Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2025, 1–18

Mohamed Dilmi, Mourad Dilmi

EXISTENCE, UNIQUENESS AND ASYMPTOTIC ANALYSIS FOR A BOUNDARY VALUE PROBLEM WITH RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE

Abstract. In this paper, we investigate the existence, uniqueness and asymptotic analysis for solutions of a boundary value problem involving the Riemann–Liouville fractional time-derivative of order $\alpha \in]1, \frac{3}{2}[$ within an open, bounded domain $Q \subset \mathbb{R}^r$ ($r = 2$ or 3). We apply the Faedo–Galerkin method to prove the existence and uniqueness of solutions to the given problem. An asymptotic behavior of the solution to the problem posed in a thin domain is also presented.

2020 Mathematics Subject Classification. 35R11, 35R35, 78M35.

Key words and phrases. Asymptotic analysis, Faedo–Galerkin method, fractional calculus, Riemann–Liouville derivative.

1 Introduction

The theory of fractional calculus plays an important role in many fields of pure and applied mathematics. Fractional integrals and derivatives, in association with different integral transforms, are used to solve different types of differential and partial differential equations.

Fractional differential equations, both ordinary and partial, are crucial for providing a more precise description of physical phenomena compared to classical integer-order differential equations. In fact, fractional differential equations are real-order expansions of differential equations. One reason for the need for fractional differential equations is the fact that many phenomena cannot be modeled as integer differential equations.

Many problems of mathematical physics and engineering such as aerodynamics, electrodynamics of complex media, viscoelastic materials, polymer physics, viscous damping and seismic analysis [15, 19, 24, 26] have been successfully modeled in recent years by fractional differential equations (FDEs). So, it is very important to find efficient methods for solving FDEs. Various researchers have introduced new methods in the literature. These methods include the homotopy analysis method (HAM) [18] and the variational iteration method (VIM) [23].

A new iterative method was presented by Jafari [17]. This technique solves many types of nonlinear equations such as ordinary and partial differential equations of integer and fractional order. Bhalekar and Daftardar–Gejji used it for solving fractional boundary value problem and evolution equations [10].

Recently, the topic of existence, uniqueness, and methods of analytic and numerical solutions for linear and nonlinear fractional differential equations has been addressed by many researchers in various aspects. For example, in [22], Muslim studied a fractional integro-differential equation of order $\alpha \in (1, 2]$ with deviated argument in a separable Hilbert space by using the α-order cosine family of linear operators and the Banach fixed point theorem.

In [9], Chadha et al. applied the Banach fixed point theorem and analytic semigroup theory to study a fractional differential equation of Sobolev type with nonlocal initial conditions in an arbitrary separable Hilbert space.

The theoretical analysis such as stability, consistency and convergence of the time-fractional diffusion-wave equation with the Caputo fractional derivative are studied in [4]. In [27], Van Bockstal investigated the existence and uniqueness of a weak solution for a non-autonomous time-fractional diffusion equation with space-dependent variable order. In [25], Ruzhanskyet et al. investigated a time-nonlocal problem for the integro-differential diffusion-wave equation on the Heisenberg group. By applying a method based on the Lax–Milgram theorem, Anakira et al. in [7] established sufficient conditions for the existence and uniqueness of the solution for a class of initial boundary value problems with Dirichlet conditions related to a category of fractional-order partial differential equations. In [3], the Cauchy problem for a 2×2 system of weakly coupled semi-linear fractional wave equations is investigated. In addition, the L_∞ decay estimates for global solutions are examined. Allikhanov [5] established the existence of solutions to boundary value problems for time-fractional diffusion-wave equations using energy inequalities. Kirane et al. [2] studied the time-fractional reaction-diffusion equation and obtained globally bounded solutions with suitable assumptions on initial data. Zhou and Peng [28] established the existence and uniqueness of a weak solution of the fractional Navier–Stokes equation using the Galerkin approximation method. For numerical methods for the fractional partial differential equations, we refer the reader to [1, 16].

The analysis of the properties of thin structures and the processes occurring on them is a pivotal area of study in engineering and materials design. Furthermore, understanding how microgeometry affects the overall properties of a material is a crucial aspect of this field of study. For a comprehensive list on this topic, see [11] and the references therein.

This work makes a notable contribution to the existing literature by providing a comprehensive analysis of fractional partial differential equations (FPDEs) within a bounded domain, with a particular focus on thin domains.

The study of the behavior of solutions to partial differential equations (PDEs) in the context of thin domains has been widely explored in the literature by various authors. However, most of these studies focus on PDEs with classical derivatives, as seen in the works such as [12, 13] and the references therein. In the case of fractional partial differential equations, there is only one work (see [14]), where the authors studied the asymptotic analysis of the solution for a viscoelastic problem with a fractional derivative in the Riemann–Liouville sense and nonlinear friction of the Tresca type in a three-dimensional thin domain.

