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$p(x)$-KIRCHHOFF TYPE PROBLEMS WITHOUT (AR)-CONDITION

Abstract. In this paper, we study the following $p(x)$-Kirchhoff problem

$$\begin{cases} -M\left(\int\limits_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)\Delta_{p(x)} u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and the nonlinear term $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. Using the mountain pass theorem with the Cerami condition, we give a result on the existence of at least one nontrivial solution without assuming the (**AR**)-condition. Next, Employing the fountain theorem, we show the existence of infinitely many solutions of the above problem.

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1 Introduction and main results

In recent years, there has been a lot of interest in differential equations and variational problems with nonstandard $p(x)$-growth conditions. It illuminates a wide range of applications in a variety of fields, including elastic mechanics, electro-rheological fluid dynamics and image processing [18, 19].

The purpose of this paper is to study the existence of nontrivial weak solutions for Kirchhoff type equations involving the $p(x)$-Laplacian with Dirichlet boundary condition

$$\begin{cases} -M\left(\int\limits_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)\Delta_{p(x)} u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p \in C_+(\overline{\Omega})$, $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplacian operator and the nonlinear term $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

Critical point theory has become one of the most important tools for determining solutions to elliptic equations of variational type since the original work of A. Ambrosetti and P. H. Rabinowitz [2]. In particular, the elliptic problem (1.1) has been intensively studied for many years. The key ingredient to obtain the existence of solutions for superlinear problems is the condition introduced by A. Ambrosetti and P. H. Rabinowitz ($(\mathbf{AR})$-condition for short).

Many authors have lately investigated problem (1.1), and a plenty of results have been obtained. Let us review some previous results that led us to this study. By means of critical point theorems, G. Dai and R. Hao [6] obtained the results on the existence and multiplicity of solutions for problem (1.1), where the nonlinear term g satisfies the $(\mathbf{AR})$-condition:

$(\mathbf{AR})$ there exist $T > 0$, $\theta > p^+$ such that for $|t| \geq T$ and a.e. $x \in \Omega$, $0 < \theta G(x,t) \leqslant tg(x,t)$, where $G(x,t) = \int\limits_0^t g(x,s)\, ds$.

In [3], M. Avci studied problem (1.1) in the particular case when $M \equiv 1$:

$$\begin{cases} -\Delta_{p(x)} u = \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and he established the existence and multiplicity of solutions to the above problem when the nonlinear term g does not satisfy the $(\mathbf{AR})$-condition.

In addition, under the $(\mathbf{AR})$-condition and some weaker assumptions, Afrouzi et al. in [1] proved that problem (1.1) has at least one nontrivial solution or infinitely many solutions. Their approach was based on the Mountain Pass Theorem and Fountain Theorem.

To state our results, we make the subsequent hypotheses on M and g:

(M_0) $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function.

(M_1) There exist $m_2 \geq m_1 > 0$ and $\beta \geq \alpha > 1$ such that for all $t \in \mathbb{R}_+$,

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\beta-1}.$$

(g_1) There exist $c_1 \geq 0$ and $\gamma \in C_+(\overline{\Omega})$ with $\gamma(x) < p^*(x)$ for each $x \in \overline{\Omega}$ such that

$$|g(x,t)| \leq c_1\big(1 + |t|^{\gamma(x)-1}\big) \;\text{ for all }\; (x,t) \in \Omega \times \mathbb{R},$$

where

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

(g_2) $g(x,t) = \circ(|t|^{\alpha p^+ - 1})$ as $t \to 0$ for $x \in \Omega$ uniformly, where α comes from (M_1) and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$.

(g_3) $\liminf_{|t|\to\infty} \frac{G(x,t)}{|t|^{\beta p^+}} = +\infty$ uniformly a.e $\forall\, x \in \Omega$, where $G(x,t) = \int_0^t g(x,s)\,ds$ and β comes from (M_1).

(g_4) There exists a positive constant $C_0 > 0$ such that $\mathcal{G}(x,t) \le \mathcal{G}(x,s) + C_0$ for any $x \in \Omega$, $0 < t < s$ or $s < t < 0$, where $\mathcal{G}(x,t) := tg(x,t) - \beta p^+ G(x,t)$.

(g_5) $g(x - t) = -g(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.

As is known, the main role in utilizing the famous Ambrosetti–Rabinowitz type conditions is to ensure the boundedness of the Palais–Smale type sequences of the corresponding functional, since this condition sometimes may be very restrictive and excludes many interesting nonlinearities. Indeed, there are several functions which are superlinear at infinity and at the origin but do not satisfy **(AR)**-condition. For example, the function

$$g(x,t) = |t|^{\beta p^+ - 2} t \ln(1 + |t|) + \frac{1}{\beta p^+} \frac{|t|^{\beta p^+ - 1} t}{1 + |t|}$$

does not satisfy the **(AR)**-condition, but it satisfies our conditions (g_1)–(g_5).

