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Nadhém Echi, Ahmed Hachani

STABILITY AND OBSERVER-BASED CONTROL
OF SWITCHED NONLINEAR TIME-DELAY SYSTEMS
WITH INPUT SATURATION

Abstract. This paper investigates a class of switched nonlinear time-delay systems subject to actuator saturation. A Lyapunov–Krasovskii functional approach is first employed to establish exponential stability conditions for the open-loop system, providing insight into its intrinsic dynamic behavior. To address practical actuator saturation constraints, a mode-dependent state-feedback controller is then designed. Since the full system state is generally unavailable for measurement, an observer-based control scheme is developed to reconstruct the state from the available output signals. By combining a mode-dependent average dwell-time strategy with an appropriate Lyapunov–Krasovskii functional, sufficient conditions are derived to ensure the exponential stability of the resulting closed-loop system. Finally, the effectiveness of the proposed approach is demonstrated through a numerical example.

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1 Introduction

Switched systems, a special class of hybrid systems consisting of multiple subsystems governed by a switching law, have received considerable attention because of their broad applications in control, communication, and signal processing [12]. They have also become increasingly important owing to the growing performance requirements of modern engineering systems. In particular, switched systems provide a natural basis for modeling complex dynamic processes operating in multiple modes, such as network control systems, power electronics, and fault-tolerant systems, where abrupt structural changes are unavoidable.

Stability is a key research concern for switched systems. Asymptotic stability under arbitrary switching has been extensively investigated using common and multiple quadratic Lyapunov functions; see, for example, [11, 13] and the references therein. Nevertheless, ensuring stability while achieving satisfactory control performance remains challenging, especially in the presence of practical constraints.

However, delay-switched systems, one common form of such systems, have also attracted considerable attention due to the detrimental effects of time delays, often unavoidable in real systems, on their stability. The analysis and control of delay-switched systems have become a subject of research in recent decades. Delays are common in practical engineering applications and can lead to system degradation, oscillations, or instability, for example, in aircraft control systems [17] and multi-agent systems [18]. Consequently, the analysis of stability in the presence of time delays becomes a fundamental step prior to any control design. Moreover, real-world systems are often subject to modeling uncertainties, external disturbances, and nonlinear perturbations, further complicating the stability and control problem.

In [9], the exponential stability and robust stabilization of switched nonlinear systems with variable delay was investigated using the mean residence time method combined with the Gronwall–Bellman inequality. A modified Lyapunov–Krasovskii functional was proposed to derive less conservative stability criteria. In [21], observer-based control with trigger control was studied for switched systems with external disturbances, nonlinearities, and time delays. A state observer for estimating the system state and a general trigger control strategy were proposed. In [2], the authors focused on linear and weakly nonlinear switched systems subject to delays, and developed multiple exponential stability criteria through the application of multiple Lyapunov functions and the dwell-time approach.

The stability of switched nonlinear time-delay systems comprising both stable and unstable subsystems was investigated in [15]. By combining an inequality technique with the average dwell-time approach, the authors derived several stability conditions that relax the conventional requirements that all subsystem matrices be pairwise commutative and Hurwitz stable. In [16], the asymptotic stability of switched linear systems was established by investigating the upper and lower bounds on the time delay and the average dwell time. The input-to-state stability of switched delay systems was investigated in [23] by introducing a Lyapunov–Krasovskii function with an indeterminate derivative. In [20], the authors investigated the stability and controller design of switched delay systems with input saturation. By employing a Lyapunov–Krasovskii functional and designing a state-feedback controller, they established sufficient conditions to guarantee exponential stability.

Although the above works have significantly advanced the theoretical development of switched time-delay systems, several important issues remain insufficiently addressed. In particular, the combined effects of nonlinear perturbations, actuator saturation, and time-varying delays with bounded rates of change have not yet been fully investigated within a unified framework. While the linear time-delay switched system studied in [20] provides a solid foundation for stability study, it has several serious constraints. The most significant is that it does not take into account the effects of nonlinear disturbances, ignores the saturation limitations in actuators, and considers only bounded delays, but without restrictions on the rate of their change. Such simplifications give rise to important questions: in the presence of nonlinear uncertainty in the system? How does actuator saturation affect the operation of a closed-loop system? What impact do these questions have? Answering these questions is essential for improving the reliability, fault tolerance, and applicability of switchable control systems in real-world engineering settings. These questions are the driving force behind the research presented in this article.

The present work is a natural extension of the study proposed in [20], applied to a class of switched nonlinear time-delay systems with input saturation, and developed on the basis of a Lyapunov–Krasovskii functional. First, we investigate the exponential stability of the considered switched nonlinear time-delay system without control input and derive a set of conditions under which it is exponentially stable. This analysis highlights the influence of key system parameters, such as delay bounds, Lipschitz constants, and mean dwell time, on the intrinsic stability properties of an open-loop system. These results provide valuable insight into the inherent characteristics of an open-loop system, which is fundamental input for further controller design.

At the same time, in practical systems, actuators are inevitably subject to saturation limits, which creates additional challenges. We then propose a mode-dependent state feedback controller that takes actuator saturation into account. Since it is often impossible or too expensive to directly measure all state variables, we also construct a state observer to reconstruct the full system state based on available output measurements. Finally, we derive sufficient stability conditions for the closed-loop system using an approach based on the mode-dependent mean sojourn time, combined with the appropriate Lyapunov–Krasovskii functional.

The main contributions of this paper can be summarized as follows:

- A novel stability analysis framework for switched nonlinear time-delay systems with actuator saturation based on a Lyapunov–Krasovskii functional and average dwell-time theory.
- Explicit consideration of nonlinear Lipschitz perturbations and time-varying delays, leading to less conservative conditions for exponential stability.
- Design of an observer-based mode-dependent state feedback controller that ensures closed-loop stability under saturation constraints.
- Systematic investigation of the impact of design parameters on system robustness and performance.

For the sake of completeness, we compare our results with those presented in related studies. Wang et al. [10] and Liu et al. [19] studied uncertain switched systems with interval time-varying delays using full-state feedback control, but they did not consider actuator saturation or nonlinear disturbances. More recently, [20] studied switched time-delay systems with input saturation by employing an improved Lyapunov–Krasovskii functional and the inverse convex approach. In contrast, the present paper addresses both actuator saturation and nonlinear Lipschitz perturbations through an observer-based control design.

The remainder of the paper is organized as follows. Section 2 presents the preliminary definitions, some auxiliary results, and the system description. Section 3 is devoted to the stability analysis, the design of the state-feedback controller and state observer, and the development of the observer-based control scheme. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed approach.

Notation 1.1. The notations used in this paper are standard. The set of real $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$, and $\mathbb{R}^n$ represents the n -dimensional Euclidean space. For any matrix M , M^T denotes its transpose. The symbol $*$ is used to denote the entries implied by symmetry in symmetric block matrices or large matrix expressions, and $\text{diag}\{\dots\}$ represents a diagonal matrix. The Euclidean norm of a vector x is written as $\|x\|$. For a symmetric matrix P , $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote its largest and smallest eigenvalues, respectively. Finally, I_n denotes the n -dimensional identity matrix.