This paper examines a boundary value problem for fractional partial differential equations. Let Q be a bounded open subset of $\mathbb{R}^r$ ($r = 2$ or 3) with a sufficiently regular boundary $\partial Q = Q_1 \cup Q_2$. In the domain Q, we study the following boundary value problem for a partial differential equation with a fractional time-derivative and Dirichlet–Fourier boundary conditions:

$$\left.\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} - \mu(x)\Delta u = f(x,t) \ \text{ in } Q\times\,]0,T[\,, \\ u(x,t) = 0 \ \text{ on } Q_1\times\,]0,T[\,, \\ \exists\, l \in \mathbb{R}_+^* : \ \frac{\partial u}{\partial n} + lu = 0 \ \text{ on } Q_2\times\,]0,T[\,, \end{aligned}\right\} \tag{1.1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}(\,\cdot\,)$ represents the Riemann–Liouville fractional derivative of order α with respect to time t, with $\alpha \in]1, \frac{3}{2}[$, $\mu(x) \in L_+^\infty(Q)$ is a spatially-dependent coefficient and $\frac{\partial u}{\partial n}$ indicates the derivative compared to the external normal on the boundary Q_2.

Moreover, problem (1.1) is equipped with the following initial conditions:

$$u(x,0) = 0, \ \ \frac{\partial^{\alpha-1} u(x,0)}{\partial t^{\alpha-1}} = 0, \ \ \alpha \in \left]1, \frac{3}{2}\right[, \ \ \forall\, x \in Q. \tag{1.2}$$

Equation (1.1) describes a system where the state u evolves through a combination of fractional time dynamics and spatial diffusion processes. This equation illustrates the relationship between diffusion, which accounts for spatial dispersion, and temporal wave-like oscillations, while also incorporating a memory effect that links the system's future behavior to its entire historical context. Such an equation is particularly useful for modeling phenomena in materials with time-dependent, nonlocal properties, such as viscoelastic substances, anomalous diffusion processes, and damped wave propagation (see [19]). The boundary value problem (1.1), (1.2) also has practical applications in various domains, including polymer physics and other fields involving complex material behavior.

The main purpose of the paper is to prove the existence and uniqueness of the weak solution for the boundary value problem (1.1), (1.2) and to study the asymptotic behavior of the solution when problem (1.1), (1.2) is posed in a two-dimensional thin domain.

The rest of the paper is organized as follows. In Section 2, we present some basic definitions and properties of fractional derivatives. In Section 3, we derive the variational formulation of the problem and prove the theorem of the existence and uniqueness of the weak solution by the classic Faedo–Galerkin method. In Section 4, we assume that the problem is posed in a thin domain $Q^\varepsilon \subset \mathbb{R}^2$, and we seek the behavior of the solution when the small parameter ε tends to zero. For this purpose, we use the technique of the change of variable to establish some estimates independent of the parameter ε. These estimates will be useful in proving the convergence results and the limit problem.

2 Preliminaries

Let $\Gamma(\,\cdot\,)$ denote the Gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1}\, dt.$$

Definition 2.1 ([19])**.** The left and right Riemann–Liouville fractional integrals of order α ($\alpha > 0$) of a function $u \in L^1(]0,T[)$ are defined by

$${}_0I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau,$$

$$ {}_tI_T^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\tau - t)^{\alpha-1} u(\tau)\, d\tau. $$

Definition 2.2 ([8]). The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$ of a function $u \in L^1(]0,T[)$ are given by

$$ {}_0^R D_t^{\alpha} u(t) = \frac{d^n}{dt^n} \int_0^t g_{n-\alpha}(t-\tau) u(\tau)\, d\tau, $$

$$ {}_t^R D_T^{\alpha} u(t) = \frac{(-1)^n d^n}{dt^n} \int_t^T g_{n-\alpha}(\tau - t) u(\tau)\, d\tau, $$

where

$$ n = [\alpha] + 1 \quad \text{and} \quad g_{\alpha}(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } \ t > 0, \\ 0 & \text{if } \ 0 \le t. \end{cases} $$

The Riemann–Liouville derivatives satisfy the following properties (see [24]):

1) The composition of the classical derivative with the fractional derivative and the reverse is given by

$$ \frac{d^n}{dt^n} \left({}_0^R D_t^{\alpha} u(t) \right) = {}_0^R D_t^{\alpha+n} u(t), \quad \forall\, n \in \mathbb{N}, $$

$$ {}_0^R D_t^{\alpha} \Big(\frac{d^n}{dt^n}\, u(t) \Big) = {}_0^R D_t^{\alpha+n} u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0) t^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)}\,, \quad \forall\, n \in \mathbb{N}^*. $$