Remark 1.1. Notice that the condition (g_4) is a consequence of the following condition $(g_4)'$, which was firstly introduced by Miyagaki and Souto [17] and developed by Li and Yang [16] and C. Ji [14]:

$(g_4)'$ There exists $t_0 > 0$ such that for all $x \in \Omega$,

$$\frac{g(x,t)}{|t|^{\beta p^+ - 2} t} \ \text{ is increasing in } \ t \ge t_0 \ \text{ and decreasing in } \ t \le -t_0.$$

Now, we present the main results of this paper.

Theorem 1.1. *It is assumed that* (M_0), (M_1), (g_1), (g_2), (g_3) *and* (g_4) *are satisfied. If* $\gamma^- > \alpha p^+$, *then problem* (1.1) *has at least one nontrivial solution.*

Theorem 1.2. *Suppose that* (M_0), (M_1), (g_1), (g_3), (g_4) *and* (g_5) *are satisfied. If* $\gamma^- > \alpha p^+$, *then problem* (1.1) *possesses infinitely many solutions with unbounded energy.*

2 Preliminaries

To study problem (1.1), we need the following preliminary results. For more details, we refer to [7, 9–11, 15] and the references therein.

For

$$p \in C_+(\overline{\Omega}) := \Big\{ p \in C(\overline{\Omega}) : \ \ p^- := \inf_{x\in\overline{\Omega}} p(x) > 1 \Big\},$$

we designate the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \text{ is measurable and } \ \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \right\}$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 : \ \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)}\,dx \le 1 \right\}.$$

Proposition 2.1 ([7])**.**

(1) *The variable exponent Lebesgue space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is defined as the dual space* $L^{q(x)}(\Omega)$, *where* $q(x)$ *is conjugate to* $p(x)$, *i.e.,* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\Big|\int\limits_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

(2) *If* $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ *for all* $x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

On $L^{p(x)}(\Omega)$, we define the modular $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ as follows:

$$\rho(u) = \int\limits_\Omega |u|^{p(x)}\,dx.$$

The relation between ρ and $|\cdot|_{p(x)}$ is established by the following result.

Proposition 2.2 ([9])**.** *For* $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \dots,$ *we have*

(1) $|u|_{p(x)} < 1\ (= 1;\ > 1) \Longleftrightarrow \rho(u) < 1\ (= 1;\ > 1)$;

(2) *for* $u \ne 0$, $|u|_{p(x)} = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(3) $|u|_{p(x)} > 1 \Longrightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$;

(4) $|u|_{p(x)} < 1 \Longrightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$.

(5) *The following statements are equivalent to each other:*

  (a) $\lim\limits_{n\to\infty} |u_n - u|_{p(x)} = 0$;

  (b) $\lim\limits_{n\to\infty} \rho(u_n - u) = 0$;

  (c) $u_n \to u$ *in measure in* Ω *and* $\lim\limits_{n\to\infty} \rho(u_n) = \rho(u)$.

(6) $\lim\limits_{n\to\infty} |u_n|_{p(x)} = \infty \Longleftrightarrow \lim\limits_{n\to\infty} \rho(u_n) = \infty$.

The generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : \ \ |\nabla u| \in L^{p(x)}(\Omega)\Big\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 2.3 ([11])**.**

(1) *The spaces* $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

(2) *There is a constant* $C > 0$ *such that*

$$|u|_{p(x)} \le C\|u\| \ \textit{ for all } \ u \in W_0^{1,p(x)}(\Omega).$$

(3) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$, *then there is a continuous (compact) embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.4 ([12]). *The functional $I : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by*

$$I(u) = \int\limits_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx$$

is continuously Fréchet differentiable and $I'(u) = -\Delta_{p(x)} u$ for all $u \in W_0^{1,p(x)}(\Omega)$, and we have:

(1) *I is a convex functional.*

(2) *$I' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is a bounded homeomorphism and a strictly monotone operator.*

(3) *I' is a mapping of type (S_+).*

(4) *I is weakly lower semi-continuous.*

From now on, we denote by $Y = W_0^{1,p(x)}(\Omega)$, $Y^* = (W_0^{1,p(x)}(\Omega))^*$ the dual space and by $\langle \cdot\,, \cdot \rangle$, the dual pair. Notice that problem (1.1) has a variational structure, in fact, its solutions can be searched as critical points of the energy functional $J : Y \to \mathbb{R}$ given by

$$J(u) = \phi(u) - \psi(u),$$

where

$$\phi(u) = \widehat{M}\bigg(\int\limits_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\bigg) \;\text{ and }\; \psi(u) = \int\limits_\Omega G(x,u)\, dx.$$

