of the solution is treated using the semi-group approach and then an exponential decay rate is obtained. In [8], a vibration control for a flexible satellite subject to input constraint and external disturbance $d(t)$ is considered and the vibration of the satellite is regulated by a control law design (for more detail, see [9,10]).

Viscoelastic materials have variable characteristics depending on the time and state of memory, making the physical phenomenon dissipative. The use of a viscoelastic term in new evolution mathematical models is becoming increasingly widespread due to the advantages it has over qualitative information for materials. Its effects are clear in the behavior of the solutions and in the rate of growth during evolution. For viscoelastic problems, we mention the results in [6], where a flexible viscoelastic satellite is studied. Under certain conditions on the relaxation function, the author established a stability results for the system.

Motivated by the above results, we investigate system (1.1)–(1.3) and obtain a new result regarding the existence of classical solutions.

This paper is structured as follows. In Section 2, we state some auxiliary results and useful tools. In Section 3, we demonstrate the existence of at least one classical solution for the problem (1.1)–(1.3). Besides, in Section 4, we prove the existence of at least two non-negative classical solutions. In Section 5, we give an illustrative example of the obtained results.

2 Preliminary results

Let $\mathcal{X}$ be a real Banach space. We recall the definitions of compact and completely continuous mappings in Banach spaces.

Definition 2.1. Let $K : M \subset \mathcal{X} \to \mathcal{X}$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of $\mathcal{X}$. The map K is said to be completely continuous if it is continuous and maps any bounded set to a relatively compact set.

Proposition 2.1 ([1])**.** *Let $\mathcal{C} \subset E$ be a closed, convex subset, $0 \in \mathcal{U} \subset \mathcal{C}$, where $\mathcal{U}$ is an open set. Let $f : \overline{\mathcal{K}} \to \mathcal{C}$ be a compact and continuous map. Then:*

(a) *either f has a fixed point in $\overline{\mathcal{K}}$,*

(b) *or there exist $x \in \partial\mathcal{K}$ and $\beta \in (0,1)$ such that $x = \beta f(x)$.*

We will use the next iterative method which is a consequence of Proposition 2.1.

Theorem 2.1. *Let $\mathcal{E}$ be a Banach space, $\mathcal{Y}$ be a closed convex subset of $\mathcal{E}$, and let $\mathcal{K}$ be any open subset of $\mathcal{Y}$ with $0 \in \mathcal{U}$. Consider two operators T and S,*

$$Tx = \varepsilon x, \quad x \in \overline{\mathcal{K}},$$

for $\varepsilon > 0$ and $S : \overline{\mathcal{K}} \to \mathcal{E}$, such that

(i) *the operator* $I - S : \overline{\mathcal{K}} \to \mathcal{Y}$, *is continuous, compact,*

(ii) *in addition,*

$$\{x \in \overline{\mathcal{K}} : \ x = \beta(I-S)x, \ x \in \partial\mathcal{K}\} = \varnothing, \ \ \forall\, \beta \in \left(0, \frac{1}{\varepsilon}\right).$$

Then there exists $x^* \in \overline{\mathcal{K}}$ *such that*

$$Tx^* + Sx^* = x^*.$$

Proof. We find that the operator

$$\frac{1}{\varepsilon}(I-S) : \overline{\mathcal{K}} \to \mathcal{Y}$$

is continuous and compact. Suppose that there exist $x_0 \in \partial\mathcal{K}$ and $\mu_0 \in (0,1)$ such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I-S)\,x_0,$$

that is,

$$x_0 = \beta_0\,(I-S)x_0,$$

where

$$\beta_0 = \mu_0 \frac{1}{\varepsilon} \in \left(0, \frac{1}{\varepsilon}\right).$$

This contradicts condition (ii). From the Leray–Schauder nonlinear alternative, it follows that there exists $x^* \in \overline{\mathcal{K}}$ such that

$$x^* = \frac{1}{\varepsilon}(I-S)\,x^*,$$

or

$$\varepsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

The proof of Theorem 2.1 is now completed. □

Definition 2.2. A map $K : \mathcal{X} \to \mathcal{Y}$ is said expansive if there exists a positive constant $a > 1$ such that

$$\|Ku - Kv\|_{\mathcal{Y}} \geq a\|u - v\|_{\mathcal{X}}, \ \ \forall\, u, v \in \mathcal{X},$$

where $\mathcal{X}$, $\mathcal{Y}$ are the real Banach spaces.

Now, we recall the definition for a cone in a Banach space.

Definition 2.3. A closed, convex set $\mathcal{P}$ in $\mathcal{X}$ is called a cone if

1. $\lambda x \in \mathcal{P}$, $\forall\, \lambda \geq 0$ and for any $x \in \mathcal{P}$,
2. $x, -x \in \mathcal{P} \implies x = 0$.

Let $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$.