2) The composition of the fractional derivative is given by

$$ {}_0^R D_t^{\beta} ({}_0^R D_t^{\alpha} u(t)) = ({}_0^R D_t^{\alpha+\beta} u(t)) - \sum_{k=1}^{m} \left[{}_0^R D_t^{\alpha-k} u(t) \right]_{t=0} \Big(\frac{t^{-\beta-k}}{\Gamma(1-\beta-k)} \Big), $$

where $n-1 < \beta < n$, $m-1 < \alpha < m$, and $n, m \in \mathbb{N}^*$. The conditions

$$ \left[{}_0^R D_t^{\alpha-k} u(t) \right]_{t=0} = 0, \quad k = 1, \dots, m, $$

are equivalent to the conditions

$$ (u(0))^{(k)} = 0, \quad k = 0, \dots, m-1. $$

Lemma 2.1. *The Riemann–Liouville derivative verifies*

$$ \left[{}_0^R D_t^{\alpha} u(t) \right] \left[{}_0^R D_t^{\alpha-1} u(t) \right] = \frac{1}{2} \frac{d}{dt} \left[{}_0^R D_t^{\alpha-1} u(t) \right]^2, \quad n-1 < \alpha < n, \quad n \in \mathbb{N}^*. $$

Indeed,

$$ \begin{aligned} & \left[{}_0^R D_t^{\alpha} u(t) \right] \left[{}_0^R D_t^{\alpha-1} u(t) \right] \\ &\quad = \left[\frac{d^n}{dt^n} \int_0^t g_{n-\alpha}(t-\tau) u(\tau)\, d\tau \right] \left[\frac{d^{n-1}}{dt^{n-1}} \int_0^t g_{n-1-(\alpha-1)}(t-\tau) u(\tau)\, d\tau \right] \\ &\quad = \frac{d}{dt} \left[\frac{d^{n-1}}{dt^{n-1}} \int_0^t g_{n-\alpha}(t-\tau) u(\tau)\, d\tau \right] \left[\frac{d^{n-1}}{dt^{n-1}} \int_0^t g_{n-\alpha}(t-\tau) u(\tau)\, d\tau \right] \\ &\quad = \frac{1}{2} \frac{d}{dt} \left[{}_0^R D_t^{\alpha-1} u(t) \right]^2. \end{aligned} $$

3 Variational formulation of the problem

For obtaining the variational formulation of the problem, we use the following notations. We denote by (u,v) the scalar product in $L^2(Q)$ i.e. $(u,v)=\int\limits_Q uv\,dx$; $H^1(Q)$ is the Sobolev space defined by

$$H^1(Q)=\Big\{u\in L^2(Q):\ \ \frac{\partial u}{\partial x_j}\in L^2(Q),\ \ j=1,2,\dots,r\Big\},$$

$H_0^1(Q)$ is the closure of $D(Q)$ in $H^1(Q)$ and $H^{-1}(Q)$ is the dual space of $H_0^1(Q)$.

Let X be a Banach space equipped with the norm $\|\cdot\|_X$. We denote by $L^2(0,T;X)$ the space of functions $u:]0,T[\to X$ such that $u(t)$ is measurable with respect to dt. This space is a Banach space equipped with the norm

$$\|u\|_{L^2(0,T;X)}=\Big(\int\limits_0^T\|u(s)\|_X^2\,ds\Big)^{\frac{1}{2}}.$$

Let $L^\infty(0,T;X)$ denote the space of functions $u:]0,T[\to X$ which are measurable and $u\in L^\infty(]0,T[)$. This space is a Banach space equipped with the norm

$$\|u\|_{L^\infty(0,T;X)}=\operatorname*{ess\,sup}_{s\in(0,T)}\|u(s)\|_X.$$

We multiply equation (1.1) by φ, where $\varphi\in H^1(Q)$, and using Green's formula, we obtain the following variational problem:

$$\left.\begin{array}{c}\text{Find }\ u\in K\ \text{ where }\ \dfrac{\partial^{\alpha-1}u}{\partial t^{\alpha-1}}\in K,\ \ \forall\,t\in]0,T[\ \text{ such that}\\[2ex] \Big(\dfrac{\partial^\alpha u}{\partial t^\alpha},\varphi\Big)+a_\mu(u,\varphi)+l\displaystyle\int\limits_\omega u\varphi\,d\omega=(f,\varphi),\ \ \forall\,\varphi\in K,\\[2ex] u(x,0)=0,\quad \dfrac{\partial^{\alpha-1}u(x,0)}{\partial t^{\alpha-1}}=0,\end{array}\right\}\tag{3.1}$$

where

$$K=\big\{v\in H^1(Q):\ v=0\text{ on }Q_1\big\},$$

$$a_\mu(u,\varphi)=\sum_{i,j=1}^r\int\limits_Q\mu(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_i}\,dx,\ \ (f,\varphi)=\int\limits_Q f\varphi\,dx.$$