Then we have the following

Proposition 2.5 ([8, Proposition 3.1]). *If the assumptions (M_1) and (g_1) hold, then the following statements are true:*

(1) *$\widehat{M} \in C^1([0,+\infty[) \cap C^0(]0,+\infty[)$, $\widehat{M}(0) = 0$, $\widehat{M}'(t) = M(t)$ for any $t > 0$ and $\widehat{M}$ is strictly increasing on $[0,+\infty[$.*

(2) *$J, \phi, \psi \in C^0(Y)$, $J(0) = \phi(0) = \psi(0) = 0$, $\phi \in C^1(Y \setminus \{0\})$, $\psi \in C^1(Y)$, $J \in C^1(Y \setminus \{0\})$. For every $u \in Y \setminus \{0\}$ and $v \in Y$,*

$$\langle J'(u), v \rangle = M\bigg(\int\limits_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\bigg) \int\limits_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx - \int\limits_\Omega g(x,u) v\, dx$$

holds. Thus $u \in Y \setminus \{0\}$ is a weak solution of (1.1) if and only if u is a nontrivial critical point of J.

(3) *The functionals $\phi, J : Y \to \mathbb{R}$ are sequentially weakly lower semi-continuous.*

(4) *The mapping $\psi' : Y \to Y^*$ is sequentially weakly-strongly continuous. For any open set $K \subset Y \setminus \{0\}$ with $\overline{K} \subset Y \setminus \{0\}$, the mappings $\phi', J' : \overline{K} \to Y^*$ are bounded, and are of type (S_+).*

Next, we give the definition of the Cerami condition, which was introduced by G. Cerami in [5].

Definition 2.1. Let $(X, \|\cdot\|)$ be a real Banach space, $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that J satisfies the Cerami c condition (we denote (C_c)-condition) if:

(C_1) any bounded sequence $(u_n) \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ has a convergent subsequence;

(C_2) there exist the constants $\alpha, r, \beta > 0$ such that

$$\|J'(u)\| \|u\| \geq \beta, \;\; \forall\, u \in J^{-1}\big([c-\alpha, c+\alpha]\big) \;\text{ with }\; \|u\| \geq r.$$

If J the (C_c)-condition is satisfied for every $c \in \mathbb{R}$, we say that J satisfies the (C)-condition.

Remark 2.1. It is clear from the above definition that if J satisfies the (PS)-condition, then it satisfies the (C)-condition. However, there are the functionals that satisfy the (C)-condition but do not satisfy the (PS)-condition (see [5]). Consequently, the (C)-condition is weaker than the (PS)-condition.

Now, we present the following theorems which will play a fundamental role in the proof of the main theorems. First of all, let us recall the Mountain Pass Theorem which we use in the proof of Theorem 1.1.

Theorem 2.1 ([4]). *Let X be a real Banach space and let $J : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$ that satisfies the (C)-condition, $J(0) = 0$, and the following conditions hold:*

(1) *There exist positive constants ρ and α such that $J(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.*

(2) *There exists a function $e \in X$ such that $\|e\| > \rho$ and $J(e) \leq 0$.*

Then the functional J has a critical value $c \geq \alpha$, that is, there exists $u \in X$ such that $J(u) = c$ and $J'(u) = 0$ in X^*.

To prove Theorem 1.2, we apply the Fountain theorem [20].

Let X be a real, separable and reflexive Banach space. It is known [21] that there exist $\{e_j\}_{j\in\mathbb{N}} \subset X$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{span\{e_j : \ j = 1, 2, \dots\}}, \quad X^* = \overline{span\{e_j^* : \ j = 1, 2, \dots\}},$$

and $\langle e_i^*, e_j\rangle = 1$ if $i = j$, $\langle e_i^*, e_j\rangle = 0$ if $i \neq j$.

We denote

$$X_j = span\{e_j\}, \ \ Y_k = \bigoplus_{j=1}^{k} X_j \ \text{ and } \ Z_k = \overline{\bigoplus_{j=k}^{+\infty} X_j}.$$

Theorem 2.2. *Assume that X is a Banach space and let $J : X \to \mathbb{R}$ be an even functional of class $C^1(X, \mathbb{R})$ satisfying the (C)-condition. For every $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that:*

(A_1) *$b_k := \inf\big\{J(u) : \ u \in Z_k, \ \|u\| = r_k\big\} \to +\infty$ as $k \to +\infty$;*

(A_2) *$a_k := \max\big\{J(u) : \ u \in Y_k, \ \|u\| = \rho_k\big\} \leq 0$.*

Then J has a sequence of critical values tending to $+\infty$.

3 Proofs of main results

First of all, we begin by showing that the (C_c)-condition holds.