Theorem 2.2 ([3,13])**.** *Let* $\mathcal{P}$ *be a cone of a Banach space* $\mathcal{E}$, $\digamma$ *be a subset of* $\mathcal{P}$ *and let* $\mathcal{K}_1$, $\mathcal{K}_2$ *and* $\mathcal{K}_3$ *be three open bounded subsets of* $\mathcal{P}$ *such that*

$$\overline{\mathcal{K}}_1 \subset \overline{\mathcal{K}}_2 \subset \mathcal{K}_3$$

and $0 \in \mathcal{K}_1$. *Let* $T : \digamma \to \mathcal{P}$ *be an expansive mapping and* $S : \overline{\mathcal{K}}_3 \to E$ *be a completely continuous mapping and*

$$S(\overline{\mathcal{K}}_3) \subset (I-T)(\digamma).$$

Assume that

$$(\mathcal{K}_2 \setminus \overline{\mathcal{K}}_1) \cap \digamma \neq \varnothing,$$
$$(\mathcal{K}_3 \setminus \overline{\mathcal{K}}_2) \cap \digamma \neq \varnothing,$$

and there exists $u_0 \in \mathcal{P}^*$ *such that*

(i) *$Sx \neq (I-T)(x-\beta u_0)$, $\forall\, \beta > 0$, $x \in \partial\mathcal{K}_1 \cap (\digamma + \beta u_0)$,*

(ii) *$\exists\, \epsilon \geq 0$ such that $Sx \neq (I-T)(\beta x)$ $\forall\, \beta \geq 1+\epsilon$, $x \in \partial\mathcal{K}_2$, $\beta x \in \digamma$,*

(iii) *$Sx \neq (I-T)(x-\beta u_0)$, $\forall\, \beta > 0$, $x \in \partial\mathcal{K}_3 \cap (\digamma + \beta u_0)$.*

Then $T+S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial\mathcal{K}_2 \cap \digamma \quad \text{and} \quad x_2 \in (\overline{\mathcal{K}}_3 \setminus \overline{\mathcal{K}}_2) \cap \digamma,$$

or

$$x_1 \in (\mathcal{K}_2 \setminus \mathcal{K}_1) \cap \digamma \quad \text{and} \quad x_2 \in (\overline{\mathcal{K}}_3 \setminus \overline{\mathcal{K}}_2) \cap \digamma.$$

3 Existence of at least one solution

Assume that the space $\mathcal{C}^2([0,\infty), \mathcal{C}^4([0,l]))$ is defined with

$$\begin{aligned}\|y\|_1 = \max\Big\{ &\sup_{t\geq 0,\ x\in[0,l]} |y|, \sup_{t\geq 0,\ x\in[0,l]} |y_t|, \sup_{t\geq 0,\ x\in[0,l]} |y_{tt}|, \\ &\sup_{t\geq 0,\ x\in[0,l]} |y_x|, \sup_{t\geq 0,\ x\in[0,l]} |y_{xx}|, \sup_{t\geq 0,\ x\in[0,l]} |y_{xxx}|, \sup_{t\geq 0,\ x\in[0,l]} |y_{xxxx}| \Big\},\end{aligned}$$

provided it exists. Define

$$\mathcal{X} = \left(\mathcal{C}^2([0,\infty), \mathcal{C}^4([0,l]))\right)^{15}$$

with

$$\|y\| = \max_{j\in\{1,\ldots,15\}} \|y_j\|_1, \quad y = (y_1, \ldots, y_{15}),$$

provided it exists. For

$$y \in \mathcal{X}, \quad y = (y_1, \ldots, y_{15}),$$

introduce the operators

$$\begin{aligned}
S_{11}(y) &= \rho A y_{1tt} + EI y_{1xxxx} - EI \int_0^t \zeta(t-s) y_{1xxxx}(s,x)\, ds - f_1, \\
S_{12}(y) &= \rho A y_{2tt} + EI y_{2xxxx} - EI \int_0^t \zeta(t-s) y_{2xxxx}(s,x)\, ds - f_2, \\
S_{13}(y) &= y_1\Big(x = \frac{l}{2}, t\Big), \\
S_{14}(y) &= y_{2x}\Big(x = \frac{l}{2}, t\Big), \\
S_{15}(y) &= y_{1xx}(x = 0, t), \\
S_{16}(y) &= y_{2xx}(x = l, t), \\
S_{17}(y) &= y_{1xxx}(x = 0, t), \\
S_{18}(y) &= y_{2xxx}(x = l, t), \\
S_{19}(y) &= y_1\Big(x = \frac{l}{2}, t\Big) - y_2\Big(x = \frac{l}{2}, t\Big), \\
S_{110}(y) &= y_1\Big(x = \frac{l}{2}, t\Big) - y_3\Big(x = \frac{l}{2}, t\Big),
\end{aligned}$$

$$S_{111}(y)(t,x) = m y_{3tt}\Big(x = \frac{l}{2}, t\Big) - y_4(t) - EI y_{1xxx}\Big(x = \frac{l}{2}, t\Big) + EI \int_0^t \zeta(t-s) y_{1xxx}\Big(x = \frac{l}{2}, s\Big)\, ds$$

$$+ EIy_{2xxx}\Big(x=\frac{l}{2},t\Big) - EI\int\limits_0^t \zeta(t-s)y_{2xxx}\Big(x=\frac{l}{2},s\Big)\,ds - d\Big(x=\frac{l}{2},t\Big),$$

$$\begin{aligned}
S_{112}(y) &= y_1(x,t=0) - y_0(x),\\
S_{113}(y) &= y_{1t} - y_1(x),\\
S_{114}(y) &= y_2(x,t=0) - z_0(x),\\
S_{115}(y) &= y_{2t}(x,t=0) - z_1(x),\\
S_1(y) &= \big(S_{111}(y),\dots,S_{115}(y)\big),\quad t\ge 0,\quad x\in[0,l].
\end{aligned}$$