Remark 3.1. Note that if u satisfies the conditions

$$u\in L^2(0,T;H^1(Q)),\qquad \frac{\partial u}{\partial t}\in L^2(0,T;L^2(Q)),$$

$$\frac{\partial^{\alpha-1}u}{\partial t^{\alpha-1}}\in L^\infty(0,T;H^1(Q)),\qquad \frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial}{\partial t}\Big(\frac{\partial^{\alpha-1}u}{\partial t^{\alpha-1}}\Big)\in L^2(0,T;L^2(Q)),$$

then, according to reference [21], it follows that $u\in C([0,T];L^2(Q))$ and $\frac{\partial^{\alpha-1}u}{\partial t^{\alpha-1}}\in C([0,T];L^2(Q))$.

Therefore, the initial conditions (1.2) make sense.

Theorem 3.1. *Assume that*

$$f,\ \frac{\partial^{\alpha-1}f}{\partial t^{\alpha-1}}\in L^2(0,T;L^2(Q)).$$

Then there exists a unique solution u of (3.1) such that

$$u\in L^2(0,T;H^1(Q)),$$

$$\frac{\partial^{\alpha-1}u}{\partial t^{\alpha-1}}\in L^\infty(0,T;H^1(Q)),$$

$$\frac{\partial^\alpha u}{\partial t^\alpha}\in L^2(0,T;L^2(Q)),$$

$$\frac{\partial u}{\partial t}\in L^2(0,T;L^2(Q)).$$

Proof. (A) *The uniqueness of the solution*

Let u_1, u_2 be two solutions of (3.1). Taking $\varphi = \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}}$ (resp. $\varphi = \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}}$) in the equation for u_1 (resp. for u_2) and adding, we get

$$\Big(\frac{\partial^{\alpha}u_1}{\partial t^{\alpha}}, \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}}\Big) + a_{\mu}\Big(u_1, \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}}\Big) + l\int\limits_{Q_2} u_1\Big(\frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}}\Big)\,dQ_2 = \Big(f, \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}}\Big), \tag{3.2}$$

$$\Big(\frac{\partial^{\alpha}u_2}{\partial t^{\alpha}}, \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}}\Big) + a_{\mu}\Big(u_2, \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}}\Big) + l\int\limits_{Q_2} u_2\Big(\frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}}\Big)\,dQ_2 = \Big(f, \frac{\partial^{\alpha-1}u_1}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1}u_2}{\partial t^{\alpha-1}}\Big). \tag{3.3}$$

Now, we put $W = u_1 - u_2$. Consequently, combining equations (3.2) and (3.3) leads to

$$\Big(\frac{\partial^{\alpha}W}{\partial t^{\alpha}}, \frac{\partial^{\alpha-1}W}{\partial t^{\alpha-1}}\Big) + a_{\mu}\Big(W, \frac{\partial^{\alpha-1}W}{\partial t^{\alpha-1}}\Big) + l\int\limits_{Q_2} W\,\frac{\partial^{\alpha-1}W}{\partial t^{\alpha-1}}\,dQ_2 = 0. \tag{3.4}$$

On the other hand, by Lemma 2.1, we have

$$\Big(\frac{\partial^{\alpha}W}{\partial t^{\alpha}}, \frac{\partial^{\alpha-1}W}{\partial t^{\alpha-1}}\Big) = \frac{1}{2}\,\frac{d}{dt}\,\Big\|\frac{\partial^{\alpha-1}W}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q)}, \tag{3.5}$$

and, by inequality (5.21) from [8], we find that

$$\int\limits_0^t a_{\mu}\Big(W(s), \frac{\partial^{\alpha-1}W(s)}{\partial t^{\alpha-1}}\Big)\,ds \ge \mu_* g_{2-\alpha}(T)\int\limits_0^t \|W(s)\|^2_{H^1(Q)}\,ds, \tag{3.6}$$
$$\int\limits_0^t\int\limits_{Q_2} W(s)\cdot\frac{\partial^{\alpha-1}W(s)}{\partial t^{\alpha-1}}\,dQ_2\,ds \ge g_{2-\alpha}(T)\int\limits_0^t \|W(s)\|^2_{L^2(Q_2)}\,ds.$$

Then, integrating equality (3.4) over $]0,t[$ and using (3.5), (3.6), we get

$$\frac{1}{2}\,\Big\|\frac{\partial^{\alpha-1}W(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q)} + \mu_* g_{2-\alpha}(T)\int\limits_0^t \|W(s)\|^2_{H^1(Q)}\,ds + g_{2-\alpha}(T)l\int\limits_0^t \|W(s)\|^2_{L^2(Q_2)}\,ds \le 0,$$

which implies that

$$W(s) = 0, \quad \forall\, s \in\,]0,T[\,.$$

This concludes the uniqueness of the solution.