Lemma 3.1. *Under the assumptions (M_0), (M_1), (g_1), (g_3) and (g_4), J satisfies the (C_c)-condition with $c \neq 0$.*

Proof. It is first proved that J satisfies the first assertion of the (C_c)-condition. Let $(u_n) \subset Y$ be bounded such that $J(u_n) \to c$, $c \in \mathbb{R}^*$ and $J'(u_n) \to 0$. Since $J(0) = 0$ and $J(u_n) \to c \neq 0$, there exists $\varepsilon > 0$ sufficiently small such that for n large enough, $\|u_n\| > \varepsilon$.

Denote $K = \{u \in Y : \ \|u\| > \varepsilon\}$, then $u_n \in K$ for n large enough. As (u_n) is bounded in Y, then up to a subsequence, still denoted by (u_n), we obtain $u_n \in K$ and $u_n \rightharpoonup u$. Using the fact that $J'(u_n) \to 0$, we have $J'(u_n)(u_n - u) \to 0$. Since $J' : \overline{K} \to Y^*$ is of type (S_+) in view of Proposition 2.5, we obtain $u_n \to u \in \overline{K}$.

Now, check that J satisfies the second assertion of the (C_c)-condition. Arguing by contradiction, let us suppose that there exist $c \in \mathbb{R}^*$ and $(u_n) \subset Y$ satisfying

$$J(u_n) \to c, \ \ \|u_n\| \to +\infty \ \text{ and } \ \|J'(u_n)\|\|u_n\| \to 0.$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Up to a subsequence, for $v \in Y$, we may assume that

$$\begin{aligned} v_n &\rightharpoonup v \ \text{ in } Y, \\ v_n &\to v \ \text{ in } L^{\gamma(x)}(\Omega), \\ v_n(x) &\to v(x) \ \text{ a.e. } \ x \in \Omega. \end{aligned}$$

Let $\omega_0 = \{x \in \Omega : \ v(x) \neq 0\}$. Then, for $x \in \omega_0$, we have

$$\lim_{n\to+\infty} v_n(x) = \lim_{n\to+\infty} \frac{u_n(x)}{\|u_n\|} = v(x) \neq 0.$$

This means that

$$|u_n(x)| = |v_n(x)| \|u_n\| \to +\infty \ \text{ a.e. in } \omega_0 \ \text{ as } \ n \to +\infty.$$

Hence, by (g_3), it follows that for each $x \in \omega_0$, we obtain

$$\lim_{n\to+\infty} \frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} \frac{|u_n(x)|^{\beta p^+}}{\|u_n\|^{\beta p^+}} = \lim_{n\to+\infty} \frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+} = +\infty. \tag{3.1}$$

Also, from (g_3), we can find $t_1 > 0$ such that

$$\frac{G(x,t)}{|t|^{\beta p^+}} > 1, \ \ \forall\, x \in \Omega, \ \ |t| > t_1. \tag{3.2}$$

Since $G(x,\cdot)$ is continuous on $[-t_1, t_1]$, there exists a positive constant c_4 such that

$$|G(x,t)| \le c_4, \ \ \forall\, (x,t) \in \Omega \times [-t_1, t_1]. \tag{3.3}$$

Then, by (3.2) and (3.3) , we deduce that there is a constant $c_5 \in \mathbb{R}$ such that

$$G(x,t) \ge c_5, \ \ \forall\, (x,t) \in \Omega \times \mathbb{R}.$$

From this we conclude that

$$\frac{G(x,u_n) - c_5}{\|u_n\|^{\beta p^+}} \ge 0, \ \ \forall\, x \in \Omega, \ \ \forall\, n \in \mathbb{N},$$

which implies that

$$\frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+} - \frac{c_5}{\|u_n\|^{\beta p^+}} \ge 0, \ \ \forall x \in \Omega, \ \ \forall\, n \in \mathbb{N}. \tag{3.4}$$

Choosing $\|u_n\| > 1$ for a sufficiently large n, in view of (M_1), we have

$$\begin{aligned} c = J(u_n) + \circ_n(1) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx \right) - \int_\Omega G(x,u_n)\, dx + \circ_n(1) \\ &\ge \frac{m_1}{\alpha (p^+)^\alpha} \|u_n\|^{\alpha p^-} - \int_\Omega G(x,u_n)\, dx + \circ_n(1), \end{aligned}$$

which implies that

$$\int_\Omega G(x,u_n)\, dx \ge \frac{m_1}{\alpha (p^+)^\alpha} \|u_n\|^{\alpha p^-} - c + \circ_n(1) \to +\infty \ \text{ as } \ n \to +\infty, \tag{3.5}$$

where and in what follows, $\circ_n(1)$ denotes a quantity which tends to zero as $n \to +\infty$.