2 System description and preliminary results

This paper studies a switched nonlinear time-delay system, exploring its stability and control under varying switching rules and delays. Consider the following class of switched nonlinear time-delay systems:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + C_{\sigma(t)}x(t - d(t)) + B_{\sigma(t)}\text{sat}(u(t)) + f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)), \\ y(t) = D_{\sigma(t)}x(t), \\ x(\theta) = \phi(\theta), \quad \theta \in [-h_2, 0], \end{cases} \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the system input, and $\phi(\theta)$ is a continuous initial function. $\sigma(t) : [0, \infty) \rightarrow \mathcal{M} = \{1, 2, \dots, N\}$ denotes the switched signal, $\mathcal{M}$ denotes the modal switching signals, $N > 1$ is the number of subsystems. The saturation function is defined as follows:

$$\text{sat}(u(t)) = \begin{bmatrix} \text{sat}(u_1(t)) \\ \text{sat}(u_2(t)) \\ \vdots \\ \text{sat}(u_m(t)) \end{bmatrix},$$

with $-1 \leq \text{sat}(u_i(t)) \leq 1$. $A_{\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$ and $D_{\sigma(t)}$ are constant matrices that depend on the switching mode, with appropriate dimensions. The delay function $d(t)$ is assumed to be continuously differentiable satisfying

$$0 \leq h_1 \leq d(t) \leq h_2, \quad -d_{\max} < \dot{d}(t) \leq d_{\max} < 1, \quad (2.2)$$

where h_1 , h_2 , and $d_{\max}$ are positive constants. To finalize the system description, the following assumptions are taken into account.

Assumption 2.1. For each mode $\sigma \in \mathcal{M}$, let $f_\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ denote the nonlinear perturbation associated with the σ th subsystem. It is assumed that $f_\sigma(t, 0, 0, u_{\sigma(t)}(t)) = 0$ for all admissible control inputs $u_{\sigma(t)}(t)$.

Moreover, the nonlinear function f_σ satisfies a uniform Lipschitz condition. More precisely, there exist the constants $c_{1\sigma}$, $c_{2\sigma} \geq 0$ such that

$$\|f_{\sigma(t)}(t, x, x_d, u_{\sigma(t)}(t)) - f_{\sigma(t)}(t, \bar{x}, \bar{x}_d, u_{\sigma(t)}(t))\| \leq c_{1\sigma}\|x - \bar{x}\| + c_{2\sigma}\|x_d - \bar{x}_d\| \quad (2.3)$$

for all $t \geq 0$ and $x, \bar{x}, x_d, \bar{x}_d \in \mathbb{R}^n$.

Assumption 2.2. The control input $u(t)$ is assumed to be piecewise continuous and bounded on $[0, +\infty)$. Moreover, due to actuator saturation, the applied control satisfies the condition $\text{sat}(u(t)) \in [-1, 1]^m$.

Remark 2.1. This condition ensures that the function f_σ remains well-defined and satisfies certain boundedness constraints in each time interval corresponding to a specific switching mode. More precisely, it ensures that f_σ does not grow uncontrollably and that its behavior remains continuous across different modes. By enforcing a uniform Lipschitz condition, the variations of the function with respect to the state, the delayed state, and the control input are systematically constrained, preventing instability and ensuring the well-posedness of the system dynamics under switching conditions.

Remark 2.2. The function $f_i : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ denotes the nonlinear perturbation associated with the i th subsystem. It is assumed to satisfy the equilibrium condition $f_i(t, 0, 0, u_i(t)) = 0$ for all $t \geq 0$, and to fulfill the Lipschitz condition stated in Assumption 2.1.

Remark 2.3. In the literature on switched systems, it is commonly assumed that switching instants are synchronized with the active subsystems, ensuring that the control input matches the currently active subsystem and thereby facilitating both stability analysis and controller synthesis; however, perfect synchronization is often difficult to achieve in practice, and any deviation from this assumption may degrade closed-loop performance.

Definition 2.1 ([6, 7]). Let $\phi \in C([-h_2, 0], \mathbb{R}^n)$ be a continuous initial function. A function $x : [t_0 - h_2, T_{\max}) \rightarrow \mathbb{R}^n$, where $t_0 \geq 0$ and $T_{\max} \in (t_0, +\infty]$, is called a solution of system (2.1) if:

1. $x(t)$ is continuous on $[t_0 - h_2, T_{\max})$ and absolutely continuous on every interval between two consecutive switching instants;
2. $x(t_0 + \theta) = \phi(\theta)$ for all $\theta \in [-h_2, 0]$;
3. $x(t)$ satisfies system (2.1) almost everywhere on $[t_0, T_{\max})$.

Moreover, this solution is defined on the maximal interval of existence $[t_0, T_{\max})$, where $T_{\max}$ denotes the maximal existence time.

Definition 2.2 ([14]). A switching signal σ is said to have an average dwell time τ_a if the following condition holds:

$$N_\sigma(\mathcal{T}, t) \leq N_0 + \frac{\mathcal{T} - t}{\tau_a}, \quad \forall \mathcal{T} \geq t \geq 0,$$

where $\tau_a > 0$, $N_0 \geq 0$ is the chatter bound, and $N_\sigma(\mathcal{T}, t)$ denotes the number of switchings occurring in the interval $[t, \mathcal{T})$.

Prior to finalizing this section, we present the following auxiliary lemmas that will be essential for proving our main theorems.

Lemma 2.1 ((Schur Complement) [3]). *For a Hermitian matrix M , the inequality*

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} < 0$$

holds if and only if one of the following conditions is satisfied:

1. $M_{11} < 0$ and $M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0$,
2. $M_{22} < 0$ and $M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0$.

Lemma 2.2 ([22]). *There are 2^m diagonal matrices $D_j \in \mathbb{R}^{m \times m}$ with diagonal entries equal to either 0 or 1, where $j \in Q$, and let $D_j^- = I - D_j$. Let the scalars η_j satisfy $0 \leq \eta_j \leq 1$ and $\sum_{j=1}^{2^m} \eta_j = 1$. Then*

$$\sum_{j=1}^{2^m} \eta_j (D_j + D_j^-) = I.$$

The symbol Q denotes the index set associated with the convex representation of the saturation non-linearity, specifically, $Q = \{1, 2, \dots, 2^m\}$, and a bounded set is defined as $\mathcal{L}(H) = \{x(t) \in \mathbb{R}^n : |h_s^T x(t)| \leq 1, s \in \{1, 2, \dots, m\}\}$, where h_s^T is the s -th row of $H \in \mathbb{R}^{m \times n}$. For $x(t) \in \mathbb{R}^n$ and $K \in \mathbb{R}^{m \times n}$, if $x(t) \in \mathcal{L}(H)$, then $\text{sat}(Kx(t)) \in \text{cor}\{D_j Kx(t) + D_j^- Hx(t), j \in Q\}$, where $\text{cor}\{\cdot\}$ denotes the convex hull.

3 Main results

3.1 Stability analysis

We first investigate the stability of system (2.1) in the absence of a control input (i.e., for $u = 0$). In this context, we derive a set of sufficient conditions under which the system exhibits exponential stability. These conditions provide valuable insight into the intrinsic dynamics of the uncontrolled system and serve as a foundation for the subsequent control design.