(B) *The existence of the solution*

To show the existence of the solution, we use the Faedo–Galerkin method. Let $\{K_m\}$ be a family of finite dimensional spaces. This method introduces a sequence (w_j) of functions having the following properties:

* $w_j \in K$, $\forall\, j = 1, \ldots, m$.
* The family $\{w_1, w_2, \ldots, w_m\}$ is linearly independent.
* The family $K_m = [w_1, w_2, \ldots, w_m]$ generated by $\{w_1, w_2, \ldots, w_m\}$ is dense in K.

Let $u_m = u_m(t)$ be an approximate solution such that

$$u_m(t) = \sum_{j=1}^{m} k_{jm}(t) w_j,$$

which implies that

$$\frac{\partial^\alpha u_m(t)}{\partial t^\alpha} = \sum_{j=1}^{m} \left[\frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-\tau)^{1-\alpha} k_{jm}(\tau)\, d\tau \right] w_j,$$

where k_{jm} are determined by solving the following ordinary fractional differential equations:

$$\Big(\frac{\partial^\alpha u_m}{\partial t^\alpha}, w_j \Big) + a_\mu(u_m, w_j) + l \int_{Q_2} u_m w_j \, dQ_2 = (f, w_j), \quad 1 \le j \le m, \tag{3.7}$$

with the initial conditions

$$\left.\begin{aligned} u_m(x,0) &= 0, \\ u_m(0) &= \sum_{j=1}^{m} \gamma_{jm}(0) w_j \xrightarrow{m\to\infty} 0 \quad \text{in } K, \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{\partial^{\alpha-1} u_m(x,0)}{\partial t^{\alpha-1}} &= 0, \\ \frac{\partial^{\alpha-1} u_m(x,0)}{\partial t^{\alpha-1}} &= \sum_{j=1}^{m} \eta_{jm}(0) w_j \xrightarrow{m\to\infty} 0 \quad \text{in } K. \end{aligned}\right\}$$

Now, we establish some estimates independent of m.

The first estimate. Multiplying equation (3.7) by $\frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{1-\alpha} k_{jm}(\tau)\, d\tau$ and summing over $j = 1$ to m, we get

$$\Big(\frac{\partial^\alpha u_m}{\partial t^\alpha}, \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big) + a_\mu\Big(u_m, \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big) + l \int_{Q_2} u_m \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \, dQ_2 = \Big(f, \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big),$$

hence

$$\frac{1}{2} \frac{d}{dt} \Big\| \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big\|^2_{L^2(Q)} + a_\mu\big(u_m, \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big) + l \int_{Q_2} u_m \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \, dQ_2 = \Big(f, \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big), \tag{3.8}$$

by integrating (3.8) over $]0, t[$ and using the inequality (5.21) from [8], we deduce

$$\begin{aligned} \Big\| \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \Big\|^2_{L^2(Q)} &+ g_{2-\alpha}(T) \mu_* \int_0^t \| u_m(s) \|^2_{H^1(Q)} \, ds \\ &+ g_{2-\alpha}(T) l \int_0^t \| u_m(s) \|^2_{L^2(Q_2)} \, ds \le 2 \int_0^t \Big(f, \frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}} \Big) \, ds. \end{aligned} \tag{3.9}$$

As

$$2 \int_0^t \Big(f(s), \frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}} \Big) \, ds \le 4 \int_0^t \| f(s) \|^2_{L^2(Q)} \, ds + \int_0^t \Big\| \frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}} \Big\|^2_{L^2(Q)} \, ds,$$

from (3.9) we deduce that

$$\left\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\right\|^2_{L^2(Q)} + \mu_* g_{2-\alpha}(T)\int_0^t \|u_m(s)\|^2_{H^1(Q)}\,ds \le 4\int_0^t \|f(s)\|^2_{L^2(Q)}\,ds + \int_0^t \left\|\frac{\partial^{\alpha-1}u_m(s)}{\partial t^{\alpha-1}}\right\|^2_{L^2(Q)} ds. \tag{3.10}$$