Note that if $y\in\mathcal{X}$ is such that

$$S_1(y)=0,\quad t\ge 0,\quad x\in[0,l],$$

then $y=y_1$, $z=y_2$, $w=y_3$ and $\eta=y_4$ solve problem (1.1)–(1.3). Set

$$\mathcal{B}_1 = \max\Big\{2\mathcal{B},\ \ (\rho A + EI + 1 + \mathcal{B})\mathcal{B},\ \ \mathcal{B}(m+2+2EI+2EI\mathcal{B})\Big\}.$$

Lemma 3.1. *Let* (H1) *hold. For $y\in\mathcal{X}$ and $\|y\|\le\mathcal{B}$, we have*

$$|S_{1j}(y)|\le\mathcal{B}_1,\quad t\ge 0,\quad x\in[0,l],\quad j\in\{1,\dots,15\}.$$

Proof. We have

$$\begin{aligned}
|S_{11}(y)| &= \Big|\rho Ay_{1tt} + EIy_{1xxxx} - EI\int\limits_0^t \zeta(t-s)y_{1xxxx}(s,x)\,ds - f_1\Big|\\
&\le \rho A|y_{1tt}| + EI|y_{1xxxx}| + EI\int\limits_0^t |\zeta(t-s)|\,|y_{1xxxx}(s,x)|\,ds + |f_1|\\
&\le \rho A\mathcal{B} + EI\mathcal{B} + \mathcal{B}^2 + \mathcal{B} \le \mathcal{B}_1,\quad t\ge 0,\quad x\in[0,l],
\end{aligned}$$

and

$$\begin{aligned}
|S_{12}(y)| &= \Big|\rho Ay_{2tt} + EIy_{2xxxx} - EI\int\limits_0^t \zeta(t-s)y_{2xxxx}(s,x)\,ds - f_1\Big|\\
&\le \rho A|y_{2tt}| + EI|y_{2xxxx}| + EI\int\limits_0^t |\zeta(t-s)|\,|y_{2xxxx}(s,x)|\,ds + |f_1|\\
&\le \rho A\mathcal{B} + EI\mathcal{B} + \mathcal{B}^2 + \mathcal{B} \le \mathcal{B}_1,\quad t\ge 0,\quad x\in[0,l],
\end{aligned}$$

and

$$\begin{aligned}
|S_{13}(y)| &= \Big|y_1\Big(x=\frac{l}{2},t\Big)\Big| \le \mathcal{B}_1,\\
|S_{14}(y)| &= \Big|y_{2x}\Big(x=\frac{l}{2},t\Big)\Big| \le \mathcal{B}_1,\\
|S_{15}(y)| &= |y_{1xx}(x=0,t)| \le \mathcal{B}_1,\\
|S_{16}(y)| &= |y_{2xx}(x=l,t)| \le \mathcal{B}_1,\\
|S_{17}(y)| &= |y_{1xxx}(x=0,t)| \le \mathcal{B}_1,\\
|S_{18}(y)| &= |y_{2xxx}(x=l,t)| \le \mathcal{B}_1,\quad t\ge 0,\quad x\in[0,l],
\end{aligned}$$

and

$$
\begin{aligned}
|S_{19}(y)| &= \left|y_1\Big(x=\frac{l}{2},t\Big) - y_2\Big(x=\frac{l}{2},t\Big)\right| \\
&\le \left|y_1\Big(x=\frac{l}{2},t\Big)\right| + \left|y_2\Big(x=\frac{l}{2},t\Big)\right| \le 2\mathcal{B} \le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l],
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{110}(y)| &= \left|y_1\Big(x=\frac{l}{2},t\Big) - y_3\Big(x=\frac{l}{2},t\Big)\right| \\
&\le \left|y_1\Big(x=\frac{l}{2},t\Big)\right| + \left|y_3\Big(x=\frac{l}{2},t\Big)\right| \le 2\mathcal{B} \le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l],
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{111}(y)| &= \Bigg| m y_{3tt}\Big(x=\frac{l}{2},t\Big) - \eta(t) - EI y_{1xxx}\Big(x=\frac{l}{2},t\Big) \\
&\quad + EI\int_0^t \zeta(t-s) y_{1xxx}\Big(x=\frac{l}{2},s\Big)\,ds + EI y_{2xxx}\Big(x=\frac{l}{2},t\Big) \\
&\quad - EI\int_0^t \zeta(t-s) y_{2xxx}\Big(x=\frac{l}{2},s\Big)\,ds - d\Big(x=\frac{l}{2},t\Big)\Bigg| \\
&\le m\left|y_{3tt}\Big(x=\frac{l}{2},t\Big)\right| + |\eta(t)| + EI\left|y_{1xxx}\Big(x=\frac{l}{2},t\Big)\right| \\
&\quad + EI\int_0^t |\zeta(t-s)|\left|y_{1xxx}\Big(x=\frac{l}{2},s\Big)\right| ds + EI\left|y_{2xxx}\Big(x=\frac{l}{2},t\Big)\right| \\
&\quad + EI\int_0^t |\zeta(t-s)|\left|y_{2xxx}\Big(x=\frac{l}{2},s\Big)\right| ds + \left|d\Big(x=\frac{l}{2},t\Big)\right| \\
&\le \mathcal{B}(m+2+EI+EI\mathcal{B}) \\
&\le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l],
\end{aligned}
$$

and

$$
|S_{112}(y)| = |y_1(x,t=0) - y_0(x)| \le |y_1(x,t=0)| + |y_0(x)| \le 2\mathcal{B} \le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l],
$$

and

$$
|S_{113}(y)| = |y_{1t} - y_1(x)| \le |y_{1t}| + |y_1(x)| \le 2\mathcal{B} \le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l],
$$

and

$$
|S_{114}(y)| = |y_2(x,t=0) - z_0(x)| \le |y_2(x,t=0)| + |z_0(x)| \le 2\mathcal{B} \le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l],
$$

and

$$
|S_{115}(y)| = |y_{2t}(x,t=0) - z_1(x)| \le |y_{2t}(x,t=0)| + |z_1(x)| \le 2\mathcal{B} \le \mathcal{B}_1, \quad t\ge 0, \quad x\in[0,l].
$$