Theorem 3.1. *For $i \in \mathcal{M}$, let $\alpha > 0$ and assume that Assumption 2.1 holds. If there exist the matrices $S_i > 0$, $H_{k,i} > 0$, and $K_{k,i} > 0$, $k = 1, 2$, such that the following linear matrix inequalities (LMIs) are satisfied:*

$$\begin{bmatrix} \Omega_{11} & S_i C_i & 0 & 0 & S_i \\ * & \Omega_{22} & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 \\ * & * & * & \Omega_{44} & 0 \\ * & * & * & * & \Omega_{55} \end{bmatrix} < 0, \quad (3.1)$$

where

$$\begin{aligned} \Omega_{11} &= S_i A_i + A_i^T S_i + I + 2\alpha S_i + H_{1i} + H_{2i}, & \Omega_{22} &= e^{-2\alpha h_1} ((1 + d_{\max}) K_{2i} - (1 - d_{\max}) K_{1i} + e^{2\alpha h_1} I), \\ \Omega_{33} &= e^{-2\alpha h_1} (K_{1i} - H_{1i}), & \Omega_{44} &= -e^{-2\alpha h_2} (H_{2i} + K_{2i}), & \Omega_{55} &= -(c_{1\sigma}^2 + c_{2\sigma}^2)^{-1} I, \end{aligned}$$

then system (2.1) for $u = 0$ is exponentially stable, provided that the switching signal satisfies the average dwell time condition

$$\tau_a > \tau_a^* = \frac{\ln \mu}{\alpha}, \quad (3.2)$$

where $\mu \geq 1$ satisfies the following inequalities: $S_i \leq \mu S_j$, $H_{k,i} \leq \mu H_{k,j}$, $K_{k,i} \leq \mu K_{k,j}$ for $k = 1, 2$, $\forall i, j \in \mathcal{M}$.

Proof. Consider the following Lyapunov–Krasovskii functional

$$V_{\sigma(t)}(t) = V_{1\sigma(t)}(t) + V_{2\sigma(t)}(t) + V_{3\sigma(t)}(t),$$

where

$$V_{1\sigma(t)}(t) = x^T(t) S_{\sigma(t)} x(t), \quad (3.3)$$

$$V_{2\sigma(t)}(t) = \int_{t-h_1}^t e^{2\alpha(s-t)} x^T(s) H_{1\sigma(t)} x(s) ds + \int_{t-h_2}^t e^{2\alpha(s-t)} x^T(s) H_{2\sigma(t)} x(s) ds, \quad (3.4)$$

$$V_{3\sigma(t)}(t) = \int_{t-d(t)}^{t-h_1} e^{2\alpha(s-t)} x^T(s) K_{1\sigma(t)} x(s) ds + \int_{t-h_2}^{t-d(t)} e^{2\alpha(s-t)} x^T(s) K_{2\sigma(t)} x(s) ds. \quad (3.5)$$

Between two consecutive switching instants t_m and t_{m+1} , the switching signal $\sigma(t)$ remains constant. Therefore, for all $t \in [t_m, t_{m+1})$, the active subsystem is denoted by $i = \sigma(t)$, and the corresponding Lyapunov–Krasovskii functional can be analyzed on this interval.

Taking the time derivative of (3.3) along the trajectories of system (2.1) yields

$$\begin{aligned} \dot{V}_{1i}(t) = & x^T(t) (S_i A_i + A_i^T S_i) x(t) + x^T(t) S_i C_i x(t-d(t)) \\ & + x^T(t-d(t)) C_i^T S_i x(t) + 2x^T(t) S_i f_i(t, x(t), x(t-d(t)), u_i(t)). \end{aligned} \quad (3.6)$$

Since $f(t, 0, 0, u_i(t)) = 0$, using condition (2.3) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & 2x^T S_i f_i(t, x(t), x(t-d(t)), u_i(t)) \\ & \leq 2\|x\| \|S_i\| \|f_i(t, x(t), x(t-d(t)), u_i(t))\| \leq 2\|x\| \|S_i\| (c_{1i}\|x\| + c_{2i}\|x(t-d(t))\|) \\ & \leq c_{1i}^2 x^T S_i S_i x + x^T(t) x(t) + c_{2i}^2 x^T S_i S_i x + x^T(t-d(t)) x(t-d(t)). \end{aligned} \quad (3.7)$$

Now, using (3.6) and (3.7), the derivative of V_{1i} can be rewritten as follows:

$$\begin{aligned} \dot{V}_{1i}(t) \leq & x^T(t) \{S_i A_i + A_i^T S_i + (c_{1i}^2 + c_{2i}^2) S_i S_i + I\} x(t) \\ & + x^T(t) S_i C_i x(t-d(t)) + x^T(t-d(t)) C_i^T S_i x(t) + x^T(t-d(t)) x(t-d(t)). \end{aligned} \quad (3.8)$$

Let us now derive the time derivative of (3.4) and (3.5) along the trajectories of system (2.1), respectively. We have

$$\begin{aligned} \dot{V}_{2i}(t) + 2\alpha V_{2i}(t) = & x^T(t) H_{1i} x(t) - e^{-2\alpha h_1} x^T(t-h_1) H_{1i} x(t-h_1) \\ & + x^T(t) H_{2i} x(t) - e^{-2\alpha h_2} x^T(t-h_2) H_{2i} x(t-h_2), \end{aligned} \quad (3.9)$$

$$\begin{aligned} \dot{V}_{3i}(t) + 2\alpha V_{3i}(t) = & e^{-2\alpha h_1} x^T(t-h_1) K_{1i} x(t-h_1) - e^{-2\alpha h_2} x^T(t-h_2) K_{2i} x(t-h_2) \\ & - (1-\dot{d}(t)) e^{-2\alpha d(t)} x^T(t-d(t)) K_{1i} x(t-d(t)) \\ & + (1-\dot{d}(t)) e^{-2\alpha d(t)} x^T(t-d(t)) K_{2i} x(t-d(t)). \end{aligned} \quad (3.10)$$

Hence, using (3.8), (3.9) and (3.10), we obtain

$$\dot{V}_i(t) + 2\alpha V_i(t) \leq x^T(t) \left\{ S_i A_i + A_i^T S_i + (c_{1i}^2 + c_{2i}^2) S_i S_i + I + 2\alpha S_i + H_{1i} + H_{2i} \right\} x(t)$$

$$\begin{aligned}
& + x^T(t)S_i C_i x(t-d(t)) + x^T(t-d(t))C_i^T S_i x(t) \\
& + e^{-2\alpha h_1} x^T(t-h_1)(K_{1i} - H_{1i})x(t-h_1) - e^{-2\alpha h_2} x^T(t-h_2)(H_{2i} + K_{2i})x(t-h_2) \\
& + e^{-2\alpha h_1} x^T(t-d(t))((1+d_{\max})K_{2i} - (1-d_{\max})K_{1i})x(t-d(t)) + x^T(t-d(t))x(t-d(t)).
\end{aligned}$$

This yields that

$$\dot{V}_i(t) + 2\alpha V_i(t) \leq \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h_1) \\ x(t-h_2) \end{bmatrix}^T \begin{bmatrix} \tilde{\Omega}_{11} & S_i C_i & 0 & 0 \\ * & \Omega_{22} & 0 & 0 \\ * & * & \Omega_{33} & 0 \\ * & * & * & \Omega_{44} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h_1) \\ x(t-h_2) \end{bmatrix},$$

where $\tilde{\Omega}_{11} = \Omega_{11} + (c_{1i}^2 + c_{2i}^2)S_i S_i$. Then, applying the Schur complement Lemma 2.1, we conclude that inequality (3.1) holds. Consequently, we obtain

$$\dot{V}_i(t) + 2\alpha V_i(t) < 0. \quad (3.11)$$

Without loss of generality, we choose $N_0 = 0$, and assume that at the switching instant t_k , $\sigma(t_k) = i$ and $\sigma(t_k^-) = j$, where $\sigma(t_k^-) = \lim_{\nu \rightarrow 0^-} \sigma(t_k + \nu)$. On the other hand, the Gronwall–Bellman inequality implies that (3.11) leads to

$$V_{\sigma(t_k)}(x(t_k)) \leq e^{-2\alpha(t-t_k)} V_{\sigma(t_k)}(x(t_k)). \quad (3.12)$$