Applying Gronwall's inequality to the inequality mentioned above, we obtain the following result:

$$\left\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\right\|^2_{L^2(Q)} \le c, \tag{3.11}$$

where c is a constant independent of m. Thus, from (3.10), we conclude that

$$\|u_m\|^2_{L^2(0,T;H^1(Q))} \le c. \tag{3.12}$$

The second estimate. For the second estimate, we derive (3.7) with respect to t of order $\alpha-1$ and obtain

$$\Big(\frac{\partial^{2\alpha-1}u_m}{\partial t^{2\alpha-1}}, w_j\Big) - \Big(\Big[\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big]_{t=0}\frac{(t)^{-\alpha}}{\Gamma(1-\alpha)}, w_j\Big) - \Big(\Big[\frac{\partial^{\alpha-2}u_m}{\partial t^{\alpha-2}}\Big]_{t=0}\frac{(t)^{-\alpha-1}}{\Gamma(-\alpha)}, w_j\Big) + a_\mu\Big(\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}, w_j\Big) + l\int_{Q_2}\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}} w_j\,dQ_2 = \Big(\frac{\partial^{\alpha-1}f}{\partial t^{\alpha-1}}, w_j\Big), \quad \forall\, w_j \in K_m.$$

As $u_m(0)=0$ and $[\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}]_{t=0}=0$, we find

$$\Big(\frac{\partial^{2\alpha-1}u_m}{\partial t^{2\alpha-1}}, w_j\Big) + a_\mu\Big(\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}, w_j\Big) + l\int_\omega \frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}} w_j\,d\omega = \Big(\frac{\partial^{\alpha-1}f}{\partial t^{\alpha-1}}, w_j\Big), \quad \forall\, w_j \in K_m,$$

multiplying this equation by $\frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-\tau)^{1-\alpha}k_{jm}(\tau)\,d\tau$ and then summing over $j=1$ to m, we get

$$\Big(\frac{\partial^{2\alpha-1}u_m}{\partial t^{2\alpha-1}}, \frac{\partial^\alpha u_m}{\partial t^\alpha}\Big) + a_\mu\Big(\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}, \frac{\partial^\alpha u_m}{\partial t^\alpha}\Big) + l\int_{Q_2}\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\frac{\partial^\alpha u_m}{\partial t^\alpha}\,dQ_2 = \Big(\frac{\partial^{\alpha-1}f}{\partial t^{\alpha-1}}, \frac{\partial^\alpha u_m}{\partial t^\alpha}\Big). \tag{3.13}$$

On the other hand, we have

$$\frac{\partial^{2\alpha-1}u_m}{\partial t^{2\alpha-1}} = \frac{1}{\Gamma(3-2\alpha)}\frac{d^2}{dt^2}\int_0^t (t-\tau)^{2-2\alpha}u_m(\tau)\,d\tau, \quad 1<\alpha<\frac{3}{2}.$$

So, we get

$$\begin{aligned}\Big(\frac{\partial^{2\alpha-1}u_m}{\partial t^{2\alpha-1}}, \frac{\partial^\alpha u_m}{\partial t^\alpha}\Big) &= \int_Q \Big[\frac{1}{\Gamma(3-2\alpha)}\frac{d^2}{dt^2}\int_0^t (t-\tau)^{2-2\alpha}u_m(\tau)\,d\tau\Big]\Big[\frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-\tau)^{1-\alpha}u_m(\tau)\,d\tau\Big]dx \\ &\ge \int_Q \Big[\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{d^2}{d\tau^2}u_m(\tau)\,d\tau\Big]\Big[\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{d^2}{d\tau^2}u_m(\tau)\,d\tau\Big]dx \\ &\ge \frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\left\|\frac{\partial^\alpha u_m}{\partial t^\alpha}\right\|^2_{L^2(Q)}.\end{aligned} \tag{3.14}$$

Then formula (3.13) becomes

$$\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\Big\|\frac{\partial^\alpha u_m}{\partial t^\alpha}\|^2_{L^2(Q)}+\frac{d}{dt}\Big[\mu_*\Big\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big\|^2_{H^1(Q)}+l\Big\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q_2)}\Big]\le\Big(\frac{\partial^{\alpha-1}f}{\partial t^{\alpha-1}},\frac{\partial^\alpha u_m}{\partial t^\alpha}\Big).$$

Integrating from 0 to t, we deduce

$$\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\int_0^t\Big\|\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q)}ds+\mu_*\Big\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big\|^2_{H^1(Q)}\le\int_0^t\Big(\frac{\partial^{\alpha-1}f(s)}{\partial t^{\alpha-1}},\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big)ds. \tag{3.15}$$