This completes the proof of Lemma 3.1. □

(H2) Assume that $\exists\, g\in\mathcal{C}([0,\infty))$ such that $g>0$ on $(0,\infty)$, where

$$
g(0)=0,
$$

and $\exists\, A_1>0$ so that

$$
24(1+l+l^2+l^3+l^4+l^5)(1+t+t^2)\int_0^t g(s)\,ds \le A_1, \quad t\ge 0.
$$

For $y \in \mathcal{X}$, let us introduce

$$S_2(y) = \int_0^t \int_0^x (t-s)^2 (x-\tau)^4 g(s) S_1(y)(s,\tau)\, d\tau\, ds, \quad t \geq 0, \quad x \in [0,l]. \tag{3.1}$$

Lemma 3.2. *Let* (H1) *and* (H2) *hold. If* $y \in \mathcal{X}$ *and* $\|y\| \leq \mathcal{B}$*, then*

$$\|S_2 y\| \leq A_1 \mathcal{B}_1.$$

Proof. For any $j \in \{1, \dots, 15\}$, we get

$$\begin{aligned}
|S_{2j}(y)| &= \left| \int_0^t \int_0^x (t-s)^2 (x-\tau)^4 g(s) S_{1j}(y)(s,\tau)\, d\tau\, ds \right| \\
&\leq \int_0^t \int_0^x (t-s)^2 (x-\tau)^4 g(s) |S_{1j}(y)(s,\tau)|\, d\tau\, ds \leq \mathcal{B}_1 l^5 t^2 \int_0^t g(s)\, ds \\
&\leq 24\mathcal{B}_1 (1 + l + l^2 + l^3 + l^4 + l^5)(1 + t + t^2) \int_0^t g(s)\, ds \leq A_1 \mathcal{B}_1, \quad t \geq 0, \quad x \in [0,l],
\end{aligned}$$

and

$$\begin{aligned}
|S_{2jt}(y)| &= \left| 2 \int_0^t \int_0^x (t-s) (x-\tau)^4 g(s) S_{1j}(y)(s,\tau)\, d\tau\, ds \right| \\
&\leq 2 \int_0^t \int_0^x (t-s) (x-\tau)^4 g(s) |S_{1j}(y)(s,\tau)|\, d\tau\, ds \leq 2\mathcal{B}_1 l^5 t \int_0^t g(s)\, ds \\
&\leq 24\mathcal{B}_1 (1 + l + l^2 + l^3 + l^4 + l^5)(1 + t + t^2) \int_0^t g(s)\, ds \leq A_1 \mathcal{B}_1, \quad t \geq 0, \quad x \in [0,l],
\end{aligned}$$

and

$$\begin{aligned}
|S_{2jtt}(y)| &= \left| 2 \int_0^t \int_0^x (x-\tau)^4 g(s) S_{1j}(y)(s,\tau)\, d\tau\, ds \right| \\
&\leq 2 \int_0^t \int_0^x (x-\tau)^4 g(s) |S_{1j}(y)(s,\tau)|\, d\tau\, ds \leq 2\mathcal{B}_1 l^5 \int_0^t g(s)\, ds \\
&\leq 24\mathcal{B}_1 (1 + l + l^2 + l^3 + l^4 + l^5)(1 + t + t^2) \int_0^t g(s)\, ds \leq A_1 \mathcal{B}_1, \quad t \geq 0, \quad x \in [0,l],
\end{aligned}$$

and

$$\begin{aligned}
|S_{2jx}(y)| &= \left| 4 \int_0^t \int_0^x (t-s)^2 (x-\tau)^3 g(s) S_{1j}(y)(s,\tau)\, d\tau\, ds \right| \\
&\leq 4 \int_0^t \int_0^x (t-s)^2 (x-\tau)^3 g(s) |S_{1j}(y)(s,\tau)|\, d\tau\, ds \leq 4\mathcal{B}_1 l^4 t^2 \int_0^t g(s)\, ds
\end{aligned}$$

$$\leq 24\mathcal{B}_1(1+l+l^2+l^3+l^4+l^5)(1+t+t^2)\int_0^t g(s)\,ds \leq A_1\mathcal{B}_1, \quad t\geq 0, \quad x\in[0,l],$$

and

$$\begin{aligned}|S_{2jxx}(y)| &= \left|12\int_0^t\int_0^x (t-s)^2(x-\tau)^2 g(s)S_{1j}(y)(s,\tau)\,d\tau\,ds\right| \\ &\leq 12\int_0^t\int_0^x (t-s)^2(x-\tau)^2 g(s)|S_{1j}(y)(s,\tau)|\,d\tau\,ds \leq 12\mathcal{B}_1 l^3t^2\int_0^t g(s)\,ds \\ &\leq 24\mathcal{B}_1(1+l+l^2+l^3+l^4+l^5)(1+t+t^2)\int_0^t g(s)\,ds \leq A_1\mathcal{B}_1, \quad t\geq 0, \quad x\in[0,l],\end{aligned}$$