Given $\mu \geq 1$, $S_i \leq \mu S_j$, $H_{k,i} \leq \mu H_{k,j}$, and $K_{k,i} \leq \mu K_{k,j}$ for $k = 1, 2$ and for all $i, j \in \mathcal{M}$, we obtain

$$V_{\sigma(t_k)}(t_k) \leq \mu V_{\sigma(t_k^-)}(t_k^-). \quad (3.13)$$

From (3.12) and (3.13), it follows that

$$V_{\sigma(t)}(t) \leq e^{-2\alpha(t-t_k)} \mu V_{\sigma(t_k^-)}(t_k^-). \quad (3.14)$$

Let $N_\sigma(0, t)$ denote the number of switchings over the interval $(0, t)$. Then $k = N_\sigma(t, 0) \leq \frac{t}{\tau_a}$. Iterating (3.14), we get the following result:

$$\begin{aligned}
V_{\sigma(t)}(t) & \leq e^{-2\alpha(t-t_k)} \mu V_{\sigma(t_k^-)}(t_k^-) \leq e^{-2\alpha(t-t_k)} \mu e^{-2\alpha(t_k-t_{k-1})} V_{\sigma(t_{k-1})}(t_{k-1}) \\
& = e^{-2\alpha(t-t_{k-1})} \mu V_{\sigma(t_{k-1})}(t_{k-1}) \leq \dots \leq e^{-2\alpha t} \mu^k V_{\sigma(0)}(0).
\end{aligned}$$

By Definition 2.2 and equation (3.2), in conjunction with the preceding inequality, we obtain

$$V_{\sigma(t)}(t) \leq e^{-(2\alpha - \frac{\ln \mu}{\tau_a})t} V_{\sigma(0)}(0). \quad (3.15)$$

On the other hand, since $S_{\sigma(t)}$ is symmetric positive definite, the Lyapunov functional satisfies the following inequality along the trajectories of system (2.1):

$$V_{\sigma(t)}(t) \geq x^T(t) S_{\sigma(t)} x(t) \geq \lambda_{\min}(S_{\sigma(t)}) \|x(t)\|^2. \quad (3.16)$$

Furthermore, we have

$$\begin{aligned}
V_{\sigma(0)}(0) & \leq \left(\lambda_{\max}(S_{\sigma(0)}) + h_1 \lambda_{\max}(H_{1,\sigma(0)}) + h_2 \lambda_{\max}(H_{2,\sigma(0)}) \right) \|\phi\|^2 \\
& \quad + \left((h_2 - h_1) (\lambda_{\max}(K_{1,\sigma(0)}) + \lambda_{\max}(K_{2,\sigma(0)})) \right) \|\phi\|^2 \\
& \leq (\lambda_1 + h_1 \lambda_2 + h_2 \lambda_3 + (h_2 - h_1)(\lambda_4 + \lambda_5)) \|\phi\|^2
\end{aligned} \quad (3.17)$$

where $\lambda_1 = \max_{i \in M} \{\lambda_{\max}(S_i)\}$, $\lambda_2 = \max_{i \in M} \{\lambda_{\max}(H_{1,i})\}$, $\lambda_3 = \max_{i \in M} \{\lambda_{\max}(H_{2,i})\}$, $\lambda_4 = \max_{i \in M} \{\lambda_{\max}(K_{1,i})\}$, $\lambda_5 = \max_{i \in M} \{\lambda_{\max}(K_{2,i})\}$. Moreover, From (3.15)–(3.17), we can obtain that $a \|x(t)\|^2 \leq b \|\phi\|^2 e^{-(2\alpha - \frac{\ln \mu}{\tau_a})t}$, where $a = \min_{i \in M} \{\lambda_{\min}(S_i)\}$, $b = \lambda_1 + h_1 \lambda_2 + h_2 \lambda_3 + (h_2 - h_1)(\lambda_4 + \lambda_5)$. This implies that

$$\|x(t)\| \leq \sqrt{\frac{b}{a}} \|\phi\| e^{-\frac{1}{2}(2\alpha - \frac{\ln \mu}{\tau_a})t}.$$

□

3.2 Stabilization analysis

In practical control applications, the actuators are subject to saturation limits that must be taken into account during the controller design process. Ignoring such nonlinearities may lead to performance degradation or even instability. To address this, we consider the design of a state feedback controller of the form

$$u(t) = K_\sigma x(t),$$

where $K_\sigma \in \mathbb{R}^{m \times n}$ is the feedback gain matrix associated with the active mode $\sigma \in \mathcal{M}$ of the switched system. However, due to actuator saturation, the actual input applied to the system is not $u(t)$, but rather its saturated version $\text{sat}(u(t))$.

To incorporate the effects of saturation in the stability analysis and controller synthesis, we invoke Lemma 2.2, which provides a convex representation of the saturation function. Specifically, for any $x(t) \in \mathcal{L}(H)$, the saturation function can be expressed as a convex combination of affine transformations:

$$\text{sat}(u(t)) = \text{sat}(K_\sigma x(t)) = \sum_{j=1}^{2^m} \eta_j (D_j K_\sigma + D_j^- H_\sigma) x(t). \quad (3.18)$$

This convex representation transforms the nonlinear effect of actuator saturation into a structured linear form that can be integrated into the controller synthesis process. It enables the use of Lyapunov-based techniques and LMIs conditions to design K_σ such that the closed-loop system remains stable and all trajectories remain bounded, despite the presence of saturation nonlinearity. This formulation highlights the affine dependence of the saturated input on the state, which enables the derivation of stability conditions using convex optimization tools such as LMIs. Then, substituting the convex representation of the saturated control input into the system dynamics, the following closed-loop system can be obtained:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)} x(t) + C_{\sigma(t)} x(t - d(t)) + B_{\sigma(t)} \sum_{j=1}^{2^m} \eta_j (D_j K_\sigma + D_j^- H_\sigma) x(t) \\ \quad + f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)), \\ x(\theta) = \phi(\theta), \quad \theta \in [-h_2, 0]. \end{cases} \quad (3.19)$$

Next, we establish sufficient conditions ensuring the exponential stability of the time-delay switched system (3.19) in the presence of control input. These conditions are presented below and build upon the previous analysis of the uncontrolled case.