As $\frac{\partial^{\alpha-1}f}{\partial t^{\alpha-1}}\in L^2(0,T;L^2(Q))$, using Cauchy–Shwarz's inequality, we deduce that

$$\int_0^t\Big(\frac{\partial^{\alpha-1}f(s)}{\partial t^{\alpha-1}},\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big)ds\le\int_0^t\Big\|\frac{\partial^{\alpha-1}f(s)}{\partial t^{\alpha-1}}\Big\|_{L^2(Q)}\Big\|\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big\|_{L^2(Q)}ds, \tag{3.16}$$

thus, according to (3.15) and (3.16), we get

$$\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\int_0^t\Big\|\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q)}ds+\mu_*\Big\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big\|^2_{H^1(Q)}$$
$$\le\frac{2\Gamma(3-2\alpha)}{g_{2-\alpha}(T)}\int_0^t\Big\|\frac{\partial^{\alpha-1}f(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q)}ds+\frac{g_{2-\alpha}(T)}{2\Gamma(3-2\alpha)}\int_0^t\Big\|\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q)}ds.$$

Then, we deduce that

$$\frac{g_{2-\alpha}(T)}{2\Gamma(3-2\alpha)}\int_0^t\Big\|\frac{\partial^\alpha u_m(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q)}ds+\mu_*\Big\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big\|^2_{H^1(Q)}\le\frac{2\Gamma(3-2\alpha)}{g_{2-\alpha}(T)}\int_0^t\Big\|\frac{\partial^{\alpha-1}f(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q)}ds,$$

consequently,

$$\Big\|\frac{\partial^\alpha u_m}{\partial t^\alpha}\Big\|^2_{L^2(0,T;L^2(Q))}+\Big\|\frac{\partial^{\alpha-1}u_m}{\partial t^{\alpha-1}}\Big\|^2_{H^1(Q)}\le C, \tag{3.17}$$

where

$$C=\frac{\frac{2\Gamma(3-2\alpha)}{g_{2-\alpha}(T)}}{\min\Big(\frac{g_{2-\alpha}(T)}{2\Gamma(3-2\alpha)},\mu_*\Big)}\Big\|\frac{\partial^{\alpha-1}f(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(0,T;L^2(Q))}$$

is a constant independent of m.

The last estimate. Multiplying equation (3.7) by $k'_{jm}(t)$ and summing over $j=1,\dots,m$, we obtain

$$\Big(\frac{\partial^\alpha u_m}{\partial t^\alpha},\frac{\partial u_m}{\partial t}\Big)+a_\mu\Big(u_m,\frac{\partial u_m}{\partial t}\Big)+l\int_{Q_2}u_m\frac{\partial u_m}{\partial t}\,dQ_2=\Big(f,\frac{\partial u_m}{\partial t}\Big),$$

which implies that

$$\Big(\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\Big(\frac{\partial u_m}{\partial t}\Big),\frac{\partial u_m}{\partial t}\Big)+\frac{1}{2}\frac{d}{dt}\Big[\mu_*\|u_m\|^2_{H^1(Q)}+l\|u_m\|^2_{L^2(Q_2)}\Big]=\Big(f,\frac{\partial u_m}{\partial t}\Big).$$

Integrating over $]0,t[$ and using the Cauchy–Schwarzs and Young inequalities, we get

$$g_{2-\alpha}(T)\int_0^t\Big\|\frac{\partial u_m(s)}{\partial t}\Big\|^2_{L^2(Q)}ds\le2\int_0^t\Big|\Big(f(s),\frac{\partial u_m(s)}{\partial t}\Big)\Big|ds$$
$$\le\frac{8}{g_{2-\alpha}(T)}\|f(s)\|^2_{L^2(0,T;L^2(Q))}+\frac{g_{2-\alpha}(T)}{2}\int_0^t\Big\|\frac{\partial u_m(s)}{\partial t}\Big\|^2_{L^2(Q)}ds.$$

Finally, we obtain

$$\int_0^t \left\| \frac{\partial u_m(s)}{\partial t} \right\|_{L^2(Q)}^2 ds \leq \frac{16}{(g_{2-\alpha}(T))^2} \|f\|_{L^2(0,T;L^2(Q))}^2. \tag{3.18}$$

(C) *Passage to the limit.*

According to (3.11), (3.12), (3.17) and (3.18), we can extract from u_m a sequence, also denoted by u_m, such that u_m converges weakly to u; $\frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}}$ converges weakly to χ and $\frac{\partial^{\alpha} u_m(s)}{\partial t^{\alpha}}$ converges weakly to ζ.