Similarly, using (M_1), it follows that

$$
\begin{aligned}
c &= J(u_n) + \circ_n(1)\widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx \Big) - \int_\Omega G(x,u_n)\, dx + \circ_n(1) \\
&\leq \frac{m_2}{\beta (p^-)^\beta} \|u_n\|^{\beta p^+} - \int_\Omega G(x,u_n)\, dx + \circ_n(1).
\end{aligned}
$$

Then, from this and (3.5), we conclude that

$$
\|u_n\|^{\beta p^+} \geq \frac{\beta (p^-)^\beta}{m_2} c + \frac{\beta (p^-)^\beta}{m_2} \int_\Omega G(x,u_n)\, dx - \circ_n(1) > 0. \tag{3.6}
$$

Hence $|\omega_0| = 0$. Indeed, arguing by contradiction, if $|\omega_0| \neq 0$, then, by (3.1), (3.4), (3.6) and Fatou's Lemma, we have

$$
\begin{aligned}
+\infty &= \int_{\omega_0} \lim_{n\to\infty} \frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+}\, dx - \int_{\omega_0} \frac{c_5}{\|u_n\|^{\beta p^+}}\, dx \\
&= \int_{\omega_0} \lim_{n\to\infty} \Big(\frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+} - \frac{c_5}{\|u_n\|^{\beta p^+}} \Big)\, dx \\
&\leq \liminf_{n\to\infty} \int_{\omega_0} \Big(\frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+} - \frac{c_5}{\|u_n\|^{\beta p^+}} \Big)\, dx \\
&\leq \liminf_{n\to\infty} \int_{\Omega} \Big(\frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+} - \frac{c_5}{\|u_n\|^{\beta p^+}} \Big)\, dx \\
&= \liminf_{n\to\infty} \int_{\Omega} \frac{G(x,u_n(x))}{|u_n(x)|^{\beta p^+}} |v_n(x)|^{\beta p^+}\, dx - \limsup_{n\to\infty} \int_\Omega \frac{c_5}{\|u_n\|^{\beta p^+}}\, dx \\
&= \liminf_{n\to\infty} \int_{\Omega} \frac{G(x,u_n(x))}{\|u_n\|^{\beta p^+}}\, dx \\
&\leq \liminf_{n\to\infty} \int_{\Omega} \frac{G(x,u_n(x))}{\frac{m_2}{\beta (p^+)^\beta} \int_\Omega G(x,u_n(x))\, dx - \circ_n(1)}\, dx.
\end{aligned} \tag{3.7}
$$

From (3.5) and (3.7), we obtain

$$
+\infty \leq \frac{\beta (p^+)^\beta}{m_2},
$$

which is a contradiction. Therefore, $|\omega_0| = 0$ and $v(x) = 0$ a.e. $x \in \Omega$.

Motivated by [13], we can define a sequence $(t_n) \subset \mathbb{R}$ such that

$$
J(t_n u_n) = \max_{t\in[0,1]} J(tu_n). \tag{3.8}
$$

It is clear that $t_n > 0$ and $J(t_n u_n) \geq c > 0 = J(0) = J(0, u_n)$.

If $t_n < 1$, then using $\frac{d}{dt} J(tu_n)|_{t=t_n} = 0$, we obtain

$$
\big\langle J'(t_n u_n), t_n u_n \big\rangle = 0. \tag{3.9}
$$

If $t_n = 1$, then

$$
\langle J'(u_n), u_n \rangle = \circ_n(1). \tag{3.10}
$$

Therefore, by (3.9) and (3.10), we always have

$$
\big\langle J'(t_n u_n), t_n u_n \big\rangle = \circ_n(1).
$$

On the one hand, using the conditions (g_4), (M_0) and Proposition 2.5, for all $t \in [0,1]$, we have

$$\begin{aligned}
\beta p^+ J(tu_n) &\le \beta p^+ J(t_n u_n) = \beta p^+ J(t_n u_n) - \big\langle J'(t_n u_n), t_n u_n\big\rangle + \circ_n(1)\\
&= \beta p^+ \bigg(\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}\,|\nabla t_n u_n|^{p(x)}\,dx\Big) - \int_\Omega G(x,u_n)\,dx\bigg)\\
&\quad - M\Big(\int_\Omega \frac{1}{p(x)}\,|\nabla t_n u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla t_n u_n|^{p(x)}\,dx + \int_\Omega g(x,t_n u_n)t_n u_n\,dx + \circ_n(1)\\
&= \beta p^+ \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}\,|\nabla t_n u_n|^{p(x)}\,dx\Big) - M\Big(\int_\Omega \frac{1}{p(x)}\,|\nabla t_n u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla t_n u_n|^{p(x)}\,dx\\
&\quad + \int_\Omega \mathcal{G}(x,t_n u_n)\,dx + \circ_n(1)\\
&\le \beta p^+ \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\,dx\Big) - M\Big(\int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)}\,dx\\
&\quad + \int_\Omega (\mathcal{G}(x,u_n) + C_0)\,dx + \circ_n(1)\\
&\le \beta p^+ J(u_n) - \langle J'(u_n), u_n\rangle + C_0|\Omega| \to \beta p^+ c + C_0|\Omega| \ \text{ as } \ n\to+\infty. \qquad (3.11)
\end{aligned}$$