and

$$\begin{aligned}|S_{2jxxx}(y)| &= \left|24\int_0^t\int_0^x (t-s)^2(x-\tau) g(s)S_{1j}(y)(s,\tau)\,d\tau\,ds\right| \\ &\leq 24\int_0^t\int_0^x (t-s)^2(x-\tau) g(s)|S_{1j}(y)(s,\tau)|\,d\tau\,ds \leq 24\mathcal{B}_1 l^2t^2\int_0^t g(s)\,ds \\ &\leq 24\mathcal{B}_1(1+l+l^2+l^3+l^4+l^5)(1+t+t^2)\int_0^t g(s)\,ds \leq A_1\mathcal{B}_1, \quad t\geq 0, \quad x\in[0,l],\end{aligned}$$

and

$$\begin{aligned}|S_{2jxxxx}(y)| &= \left|24\int_0^t\int_0^x (t-s)^2 g(s)S_{1j}(y)(s,\tau)\,d\tau\,ds\right| \\ &\leq 24\int_0^t\int_0^x (t-s)^2 g(s)|S_{1j}(y)(s,\tau)|\,d\tau\,ds \leq 24\mathcal{B}_1 lt^2\int_0^t g(s)\,ds \\ &\leq 24\mathcal{B}_1(1+l+l^2+l^3+l^4+l^5)(1+t+t^2)\int_0^t g(s)\,ds \leq A_1\mathcal{B}_1, \quad t\geq 0, \quad x\in[0,l].\end{aligned}$$

Thus $\|S_2y\|\leq A\mathcal{B}_1$, which completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that* (H1) *and* (H2) *hold. If* $y\in\mathcal{X}$ *satisfy*

$$S_2(y)=D, \quad t\geq 0, \quad x\in[0,l], \tag{3.2}$$

for some constant D*, then* y *is a solution to* (1.1)–(1.3).

Proof. Differentiating (3.2) three times with respect to t and five times with respect to x we find

$$g(t)S_1(y)=0, \quad t\geq 0, \quad x\in[0,l],$$

whereupon

$$S_1(y)=0, \quad t>0, \quad x\in[0,l].$$

Since $S_1 y(\,\cdot\,,\,\cdot\,) \in \mathcal{C}([0,\infty)\times[0,l])$, we get

$$0 = \lim_{t\to 0} S_1(y) = S_1(y)(0,x), \quad x \in [0,l].$$

Thus

$$S_1(y) = 0, \quad t \geq 0, \quad x \in [0,l].$$

Hence we conclude that y is a solution to (1.1)–(1.3). The proof of Lemma 3.3 is now completed. $\square$

Our main result in this section is as follows.

Theorem 3.1. *Let* (H1) *and* (H2) *hold. Then problem* (1.1)–(1.3) *has at least one solution in* $\mathcal{X}$*.*

Proof. Denote by $\widetilde{\mathcal{Y}}$, the set of all equicontinuous families in $\mathcal{X}$ related to $\|\cdot\|$. Put $\mathcal{Y} = \overline{\overline{\widetilde{\mathcal{Y}}}}$ and

$$\mathcal{K} = \big\{ y \in \mathcal{Y} : \quad \|y\| < \mathcal{B} \big\}.$$

For $y \in \overline{\mathcal{K}}$ and $\epsilon > 0$, define the operators

$$\begin{aligned} &T(y) = \epsilon y, \\ &S(y) = y - \epsilon y - \epsilon S_2(y), \quad t \geq 0, \quad x \in [0,l]. \end{aligned}$$

For $y \in \overline{\mathcal{K}}$, we have

$$\|(I-S)(y)\| = \|\epsilon y + \epsilon S_2(y)\| \leq \epsilon\|y\| + \epsilon\|S_2(y)\| \leq \epsilon\mathcal{B}_1 + \epsilon A_1 \mathcal{B}_1.$$

Then $S : \overline{\mathcal{K}} \to \mathcal{X}$ is continuous and $(I-S)(\overline{\mathcal{K}})$ is located in a compact subset of $\mathcal{Y}$. Now, suppose that there is $y \in \partial\mathcal{K}$ so that

$$y = \beta(I-S)(y),$$

or

$$y = \beta\epsilon(y + S_2(y))$$

for $\beta \in (0, \frac{1}{\epsilon})$. Then, using $S_2(y)(x, t=0) = 0$, we get

$$y(x,t=0) = \beta\epsilon(y(x,t=0) + S_2(y)(x,t=0)) = \beta\epsilon y(x,t=0), \quad x \in [0,l],$$

whereupon $\beta\epsilon = 1$, which is a contradiction. Consequently,

$$\big\{ y \in \overline{\mathcal{K}} : \quad y = \beta_1(I-S)(y), \quad y \in \partial\mathcal{K} \big\} = \varnothing$$

for any $\beta_1 \in (0, \frac{1}{\epsilon})$. Then it follows from Theorem 2.1 that the operator $T+S$ has a fixed point $y^* \in \mathcal{Y}$. Then

$$y^* = T(y^*) + S(y^*) = \epsilon y^* + y^* - \epsilon y^* - \epsilon S_2(y^*), \quad t \geq 0, \quad x \in [0,l],$$

whence

$$S_2(y^*) = 0, \quad t \geq 0, \quad x \in [0,l].$$

Then y^* is a solution to (1.1)–(1.3) by Lemma 3.3. The proof of Theorem 3.1 is now completed. $\square$

4 The existence of non-negative solutions

Let $\mathcal{B}$ and A be the same constants that appear in conditions (H1) and (H2) such that

(H3) $A_1\mathcal{B}_1 < \frac{L}{5}$, and $L > 0$ satisfy

$$r < L < R_1 \leq \mathcal{B},$$

where $r, R_1 > 0$.