Theorem 3.2. Let $\Delta_i = \sum_{j=1}^{2^m} \eta_j (D_j K_i + D_j^- H_i)$ and $\tilde{A}_i = A_i + B_i \Delta_i$. Let $\alpha > 0$ be the scalar defined in Theorem 3.1. Under Assumptions 2.1–2.2, suppose that there exist symmetric positive-definite matrices $P_i > 0$, $Q_{k,i} > 0$, and $D_{k,i} > 0$, $k = 1, 2$, satisfying the following LMIs:

$$\begin{bmatrix} \Phi_{11} & P_i C_i & 0 & 0 & P_i \\ * & \Phi_{22} & 0 & 0 & 0 \\ * & * & \Phi_{33} & 0 & 0 \\ * & * & * & \Phi_{44} & 0 \\ * & * & * & * & \Phi_{55} \end{bmatrix} < 0, \quad (3.20)$$

where

$$\begin{aligned} \Phi_{11} &= P_i \tilde{A}_i + \tilde{A}_i^T P_i + I + 2\alpha P_i + Q_{1i} + Q_{2i}, & \Phi_{22} &= e^{-2\alpha h_1} ((1 + d_{\max}) D_{2i} - (1 - d_{\max}) D_{1i} + e^{2\alpha h_1} I), \\ \Phi_{33} &= e^{-2\alpha h_1} (D_{1i} - Q_{1i}), & \Phi_{44} &= -e^{-2\alpha h_2} (Q_{2i} + D_{2i}), & \Phi_{55} &= -(c_{1\sigma}^2 + c_{2\sigma}^2)^{-1} I. \end{aligned}$$

For every switching signal $\sigma(t)$ with average dwell time satisfying (3.2), the switched system (2.1) is exponentially stabilizable via the feedback controller (3.18), where $\mu \geq 1$ is the compatibility constant between the subsystems satisfying

$$P_i \leq \mu P_j, \quad Q_{k,i} \leq \mu Q_{k,j}, \quad D_{k,i} \leq \mu D_{k,j}, \quad \text{for } k = 1, 2, \quad \forall i, j \in \mathcal{M}. \quad (3.21)$$

Proof. We propose the following Lyapunov–Krasovskii functional candidate:

$$V_{\sigma(t)}(t) = V_{1\sigma(t)}(t) + V_{2\sigma(t)}(t) + V_{3\sigma(t)}(t),$$

where

$$\begin{aligned} V_{1\sigma(t)}(t) &= x^T(t)P_{\sigma(t)}x(t), \\ V_{2\sigma(t)}(t) &= \int_{t-h_1}^t x^T(s)e^{2\alpha(s-t)}Q_{1\sigma(t)}x(s)ds + \int_{t-h_2}^t e^{2\alpha(s-t)}x^T(s)Q_{2\sigma(t)}x(s)ds, \\ V_{3\sigma(t)}(t) &= \int_{t-d(t)}^{t-h_1} e^{2\alpha(s-t)}x^T(s)D_{1\sigma(t)}x(s)ds + \int_{t-d(t)}^{t-h_2} e^{2\alpha(s-t)}x^T(s)D_{2\sigma(t)}x(s)ds. \end{aligned} \quad (3.22)$$

Taking the derivative of (3.22) along the solution of system (3.19), we have

$$\begin{aligned} \dot{V}_{1i}(t) &= x^T(t)(P_i A_i + A_i^T P_i + P_i B_i \Delta_i + \Delta_i^T B_i^T P_i)x(t) + x^T(t)P_i C_i x(t-d(t)) \\ &\quad + x^T(t-d(t))C_i^T P_i x(t) + 2x^T(t)P_i f_i(t, x(t), x(t-d(t)), u_i(t)). \end{aligned}$$

Using the results from (3.6), we have

$$\begin{aligned} \dot{V}_{1i}(t) &\leq x^T(t)\{P_i \tilde{A}_i + \tilde{A}_i^T P_i + (c_{1i}^2 + c_{2i}^2)P_i P_i + I\}x(t) + x^T(t)P_i C_i x(t-d(t)) \\ &\quad + x^T(t-d(t))C_i^T P_i x(t) + x^T(t-d(t))x(t-d(t)). \end{aligned} \quad (3.23)$$

Following the same reasoning as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} \dot{V}_{2i}(t) + 2\alpha V_{2i}(t) &= x^T(t)Q_{1i}x(t) - e^{-2\alpha h_1}x^T(t-h_1)Q_{1i}x(t-h_1) \\ &\quad + x^T(t)Q_{2i}x(t) - e^{-2\alpha h_2}x^T(t-h_2)Q_{2i}x(t-h_2), \end{aligned} \quad (3.24)$$

$$\begin{aligned} \dot{V}_{3i}(t) + 2\alpha V_{3i}(t) &= e^{-2\alpha h_1}x^T(t-h_1)D_{1i}x(t-h_1) - e^{-2\alpha h_2}x^T(t-h_2)D_{2i}x(t-h_2) \\ &\quad - (1-\dot{d}(t))e^{-2\alpha d(t)}x^T(t-d(t))D_{1i}x(t-d(t)) \\ &\quad + (1-\dot{d}(t))e^{-2\alpha d(t)}x^T(t-d(t))D_{2i}x(t-d(t)). \end{aligned} \quad (3.25)$$

Consequently, by systematically combining the key relationships derived in (3.23), (3.24), and (3.25), we arrive at the following result:

$$\begin{aligned} \dot{V}_i + 2\alpha V_i(t) &\leq x^T(t)\left\{P_i \tilde{A}_i + \tilde{A}_i^T P_i + (c_{1i}^2 + c_{2i}^2)P_i P_i + I + 2\alpha P_i + Q_{1i} + Q_{2i}\right\}x(t) \\ &\quad + x^T(t)P_i C_i x(t-d(t)) + x^T(t-d(t))C_i^T P_i x(t) \\ &\quad + e^{-2\alpha h_1}x^T(t-h_1)(D_{1i} - Q_{1i})x(t-h_1) - e^{-2\alpha h_2}x^T(t-h_2)(Q_{2i} + D_{2i})x(t-h_2) \\ &\quad + e^{-2\alpha h_1}x^T(t-d(t))((1+d_{\max})D_{2i} - (1-d_{\max})D_{1i})x(t-d(t)) + x^T(t-d(t))x(t-d(t)), \end{aligned}$$

which yields

$$\dot{V}_i(t) + 2\alpha V_i(t) \leq \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h_1) \\ x(t-h_2) \end{bmatrix}^T \begin{bmatrix} \tilde{\Phi}_{11} & P_i C_i & 0 & 0 \\ * & \Phi_{22} & 0 & 0 \\ * & * & \Phi_{33} & 0 \\ * & * & * & \Phi_{44} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h_1) \\ x(t-h_2) \end{bmatrix},$$

where $\tilde{\Phi}_{11} = \Phi_{11} + (c_{1i}^2 + c_{2i}^2)P_i P_i$. Finally, by applying the Schur complements to systematically decouple the system's matrix structure, we rigorously derive the main result encapsulated in equation (3.20). Invoking the methodology established in the proof of Theorem 3.1 and guided by the critical constraint formulated in condition (3.21), we systematically derive the stability criteria. Consequently, the closed-loop system (3.19) is rigorously demonstrated to be exponentially stable. $\square$

Remark 3.1. Lemma 2.2 plays a fundamental role in the analysis and control of systems with actuator saturation. Its importance can be summarized as follows:

1. Lemma 2.2 provides a convex representation of the nonlinear saturation function $\text{sat}(Kx(t))$ as an element of the convex hull of affine combinations involving diagonal matrices D_j and their complements D_j^- . This reformulation allows us to transform the nonlinear problem into a structure that can be addressed using linear matrix inequality (LMI) techniques.
2. It enables the decomposition of the saturated signal into a finite set of 2^m linear terms. Each matrix D_j models a specific combination of saturation activation (i.e., which components are saturated). The coefficients η_j , which satisfy the properties of a convex combination, ensure that $\text{sat}(Kx(t))$ lies within the convex hull of these linear expressions. This is particularly useful within the bounded set $\mathcal{L}(H)$.
3. Lemma 2.2 facilitates the design of stabilizing controllers in the presence of input saturation, especially for switched or time-delay systems. It allows us to apply Lyapunov-based approaches and LMI formulations to ensure closed-loop stability and to derive controllers that guarantee the boundedness and convergence of state trajectories despite actuator constraints.