Let $(r_k)_{k\in\mathbb{N}}$ be a positive sequence of real numbers such that $r_k > 1$ for any k and $r_k \to +\infty$ as $k\to+\infty$. Then it is clear that

$$\|r_k v_n\| = r_k > 1, \ \ \forall\, k,n\in\mathbb{N}.$$

On the other hand, since $v_n \to 0$ in $L^{\gamma(x)}(\Omega)$ and $v_n(x)\to 0$ a.e. $x\in\Omega$ as $n\to+\infty$, using the condition (g_1) and the Lebesgue dominated convergence theorem, we deduce for a fixed $k\in\mathbb{N}$ that

$$\int_\Omega G(x, r_k v_n)\,dx \to 0 \ \text{ as } \ n\to+\infty. \tag{3.12}$$

Since $\|u_n\|\to+\infty$ as $n\to+\infty$, we have $\|u_n\| > r_k$, which implies $\frac{r_k}{\|u_n\|}\in]0,1[$ for n large enough.

Thus from (3.8) and (3.12), we deduce for a fixed $k\in\mathbb{N}$ that

$$J(t_n u_n) \ge J\Big(\frac{r_k}{\|u_n\|}\,u_n\Big) = J(r_k v_n) \ge \frac{m_1}{\alpha(p^+)^\alpha}\,r_k^{\alpha p^-} - \int_\Omega G(x, r_k v_n)\,dx \ge \frac{m_1}{2\alpha(p^+)^\alpha}\,r_k^{\alpha p^-} \tag{3.13}$$

for any n large enough.

From (3.13), letting $n,k\to+\infty$, we obtain

$$J(t_n u_n) \to +\infty \ \text{ as } \ n\to+\infty. \tag{3.14}$$

Combining (3.11) and (3.14) gives a contradiction. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.1. By Lemma 3.1, J satisfies the (C_c)-condition in Y with $c\neq 0$. To apply Theorem 2.1, with $X = Y$, we will show that J has a mountain pass geometry.

First, we affirm that there exists $\mu, v > 0$ such that

$$J(u) \geqslant v, \ \ \forall\, u\in Y \ \text{ with } \ \|u\| = \mu. \tag{3.15}$$

In fact, since $\alpha p^+ < \gamma^- \le \gamma(x) < p^*(x)$ for all $x\in\Omega$, we have from Proposition 2.3 that $Y \hookrightarrow L^{\alpha p^+}(\Omega)$ with a continuous and compact embeddings. So, there exists c_6 such that

$$|u|_{\alpha p^+} \le c_6\|u\|, \ \ \forall\, u\in Y.$$

Let $\varepsilon > 0$ such that $\varepsilon c_6^{\alpha p^+} < \frac{m_1}{2\alpha(p^+)^\alpha}$. Using (g_1) and (g_2), it follows that

$$G(x,t) \leq \varepsilon |t|^{\alpha p^+} + C(\varepsilon)|t|^{\gamma(x)}, \;\; \forall\,(x,t) \in \Omega \times \mathbb{R}.$$

Therefore, in view of (M_1) and (3.15), for $\|u\|$ sufficiently small, we get

$$\begin{aligned} J(u) &\geqslant \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^\alpha - \varepsilon \int_\Omega |u|^{\alpha p^+}\,dx - C(\varepsilon)\int_\Omega |u|^{\gamma(x)}\,dx \\ &\geqslant \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \varepsilon c_6^{\alpha p^+}\|u\|^{\alpha p^+} - C(\varepsilon)\int_\Omega |u|^{\gamma(x)}\,dx. \end{aligned}$$