Theorem 4.1. *Assume that* (H1)–(H3) *hold. Then problem* (1.1)–(1.3) *has at least two nonnegative solutions in* $\mathcal{X}$.

Proof. Let
$$\widetilde{P} = \big\{y \in \mathcal{X} : \;\; y \geq 0 \;\text{ on }\; t \geq 0, \;\; x \in [0, L]\big\}.$$
Here, the set $\mathcal{P}$ denotes the set of all equicontinuous families in $\widetilde{P}$. For $y \in \mathcal{X}$, define the operators
$$\begin{aligned}
T_1(y) &= (1 + m_1\epsilon)y - \epsilon \,\frac{L}{10}\,,\\
S_3(y) &= -\epsilon S_2(y) - m_1\epsilon y - \epsilon \,\frac{L}{10}\,, \;\; t \geq 0, \;\; x \in [0, L],
\end{aligned}$$
where $\epsilon > 0$ and $m_1 > 0$ is large enough. The operator S_2 is given by (3.1). Any fixed point $y \in \mathcal{X}$ of $T_1 + S_3$ is a solution to (1.1)–(1.3). Let
$$\begin{aligned}
\digamma &= \mathcal{P},\\
\mathcal{K}_1 &= \mathcal{P}_r = \big\{y \in \mathcal{P} : \;\; \|y\| < r\big\},\\
\mathcal{K}_2 &= \mathcal{P}_L = \big\{y \in \mathcal{P} : \;\; \|y\| < L\big\},\\
\mathcal{K}_3 &= \mathcal{P}_{R_1} = \big\{y \in \mathcal{P} : \;\; \|y\| < R_1\big\}.
\end{aligned}$$

1. For $y_1, y_2 \in \digamma$, we have
$$\big\|T_1(y_1) - T_1(y_2)\big\| = (1 + m_1\epsilon)\|y_1 - y_2\|,$$
where $T_1 : \digamma \to \mathcal{X}$ is an expansive operator with
$$h = 1 + m_1\epsilon > 1.$$

2. For $y \in \overline{\mathcal{P}}_{R_1}$, we have
$$\|S_3(y)\| \leq \epsilon\|S_2(y)\| + m_1\epsilon\|y\| + \epsilon\,\frac{L}{10} \leq \epsilon\Big(A_1\mathcal{B}_1 + m_1R_1 + \frac{L}{10}\Big).$$
Then $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. As $S_3 : \overline{\mathcal{P}}_{R_1} \to \mathcal{X}$, is continuous, we find that $S_3(\overline{\mathcal{P}}_{R_1})$ is equicontinuous. Then $S_3 : \overline{\mathcal{P}}_{R_1} \to \mathcal{X}$ is completely continuous.

3. Let $y_1 \in \overline{\mathcal{P}}_{R_1}$. Set
$$y_2 = y_1 + \frac{1}{m_1}\,S_2(y_1) + \frac{L}{5m_1}\,.$$
Note that $S_2(y_1) + \frac{L}{5} \geq 0$ on $[0, \infty) \times [0, l]$. We have $y_2 \geq 0$ on $t \geq 0$, $x \in [0, l]$. Therefore, $y_2 \in \digamma$ and
$$-\epsilon m_1 y_2 = -\epsilon m_1 y_1 - \epsilon S_2(y_1) - \epsilon\,\frac{L}{10} - \epsilon\,\frac{L}{10}\,,$$
or
$$(I - T_1)(y_2) = -\epsilon m_1 y_2 + \epsilon\,\frac{L}{10} = S_3(y_1).$$
Then $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\digamma)$.

4. Assume that $\forall\, v_0 \in \mathcal{P}^*$, there exist $\beta \geq 0$ and
$$y \in \partial\mathcal{P}_r \cap (\digamma + \beta v_0),$$
or
$$y \in \partial\mathcal{P}_{R_1} \cap (\digamma + \beta v_0),$$
such that
$$S_3(y) = (I - T_1)(y - \beta v_0).$$

Then

$$-\epsilon S_2(y) - m_1\epsilon y - \epsilon\,\frac{L}{10} = -m_1\epsilon(y - \beta v_0) + \epsilon\,\frac{L}{10},$$

or

$$-S_2(y) = \beta m_1 v_0 + \frac{L}{5}\,.$$

Hence

$$\|S_2(y)\| = \left\|\beta m_1 v_0 + \frac{L}{5}\right\| \geq \frac{L}{5}\,,$$

which is a contradiction.