3.3 Observer design

The main goal is to provide a state observer to recover the complete system state $x(t)$ from available output measurements $y(t)$. This is because it is usually infeasible or prohibitively expensive to directly measure all state variables. Therefore, a state observer is designed for system (2.1) to address this issue using the following construction:

$$\begin{cases} \dot{\hat{x}}(t) = A_{\sigma(t)}\hat{x}(t) + C_{\sigma(t)}\hat{x}(t-d(t)) + B_{\sigma(t)}\text{sat}(u(t)) \\ \quad + f_{\sigma(t)}(t, \hat{x}(t), \hat{x}(t-d(t)), u_{\sigma(t)}(t)) + L_{\sigma(t)}(y(t) - \hat{y}(t)), \\ \hat{y}(t) = D_{\sigma(t)}\hat{x}(t) \end{cases} \quad (3.26)$$

with $\hat{x}(t)$ and $\hat{y}(t)$ as the state and output estimates, respectively, and $L_{\sigma(t)}$ the as switching observer gain. Observer (3.26) aims to achieve an exponential estimate of $x(t)$ despite the inherent complexity of the system.

Theorem 3.3. For $i \in \mathcal{M}$, let $\alpha > 0$ and assume that Assumption 2.1 holds. If there exist the matrices $\hat{P}_i > 0$, $\hat{Q}_{k,i} > 0$, and $\hat{D}_{k,i} > 0$, $k = 1, 2$, such that the following linear matrix inequalities (LMIs) are satisfied:

$$\begin{bmatrix} \Psi_{11} & \hat{P}_i C_i & 0 & 0 & \hat{P}_i \\ * & \Psi_{22} & 0 & 0 & 0 \\ * & * & \Psi_{33} & 0 & 0 \\ * & * & * & \Psi_{44} & 0 \\ * & * & * & * & \Psi_{55} \end{bmatrix} < 0, \quad (3.27)$$

where

$$\begin{aligned} \Psi_{11} &= \hat{P}_i(A_i - L_i D_i) + (A_i - L_i D_i)^T \hat{P}_i + I + 2\alpha \hat{P}_i + \hat{Q}_{1i} + \hat{Q}_{2i}, \\ \Psi_{22} &= e^{-2\alpha h_1}((1 + d_{\max})\hat{D}_{2i} - (1 - d_{\max})\hat{D}_{1i} + e^{2\alpha h_1} I), \\ \Psi_{33} &= e^{-2\alpha h_1}(\hat{D}_{1i} - \hat{Q}_{1i}), \quad \Psi_{44} = -e^{-2\alpha h_2}(\hat{Q}_{2i} + \hat{D}_{2i}), \quad \Psi_{55} = -(c_{1\sigma}^2 + c_{2\sigma}^2)^{-1} I, \end{aligned}$$

then system (3.26) is an exponential observer for system (2.1), provided that the switching signal satisfies the average dwell-time constraint

$$\tau_a > \tau_a^* = \frac{\ln \mu}{\alpha}, \quad (3.28)$$

where $\mu \geq 1$ satisfies the following inequalities:

$$\hat{P}_i \leq \mu \hat{P}_j, \quad \hat{Q}_{k,i} \leq \mu \hat{Q}_{k,j}, \quad \hat{D}_{k,i} \leq \mu \hat{D}_{k,j} \quad \text{for } k = 1, 2, \quad \forall i, j \in \mathcal{M}. \quad (3.29)$$

Proof. We define the state error as $e(t) = \hat{x}(t) - x(t)$,

$$\begin{aligned} \dot{e}(t) = & (A_{\sigma(t)} + L_{\sigma(t)}D_{\sigma(t)})e(t) + C_{\sigma(t)}e(t - d(t)) \\ & + f_{\sigma(t)}(t, \hat{x}(t), \hat{x}(t - d(t)), u_{\sigma(t)}(t)) - f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)). \end{aligned} \quad (3.30)$$

Introduce the Lyapunov-Krasovskii functional candidate:

$$V_{\sigma(t)}(e(t)) = V_{1\sigma(t)}(e(t)) + V_{2\sigma(t)}(e(t)) + V_{3\sigma(t)}(e(t)),$$

where

$$\begin{aligned} V_{1\sigma(t)}(e(t)) &= e^T(t) \hat{P}_{\sigma(t)} e(t), \\ V_{2\sigma(t)}(e(t)) &= \int_{t-h_1}^t e^{2\alpha(s-t)} e^T(s) \hat{Q}_{1\sigma(t)} e(s) ds + \int_{t-h_2}^t e^{2\alpha(s-t)} e^T(s) \hat{Q}_{2\sigma(t)} e(s) ds, \end{aligned} \quad (3.31)$$

$$V_{3\sigma(t)}(e(t)) = \int_{t-d(t)}^{t-h_1} e^{2\alpha(s-t)} e^T(s) \hat{D}_{1\sigma(t)} e(s) ds + \int_{t-h_2}^{t-d(t)} e^{2\alpha(s-t)} e^T(s) \hat{D}_{2\sigma(t)} e(s) ds. \quad (3.32)$$

Then the derivative of V_{1i} along the trajectories of (3.30) is

$$\begin{aligned} \dot{V}_{1i}(e(t)) &= e^T(t) (\hat{P}_i(A_i + L_i D_i) + (A_i + L_i D_i)^T \hat{P}_i) e(t) \\ &\quad + e^T(t) \hat{P}_i C_i e(t - d(t)) + e^T(t - d(t)) C_i^T \hat{P}_i e(t) + 2e^T(t) \hat{P}_i \tilde{f}_i \end{aligned} \quad (3.33)$$

with $\tilde{f}_i = f_i(t, \hat{x}(t), \hat{x}(t - d(t)), u_i(t)) - f_i(t, x(t), x(t - d(t)), u(t))$.

Using (3.33) and (2.3), the derivative of $V_{1\sigma(t)}$ can be rewritten as follows:

$$\begin{aligned} \dot{V}_{1i}(e(t)) &\leq e^T(t) \left\{ \hat{P}_i(A_i + L_i D_i) + (A_i + L_i D_i)^T \hat{P}_i + (c_{1i}^2 + c_{2i}^2) \hat{P}_i \hat{P}_i + I \right\} e(t) \\ &\quad + e^T(t) \hat{P}_i C_i e(t - d(t)) + e^T(t - d(t)) C_i^T \hat{P}_i e(t) + e^T(t - d(t)) e(t - d(t)). \end{aligned} \quad (3.34)$$

We now derive the time derivative of (3.31) and (3.32), respectively. We get

$$\begin{aligned} \dot{V}_{2i}(e(t)) + 2\alpha V_{2i}(e(t)) &= e^T(t) \hat{Q}_{1i} e(t) - e^{-2\alpha h_1} e^T(t - h_1) \hat{Q}_{1i} e(t - h_1) \\ &\quad + e^T(t) H_{2i} e(t) - e^{-2\alpha h_2} e^T(t - h_2) H_{2i} e(t - h_2) \end{aligned} \quad (3.35)$$

$$\begin{aligned} \dot{V}_{3i}(e(t)) + 2\alpha V_{3i}(e(t)) &= e^{-2\alpha h_1} e^T(t - h_1) \hat{D}_{1i} e(t - h_1) - e^{-2\alpha h_2} e^T(t - h_2) \hat{D}_{2i} e(t - h_2) \\ &\quad - (1 - \dot{d}(t)) e^{-2\alpha d(t)} e^T(t - d(t)) \hat{D}_{1i} e(t - d(t)) \\ &\quad + (1 - \dot{d}(t)) e^{-2\alpha d(t)} e^T(t - d(t)) \hat{D}_{2i} e(t - d(t)). \end{aligned} \quad (3.36)$$