Since $Y \hookrightarrow L^{\gamma(x)}(\Omega)$ (because $\gamma(x) < p^*(x)$), there exists $c_7 > 0$ such that

$$|u|_{\gamma(x)} \leq c_7\|u\|.$$

Thus

$$\begin{aligned} J(u) &\geqslant \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \varepsilon c_6^{\alpha p^+}\|u\|^{\alpha p^+} - C(\varepsilon)c_7^{\gamma^-}\|u\|^{\gamma^-} \\ &\geqslant \|u\|^{\alpha p^+}\Big(\frac{m_1}{\alpha(p^+)^\alpha} - \varepsilon c_6^{\alpha p^+} - C(\varepsilon)c_7^{\gamma^-}\|u\|^{\gamma^- - \alpha p^+}\Big) \\ &\geqslant \|u\|^{\alpha p^+}\Big(\frac{m_1}{2\alpha(p^+)^\alpha} - C(\varepsilon)c_7^{\gamma^-}\|u\|^{\gamma^- - \alpha p^+}\Big). \end{aligned}$$

Since $\gamma^- > \alpha p^+$, the function

$$t \longmapsto \Big(\frac{m_1}{2\alpha(p^+)^\alpha} - C(\varepsilon)c_7^{\gamma^-}t^{\gamma^- - \alpha p^+}\Big)$$

is strictly positive in a neighborhood of zero. Then there exists $\mu, v > 0$ such that

$$J(u) \geqslant v, \;\; \forall\, u \in Y \;\text{ with }\; \|u\| = \mu.$$

Next, we affirm that there exists $e \in Y$ with $\|u\| > \rho$ such that $J(e) < 0$. In fact, from (g_3) it follows that for all $T > 0$, there exists a constant $M_T > 0$, depending on T, such that

$$F(x,t) > Tt^{\beta p^+} \;\text{ a.e. }\; x \in \Omega, \;\; \forall |t| > M_T.$$

Since $G(x,\cdot)$ is continuous on $[-M_T, M_T]$, there exists a positive constant c_8 such that

$$|G(x,t)| \leq c_8, \;\; \forall\,(x,s) \in \Omega \times [-M_T, T_T].$$

Then

$$G(x,t) \geqslant Tt^{\beta p^+} - c_8, \;\text{ a.e. }\; x \in \Omega, \;\; \forall\, t \in \mathbb{R}.$$

Hence, for $w \in Y \setminus \{0\}$, $\|w\| = 1$ and $t > 1$ large enough, we obtain

$$\begin{aligned} J(tw) &\leq \frac{m_2}{\beta(p^-)^\beta}t^{\beta p^+}\left(\int_\Omega |\nabla w|^{p(x)}\,dx\right)^\beta - T\int_\Omega t^{\beta p^+}w^{\beta p^+}\,dx + c_8|\Omega| \\ &\leq \frac{m_2}{\beta(p^-)^\beta}t^{\beta p^+} - Tt^{\beta p^+}\int_\Omega w^{\beta p^+}\,dx + C \leq \frac{m_2}{\beta(p^-)^\beta}t^{\beta p^+} - Tt^{\beta p^+}|w|_{\beta p^+}^{\beta p^+} + C \\ &= \Big(\frac{m_2}{\beta(p^-)^\beta} - T|w|_{\beta p^+}^{\beta p^+}\Big)t^{\beta p^+} + C. \end{aligned}$$

As

$$\frac{m_2}{\beta(p^-)^\beta} - T|w|_{\beta p^+}^{\beta p^+} < 0$$

for $T > 0$ large enough, we deduce

$$J(tw) \to -\infty \;\text{ as }\; t \to +\infty.$$

Thus there exists $t_0 > 1$ and $e = t_0 w \in X \setminus \overline{B_\rho(0)}$ such that $J(e) < 0$. $\square$

Proof of Theorem 1.2. We check that J satisfies the assumptions of fountain Theorem 2.2. In view of Lemma 3.1, J satisfies the (C_c)-condition with $c \neq 0$. By condition (g_5), we see that J is an even functional. Then, to apply Theorem 2.2, it suffices to show that if k is large enough, then there exist $\rho_k > r_k > 0$ such that:

(A_1) $b_k := \inf\{J(u) : \ u \in Z_k, \ \|u\| = r_k\} \to +\infty$ as $k \to +\infty$.

(A_2) $a_k := \max\{J(u) : \ u \in Y_k, \ \|u\| = \rho_k\} \leq 0$.

We first give the following lemmas that will be used later.

Lemma 3.2. *If $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for all $x \in \Omega$, and we denote*

$$\alpha_k = \sup\big\{|u|_{\alpha(x)} : \ \ \|u\| = 1, \ \ u \in Z_k\big\},$$

then $\lim\limits_{k\to+\infty} \alpha_k = 0$.

Proof. Suppose by contradiction that there exist $\varepsilon > 0, k_1 > 0$ and $(u_k) \subset Z_k$ such that

$$\|u_k\| = 1 \ \text{ and } \ \|u|_{\alpha(x)} \geq \varepsilon$$

for every $k \geq k_1$. Since (u_k) is bounded in Y, there exists $u \in Y$ such that

$$u_k \underset{k\to\infty}{\rightharpoonup} u \quad \text{in } Y \ \text{ and } \ \langle e_i^*, u\rangle = \lim_{k\to\infty} \langle e_i^*, u_k\rangle = 0$$

for $i = 1, 2 \dots$.