5. Set $\epsilon_1 = \frac{2}{5m_1}$. Suppose that there exist $y_1 \in \partial\mathcal{P}_L$ and $\beta_1 \geq 1 + \epsilon_1$ such that

$$S_3(y_1) = (I - T_1)(\beta_1 y_1).$$

Moreover,

$$-\epsilon S_2(y_1) - m_1\epsilon y_1 - \epsilon\,\frac{L}{10} = -\beta_1 m_1 \epsilon y_1 + \epsilon\,\frac{L}{10}\,,$$

or

$$S_2(y_1) + \frac{L}{5} = (\beta_1 - 1)m_1 y_1.$$

From here,

$$2\,\frac{L}{5} > \left\|S_2(y_1) + \frac{L}{5}\right\| = (\beta_1 - 1)m_1\|y_1\| = (\beta_1 - 1)m_1 L$$

and

$$\frac{2}{5m_1} + 1 > \beta_1,$$

which is a contradiction.

Then system (1.1)–(1.3) has at least two solutions y^1 and y^2 such that

$$\|y^1\| = L < \|y^2\| < R_1,$$

or

$$r < \|y^1\| < L < \|y^2\| < R_1.$$

The proof of Theorem 4.1 is now completed. □

5 Example

Let

$$\rho = m = A = EI = 1, \quad \mathcal{B} = 1,$$

$$y_0(x) = y_1(x) = z_0(x) = z_1(x) = 1, \;\; x \in [0,1], \quad \zeta(t) = \frac{1}{4(1+t)^2}\,, \;\; t \geq 0,$$

and

$$f_1 = \frac{1}{1 + t^2 + x^2}\,, \quad f_2 = \frac{2}{1 + 3t^2 + 4x^2}\,,$$

$$d = \frac{1}{1 + t^2 + x^2}, \;\; t \geq 0, \;\; x \in [0,l],$$

and

$$R_1 = \frac{9}{10}\,, \quad L = \frac{3}{5}\,, \quad r = \frac{2}{5}\,, \quad m_1 = 10^{50}\,, \quad A_1 = \frac{1}{10\mathcal{B}_1}\,.$$

Then

$$\mathcal{B}_1 = 7.$$

Next,

$$r < L < R_1 < B, \quad A_1\mathcal{B}_1 < \frac{L}{5}\,.$$

i.e., (H3) holds. Take

$$h(s) = \log\frac{1+s^{22}+s^{11}\sqrt{2}}{1+s^{22}-s^{11}\sqrt{2}}\,, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\,, \quad s\in\mathbb{R}, \quad s\neq\pm1.$$

Then

$$h'(s) = \frac{(1-s^{22})22\sqrt{2}s^{10}}{(1+s^{22}-s^{11}\sqrt{2})(1+s^{22}+s^{11}\sqrt{2})}\,, \quad l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}}, \quad s\in\mathbb{R}, \quad s\neq\pm1,$$

and

$$-\infty < 120\lim_{s\to\pm\infty}(1+s+s^2)h(s) < \infty,$$
$$-\infty < 120\lim_{s\to\pm\infty}(1+s+s^2)l(s) < \infty.$$

Thus there exists $D_1 > 0$ such that

$$120(1+s+s^2)\Big(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\Big) \leq D_1, \quad s\in\mathbb{R}.$$

Since

$$\lim_{s\to\pm1} l(s) = \frac{\pi}{2}\,,$$

and by [14, pp. 707, Integral 79], we obtain

$$\int\frac{d\tau}{1+\tau^4} = \frac{1}{4\sqrt{2}}\log\frac{1+\tau\sqrt{2}+\tau^2}{1-\tau\sqrt{2}+\tau^2} + \frac{1}{2\sqrt{2}}\arctan\frac{\tau\sqrt{2}}{1-\tau^2}\,.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s+s^2)^2(1+s^{44})}\,, \quad s\in\mathbb{R},$$

and

$$g_1(t) = Q(t), \quad t\geq 0.$$

Then there exists $D_1 > 0$ such that

$$120(1+t+t^2)\int_0^t g_1(\tau)\,dz\,d\tau \leq D_1, \quad t\geq 0, \quad x\in[0,l].$$

Let

$$g(t) = \frac{A}{D_1}g_1(t), \quad t\geq 0, \quad x\in[0,l].$$

Then

$$120(1+t+t^2)\int_0^t g(\tau)\,dz\,d\tau \leq A, \quad t\geq 0, \quad x\in[0,1],$$

i.e., (H3) holds. Next,

$$\int_0^t \zeta(t-s)\,ds = \frac{1}{4}\int_0^t \frac{1}{(1+t-s)^2}\,ds = \frac{1}{4}\left(1-\frac{1}{1+t}\right) \leq 1.$$

Therefore, the problem

$$y_{tt} + y_{xxxx} - \frac{1}{4}\int_0^t \frac{1}{(1+t-s)^2}\, y_{xxxx}(s,x)\, ds = \frac{1}{1+t^2+x^2}\,, \quad x \in \Big[0, \frac{1}{2}\Big], \quad t \geq 0,$$
$$z_{tt} + z_{xxxx} - \frac{1}{4}\int_0^t \frac{1}{(1+t-s)^2}\, z_{xxxx}(s,x)\, ds = \frac{2}{1+3t^2+4x^2}\,, \quad x \in \Big[\frac{1}{2}, 1\Big], \quad t \geq 0,$$