Hence, using (3.34), (3.35) and (3.36), we obtain

$$\begin{aligned} &\dot{V}_i(e(t)) + 2\alpha V_i(e(t)) \\ &\leq e^T(t) \left\{ \hat{P}_i(A_i + L_i D_i) + (A_i + L_i D_i)^T \hat{P}_i + (c_{1i}^2 + c_{2i}^2) \hat{P}_i \hat{P}_i + I + 2\alpha \hat{P}_i + \hat{Q}_{1i} + \hat{Q}_{2i} \right\} e(t) \\ &\quad + e^T(t) S_i C_i e(t - d(t)) + e^T(t - d(t)) C_i^T \hat{P}_i e(t) \\ &\quad + e^{-2\alpha h_1} e^T(t - h_1) (\hat{D}_{1i} - \hat{Q}_{1i}) e(t - h_1) - e^{-2\alpha h_2} e^T(t - h_2) (\hat{Q}_{2i} + \hat{D}_{2i}) e(t - h_2) \\ &\quad + e^{-2\alpha h_1} e^T(t - d(t)) ((1 + d_{\max}) \hat{D}_{2i} - (1 - d_{\max}) \hat{D}_{1i}) e(t - d(t)) + e^T(t - d(t)) e(t - d(t)), \end{aligned}$$

which implies that

$$\dot{V}_i(t) + 2\alpha V_i(t) \leq \begin{bmatrix} e(t) \\ e(t - d(t)) \\ e(t - h_1) \\ e(t - h_2) \end{bmatrix}^T \begin{bmatrix} \tilde{\Psi}_{11} & S_i C_i & 0 & 0 \\ * & \Psi_{22} & 0 & 0 \\ * & * & \Psi_{33} & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t - d(t)) \\ e(t - h_1) \\ e(t - h_2) \end{bmatrix},$$

where $\tilde{\Psi}_{11} = \Psi_{11} + (c_{1i}^2 + c_{2i}^2)\hat{P}_i\hat{P}_i$. Using the Schur complement technique to break down the matrix, we can prove the main result given in (3.27), based on the idea from Theorem 3.1 and, given the specific rule in condition (3.29), we derive the stability rules. Thus, the system (3.26) turns out to be a superefficient observer for the system (2.1). $\square$

3.4 Observer-based control

In this subsection, we provide sufficient conditions to ensure the stability of system (2.1) using a mode-dependent average dwell-time approach in combination with the Lyapunov–Krasovskii functional. Consider the following observer-based control law applied to system (2.1):

$$\text{sat}(u(t)) = \text{sat}(K_\sigma \hat{x}(t)) = \sum_{j=1}^{2^m} \eta_j (D_j K_\sigma + D_j^- H_\sigma) \hat{x}(t) = \Delta_i \hat{x}(t), \quad (3.37)$$

Under the control law (3.37), the corresponding closed-loop switched system can be described as follows:

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)} \Delta_i) x(t) + C_{\sigma(t)} x(t - d(t)) + f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)) + B_{\sigma(t)} \Delta_i e(t). \quad (3.38)$$

Now, consider both the closed-loop system (3.38) and the observer error dynamics (3.30). Define the augmented state vector $z(t) = [x^T(t) \ e^T(t)]^T$. The combined dynamics of the system and the observer can then be formulated accordingly:

$$\begin{aligned} \dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A_{\sigma(t)} + B_{\sigma(t)} \Delta_i & B_{\sigma(t)} \Delta_i \\ 0 & A_{\sigma(t)} + L_{\sigma(t)} D_{\sigma(t)} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} C_{\sigma(t)} & 0 \\ 0 & C_{\sigma(t)} \end{bmatrix} \begin{bmatrix} x(t - d(t)) \\ e(t - d(t)) \end{bmatrix} \\ + \begin{bmatrix} f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)) \\ f_{\sigma(t)}(t, \hat{x}(t), \hat{x}(t - d(t)), u_{\sigma(t)}(t)) - f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)) \end{bmatrix}. \end{aligned}$$

This can be compactly expressed as follows:

$$\dot{z}(t) = \Sigma_{\sigma(t)} z(t) + \tilde{C}_{\sigma(t)} z(t - d(t)) + \Delta f_{\sigma(t)}(t, x(t), \hat{x}(t), x(t - d(t)), \hat{x}(t - d(t)), u_{\sigma(t)}(t)),$$

where

$$\begin{aligned} \Sigma_{\sigma(t)} &= \begin{bmatrix} A_{\sigma(t)} + B_{\sigma(t)} \Delta_i & B_{\sigma(t)} \Delta_i \\ 0 & A_{\sigma(t)} + L_{\sigma(t)} D_{\sigma(t)} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad \tilde{C}_{\sigma(t)} = \begin{bmatrix} C_{\sigma(t)} & 0 \\ 0 & C_{\sigma(t)} \end{bmatrix}, \\ \Delta f_{\sigma(t)} &= \begin{bmatrix} f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)) \\ f_{\sigma(t)}(t, \hat{x}(t), \hat{x}(t - d(t)), u_{\sigma(t)}(t)) - f_{\sigma(t)}(t, x(t), x(t - d(t)), u_{\sigma(t)}(t)) \end{bmatrix}. \end{aligned}$$

Theorem 3.4. *Consider the time-delay nonlinear system (2.1). Suppose that Assumption 2.1 is satisfied. For $i \in M$, let $\alpha > 0$ and $\mu \geq 1$ satisfy equality (3.28), and suppose that there exist symmetric positive definite matrices $P_i, \hat{P}_i, Q_{k,i}, \hat{Q}_{k,i}, D_{k,i}, \hat{D}_{k,i}, k = 1, 2$, satisfying equalities (3.21) and (3.29) such that LMIs (3.20) and (3.27) hold. Then system (2.1) globally exponentially stable under the observer-based feedback (3.37).*

Proof. Following a similar reasoning as in the proofs of Theorems 3.2 and 3.3, we consider the following Lyapunov–Krasovskii functional:

$$W_{\sigma(t)}(z(t)) = W_{1\sigma(t)}(z(t)) + W_{2\sigma(t)}(z(t)) + W_{3\sigma(t)}(z(t)),$$

where

$$W_{1\sigma(t)}(z(t)) = z^T(t) \tilde{P}_{\sigma(t)} z(t),$$

$$\begin{aligned}
W_{2\sigma(t)}(z(t)) &= \int_{t-h_1}^t e^{2\alpha(s-t)} z^T(s) \tilde{Q}_{1,\sigma(t)} z(s) ds + \int_{t-h_2}^t e^{2\alpha(s-t)} z^T(s) \tilde{Q}_{2,\sigma(t)} z(s) ds, \\
W_{3\sigma(t)}(z(t)) &= \int_{t-d(t)}^{t-h_1} e^{2\alpha(s-t)} z^T(s) \tilde{D}_{1,\sigma(t)} z(s) ds + \int_{t-h_2}^{t-d(t)} e^{2\alpha(s-t)} z^T(s) \tilde{D}_{2,\sigma(t)} z(s) ds.
\end{aligned}$$