Thus $u = 0$. However, we obtain

$$\varepsilon \leq \lim_{k\to\infty} |u_k|_{\alpha(x)} = |u|_{\alpha(x)} = 0,$$

which is a contradiction. □

Lemma 3.3. *For every $\gamma \in C_+(\overline{\Omega})$ and $u \in L^{\gamma(x)}(\Omega)$, there is $\zeta \in \Omega$ such that*

$$\int_\Omega |u|^{\gamma(x)}\, dx = |u|^{\gamma(\zeta)}_{\gamma(x)}.$$

(A_1) Let $u \in Z_k$ such that $\|u\| = r_k \geq 1$. It follows from the assumptions (M_1), (g_1) and Lemma 3.3 that

$$\begin{aligned}
J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}\, |\nabla u|^{p(x)}\, dx\Big) - \int_\Omega G(x,u)\, dx \\
&\geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}\, |\nabla u|^{p(x)}\, dx\right)^{\alpha} - c_1 \int_\Omega |u|^{\gamma(x)}\, dx - c_1 \int_\Omega |u|\, dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\, \|u\|^{\alpha p^-} - c_1 |u|^{\gamma(\zeta)}_{\gamma(x)} - c_5\|u\|, \ \text{ where } \ \zeta \in \Omega \\
&\geq \begin{cases} \dfrac{m_1}{\alpha(p^+)^\alpha}\, \|u\|^{\alpha p^-} - c_1 - c_5\|u\| & \text{if } \ |u|_{\gamma(x)} \leq 1 \\[2ex] \dfrac{m_1}{\alpha(p^+)^\alpha}\, \|u\|^{\alpha p^-} - c_1 \alpha_k^{\gamma^+} \|u\|^{\gamma^+} - c_5\|u\| & \text{if } \ |u|_{\gamma(x)} > 1 \end{cases} \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\, \|u\|^{\alpha p^-} - c_1 \alpha_k^{\gamma^+} \|u\|^{\gamma^+} - c_5\|u\| - c_1 \\
&= m_1\Big(\frac{1}{\alpha(p^+)^\alpha} - \frac{1}{\gamma^+}\Big) r_k^{\alpha p^-} - c_5 r_k - c_1.
\end{aligned}$$

Choose

$$r_k := \left(c_1\gamma^+\alpha_k^{\gamma^+}m_1^{-1}\right)^{\frac{1}{\alpha p^- - \gamma^+}}.$$

Since $\gamma^+ > \alpha(p^+)^\alpha$ and $\alpha_k \to 0$ as $k \to \infty$, we assert that $r_k \to +\infty$ as $kto\infty$. Consequently,

$$J(u) \to +\infty \text{ as } \|u\| \to +\infty \text{ with } u \in Z_k,$$

which implies (A_1).

(A_2) Since Y_k is finite-dimensional, all norms are equivalent. So, there exists a constant $R_k > 0$ such that for all $u \in Y_k$ with $\|u\| \geq 1$, we obtain

$$\phi(u) \leq \frac{m_2}{\beta(p^-)^\beta}\left(\int\limits_\Omega |\nabla u|^{p(x)}\,dx\right)^\beta \leq \frac{m_2}{\beta(p^-)^\beta}\,\|u\|^{\beta p^+} \leq R_k|u|_{\beta p^+}^{\beta p^+}. \tag{3.16}$$

Next, the assumption (g_3) implies that exists $C_k > 0$ such that for $|s| \geq C_k$, we have

$$G(x,s) \geq 2R_k|s|^{\beta p^+}.$$

Then, for all $(x,t) \in \Omega \times \mathbb{R}$, we get

$$G(x,t) \geq 2R_k|s|^{\beta p^+} - T_k, \tag{3.17}$$

where $T_k = \max\limits_{|s|\leq C_k} G(x,s)$.

Combining (3.16) and (3.17), for $u \in Y_k$ such that $\|u\| = \rho_k > r_k$, we conclude that

$$J(u) = \phi(u) - \int\limits_\Omega G(x,u)\,dx \leq -R_k|u|_{\beta p^+}^{\beta p^+} + T_k|\Omega| \leq -\frac{m_2}{\beta(p^-)^\beta}\|u\|^{p^+} + T_k|\Omega|.$$

Therefore, for ρ_k large enough ($\rho_k > r_k$), from the above we get

$$a_k := \max_{u\in Y_k\cap S_{\rho_k}} J(u) \leq 0.$$

The assertion (A_2) holds. This completes the proof of Theorem 1.2. □

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