subject to the boundary conditions

$$y_x\Big(x=\frac{1}{2}, t\Big) = z_x\Big(x=\frac{1}{2}, t\Big) = 0,$$
$$y_{xx}(x=0,t) = z_{xx}(x=1,t) = 0,$$
$$y_{xxx}(x=0,t) = z_{xxx}(x=1,t) = 0,$$
$$y\Big(x=\frac{1}{2}, t\Big) = z\Big(x=\frac{1}{2}, t\Big) = w\Big(x=\frac{1}{2}, t\Big),$$
$$w_{tt}\Big(x=\frac{1}{2}, t\Big) = \eta(t) + y_{xxx}\Big(x=\frac{1}{2}, t\Big) - \frac{1}{4}\int_0^t \frac{1}{(1+t-s)^2}\, y_{xxx}\Big(x=\frac{1}{2}, s\Big)\, ds$$
$$- z_{xxx}\Big(x=\frac{1}{2}, t\Big) + \frac{1}{4}\int_0^t \frac{1}{(1+t-s)^2}\, z_{xxx}\Big(x=\frac{1}{2}, s\Big)\, ds + \frac{1}{1+t^2+x^2}, \quad t \geq 0,$$

and the initial conditions

$$y(x,t=0) = 1, \quad y_t(x,t=0) = 1, \quad x \in \Big[0, \frac{1}{2}\Big],$$
$$z(x,t=0) = 1, \quad z_t(x,t=0) = 1, \quad x \in \Big[\frac{1}{2}, 1\Big],$$

satisfies all requirements of Theorem 3.1 and Theorem 4.1.

Conclusion

An analytical approach to construct a mathematical model of viscoelastic flexible satellite system based on Hamilton's principle is considered. Solving the viscoelastic Euler–Bernoulli equation by a various methods, made it possible to obtain a mathematical model of a damping flexible satellite system. New scenarios for the existence of classical solutions for the viscoelastic flexible satellite system have been identified with a few assumptions on the relaxation function ξ. Taking into account the influence of center body control when constructing a mathematical model of the system significantly affects the existence of at least two solutions under the influence of the dynamic boundary conditions. It was revealed that the classical solutions always exist. Our results need a large class of sources functions f_1, f_2.

References

[1] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*. Cambridge Tracts in Mathematics, 141. Cambridge University Press, Cambridge, 2001.

[2] M. Baghani, H. Mohammadi and R. Naghdabadi, An analytical solution for shape-memory-polymer Euler–Bernoulli beams under bending. *International Journal of Mechanical Sciences* **84** (2014), 84–90.

[3] S. Djebali and K. Mebarki, Fixed point index theory for perturbation of expansive mappings by k-set contractions. *Topol. Methods Nonlinear Anal.* **54** (2019), no. 2, 613–640.

[4] S. Georgiev and Kh. Zennir, *Multiple Fixed-Point Theorems and Applications in the Theory of ODEs, FDEs and PDEs*. Chapman and Hall/CRC, New York, 2020.

[5] S. G. Georgiev and Kh. Zennir, Classical solutions for a class of IVP for nonlinear two-dimensional wave equations via new fixed point approach. *Partial Differential Equations in Applied Mathematics* **2** (2020), 100014.

[6] S. Hamdi and A. Berkani, A new stability result for a flexible satellite system with viscoelastic damping. *Math. Methods Appl. Sci.* **45** (2022), no. 16, 10070–10098.

[7] W. He and S. S. Ge, Dynamic modeling and vibration control of a flexible satellite. *IEEE Transactions on Aerospace and Electronic Systems* **51** (2015), no. 2, 1422–1431.

[8] N. Ji and J. Liu, Vibration control for a flexible satellite with input constraint based on Nussbaum function via backstepping method. *Aerospace Science and Technology* **77** (2018), 563–572.

[9] N. Ji and J. Liu, Adaptive vibration control for flexible satellite with output constraint and unknown control direction. *J. Franklin Inst.* **357** (2020), no. 15, 10600–10625.

[10] N. Ji and J. Liu, Distributed vibration control for flexible spacecraft with distributed disturbance and actuator fault. *Journal of Sound and Vibration* **475** (2020), no. 9 June 2020, 115274.

[11] B. Lekdim and A. Khemmoudj, Control of a flexible satellite with an internal nonlinear disturbance. *Evol. Equ. Control Theory* **13** (2024), no. 1, 128–139.

[12] K. Mebarki, S. Georgiev, S. Djebali and Kh. Zennir, *Fixed Point Theorems with Applications*. CRC Press, Boca Raton, FL, 2023.

[13] A. Mouhous, S. G. Georgiev and K. Mebarki, Existence of solutions for a class of first order boundary value problems. *Arch. Math. (Brno)* **58** (2022), no. 3, 141–158.

[14] A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*. CRC Press, Boca Raton, FL, 1998

[15] B. S. Yu, D. P. Jin and H. Wen, Analytical deployment control law for a flexible tethered satellite system. *Aerospace Science and Technology* **66** (2017), 294–303.

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Authors' addresses:

Svetlin G. Georgiev
Department of Mathematics, Sorbonne University, Paris, France
E-mail: `svetlingeorgiev1@gmail.com`

Khaled Zennir
Department of Mathematics, College of Science, Qassim University, Saudi Arabia
E-mail: `k.zennir@qu.edu.sa`

Keltoum Bouhali
Department of Mathematics, College of Science, Qassim University, Saudi Arabia
E-mail: `k.bouhali@qu.edu.sa`

Aissa Boukarou
University of Science and Technology Houari Boumediene, Bab Ezzouar, Algeria
E-mail: `boukarouaissa@gmail.com`