The weighting matrices are defined as

$$\begin{aligned}
\tilde{P}_{\sigma(t)} &= \text{diag} \{P_{\sigma(t)}, \hat{P}_{\sigma(t)}\}, \\
\tilde{Q}_{1,\sigma(t)} &= \text{diag} \{Q_{1,\sigma(t)}, \hat{Q}_{1,\sigma(t)}\}, \quad \tilde{Q}_{2,\sigma(t)} = \text{diag} \{Q_{2,\sigma(t)}, \hat{Q}_{2,\sigma(t)}\}, \\
\tilde{D}_{1,\sigma(t)} &= \text{diag} \{D_{1,\sigma(t)}, \hat{D}_{1,\sigma(t)}\}, \quad \tilde{D}_{2,\sigma(t)} = \text{diag} \{D_{2,\sigma(t)}, \hat{D}_{2,\sigma(t)}\}.
\end{aligned}$$

Computing the time derivative of $W_{\sigma(t)}(z(t))$ along the trajectories of the system and applying standard inequalities, we obtain $\dot{W}_{\sigma(t)}(z(t)) + 2\alpha W_{\sigma(t)}(z(t)) \leq 0$, which guarantees the exponential stability of $z(t)$. $\square$

4 Numerical example and simulation

For illustration purposes, we consider the problem of balancing an inverted pendulum on a cart [5, 19, 20]. Even though in the conventional problem setup, this example does not have perturbations, in our case the nonlinearity is taken to be $\sin(\cdot)$ and $\tanh(\cdot)$, and the delayed state $x(t - d(t))$ arises naturally from actuation and sensing delays. Especially, we consider the switched nonlinear time-delay system (2.1) with two switching modes, whose system matrices are given by

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0.2396 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.0052 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0.8 \\ 0 & 0.21 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0.8 \\ 0 & 0.18 \end{bmatrix}.
\end{aligned}$$

In addition to the linear dynamics, a globally Lipschitz nonlinear perturbation is incorporated as follows:

$$f_i(t, x(t), x(t - d(t)), u(t)) = \begin{bmatrix} 0 \\ a_i \tanh(x_1(t)) + b_i \sin(x_2(t)) + c_i \tanh(x_2(t - d(t))) \end{bmatrix}, \quad i \in \{1, 2\}.$$

From (2.2), we can obtain $d_{\max} = 0.5$. Choose $\alpha = 0.5$, $\mu = 6.9178$, $h_1 = 0$ and $h_2 = 0.3$. Solving the LMIs (3.20) and (3.27), we get

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0.0206 & -0.1411 \\ -0.1411 & 2.5421 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0218 & -0.3709 \\ -0.3709 & 7.2191 \end{bmatrix}, \\
\hat{P}_1 &= \begin{bmatrix} 1.3685 & 0.6425 \\ 0.6425 & 1.9475 \end{bmatrix}, \quad \hat{P}_2 = \begin{bmatrix} 1.7987 & 0.1217 \\ 0.1217 & 1.6433 \end{bmatrix}.
\end{aligned}$$

The controller gains are $K_1 = [-100 \quad -15]$, $K_2 = [-8 \quad -5]$, and the observer gains are $L_1 = \begin{bmatrix} 15 & 0 \\ 0 & 20 \end{bmatrix}$, $L_2 = \begin{bmatrix} 10 & 0 \\ 0 & 12 \end{bmatrix}$.

Fig. 4.1 shows the evolution of the switching signal $\sigma(t)$, indicating that the dwell times vary among the subsystems. Figures 4.2 and 4.3 depict the state variables x_1 and x_2 along with their respective estimates $\hat{x}_1$ and $\hat{x}_2$. The close correspondence between the estimated states and their actual values demonstrates the precision and efficiency of the designed observer.

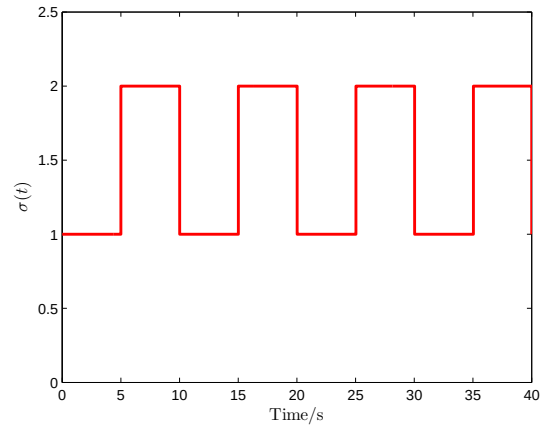
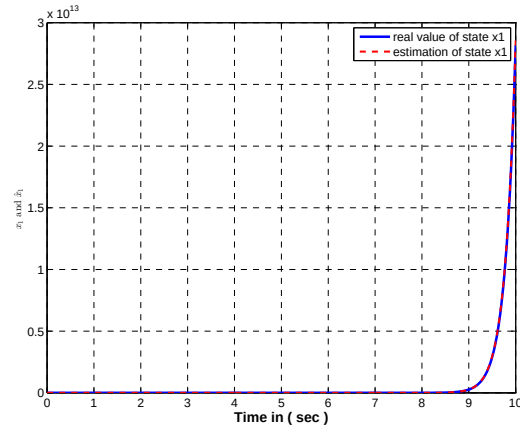
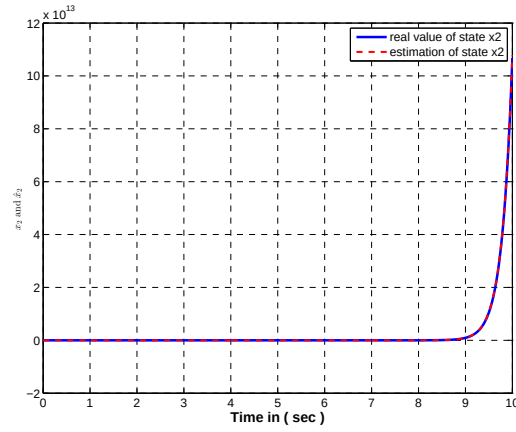


Figure 4.1: The switching signal.

Figure 4.2: State trajectories of x_1 and their estimates $\hat{x}_1$.Figure 4.3: State trajectories of x_2 and their estimates $\hat{x}_2$.

5 Conclusion

This paper explores a natural progression of earlier research by examining a type of switched nonlinear time-delay systems with input saturation. The main contributions are threefold. First, we performed an exponential stability analysis for the open-loop system, offering important insights into its dynamics. Second, to tackle the practical issues of statelessness and actuator saturation, we co-designed a mode-dependent state feedback controller and a state observer. Finally, the key result is that we derived the sufficient stability conditions for the closed-loop system using a mode-dependent average dwell-time approach along with the appropriate Lyapunov–Krasovskii functional. Numerical simulations confirm the effectiveness and practicality of the proposed synthesis method.

Future work could aim to broaden these results to a broader class of systems, particularly one-sided Lipschitz nonlinear systems, and to reduce the conservatism of the stability conditions developed for such systems [1, 4]. Another promising direction is the implementation and experimental validation of these control strategies on real-world prototypes.

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Authors' addresses:

Nadhem Echi

Gafsa University, Faculty of Sciences of Gafsa, Department of Mathematics, Zarroug Gafsa 2112 Tunisia

E-mail: echi.nadhem@fsgf.ugaf.tn

Ahmed Hachani

Sfax University, Faculty of Sciences of Sfax, Department of Mathematics, BP 1171 Sfax 3000 Tunisia

E-mail: ahmedhachnai@gmail.com