Let $\varphi \in D(]0,T[\times Q)$, using integration by parts (see [6]), we find

$$\lim_{m\to\infty} \int_0^T \Big(\frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}}, \varphi \Big) ds = \lim_{m\to\infty} \int_0^T \big(u_m(s), {}_t^R D_T^{\alpha-1} \varphi \big) ds - \big[\big((u_m(s)), {}_t I_T^{2-\alpha} \varphi(t) \big) \big]_0^T.$$

Since $u_m(0) = 0$ and u_m converges weakly to u, we get

$$\lim_{m\to\infty} \int_0^T \Big(\frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}}, \varphi \Big) ds = \int_0^T \big(u(s), {}_t^R D_T^{\alpha-1} \varphi \big) ds = \int_0^T \Big(\frac{\partial^{\alpha-1} u(s)}{\partial t^{\alpha-1}}, \varphi \Big) ds.$$

This implies that

$$\frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} \rightharpoonup \frac{\partial^{\alpha-1} u}{\partial t^{\alpha-1}} \text{ in } L^\infty(0,T;H^1(Q)).$$

To prove that $\zeta = \frac{\partial^\alpha u}{\partial t^\alpha}$, we have $\forall\, \varphi \in D(]0,T[\,\times Q)$

$$\lim_{m\to\infty} \int_0^T \Big(\frac{\partial^{\alpha} u_m(s)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} u(s)}{\partial t^{\alpha}}, \varphi \Big) ds = \lim_{m\to\infty} \int_0^T \Big(\frac{\partial^{\alpha-1} u_m(s)}{\partial t^{\alpha-1}} - \frac{\partial^{\alpha-1} u(s)}{\partial t^{\alpha-1}}, \frac{\partial \varphi}{\partial t} \Big) ds.$$

As $\frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}}$ converges weakly to $\frac{\partial^{\alpha-1} u}{\partial t^{\alpha-1}}$, we find

$$\frac{\partial^{\alpha} u_m}{\partial t^{\alpha}} \rightharpoonup \frac{\partial^{\alpha} u}{\partial t^{\alpha}} \text{ in } L^2(0,T;L^2(Q)).$$

So, we have the following convergence result:

$$\begin{aligned} u_m &\rightharpoonup u \text{ in } L^2(0,T;H^1(Q)), \\ \frac{\partial u_m}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(0,T;L^2(Q)), \\ \frac{\partial^{\alpha-1} u_m}{\partial t^{\alpha-1}} &\rightharpoonup^* \frac{\partial^{\alpha-1} u}{\partial t^{\alpha-1}} \text{ in } L^\infty(0,T;H^1(Q)), \\ \frac{\partial^{\alpha} u_m}{\partial t^{\alpha}} &\rightharpoonup \frac{\partial^{\alpha} u}{\partial t^{\alpha}} \text{ in } L^2(0,T;L^2(Q)). \end{aligned}$$

Finally, since the space K_m is dense in K, we obtain

$$\lim_{m\to\infty} \Big[\Big(\frac{\partial^\alpha u_m}{\partial t^\alpha}, \varphi \Big) + a_\mu(u_m, \varphi) + l \int_{Q_2} u_m \varphi \, dQ_2 - (f, \varphi) \Big]$$
$$= \Big(\frac{\partial^\alpha u}{\partial t^\alpha}, \varphi \Big) + a_\mu(u, \varphi) + l \int_{Q_2} u \varphi \, dQ_2 - (f, \varphi) = 0 \text{ for all } \varphi \in K.$$

Then u satisfies

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \mu(x) \Delta u = f \text{ a.e in } Q \times]0,T[\,.$$

□

4 Asymptotic behavior of the solution in a thin domain

In this section, we assume that system (1.1), (1.2) is posed in the following two-dimensional thin domain:

$$Q^\varepsilon = \{x = (x_1, x_2) \in \mathbb{R}^2,\ \ 0 < x_1 < L,\ \ 0 < x_2 < \varepsilon h(x_1)\},$$

where ε is a small parameter that will tend to zero, and $h(\cdot)$ is a function of class C^1 defined on $[0, L]$ such that

$$0 < \underline{h} = h_{min} \le h(x_1) \le h_{max} = \overline{h},\ \ \forall x_1 \in [0, L].$$

The boundary of Q^ε is $\partial Q^\varepsilon = Q_1^\varepsilon \cup Q_2^\varepsilon \cup]0, L[$, where

$$Q_1^\varepsilon = \{x \in \partial Q^\varepsilon :\ \ x_2 = \varepsilon h(x_1)\}$$

is the upper boundary and Q_2^ε is the lateral boundary.

Since the system is now posed in a thin domain, the solution is influenced by the small parameter ε. More precisely, instead of (1.1), (1.2), we consider

$$\left.\begin{aligned}
\frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha} - \mu^\varepsilon(x)\Delta u^\varepsilon = f^\varepsilon \ \text{ in } Q^\varepsilon \times]0, T[\,, \\
u^\varepsilon(x,t) = 0 \ \text{ on } (\partial Q^\varepsilon \cup \partial Q_2^\varepsilon) \times]0, T[\,, \\
\exists\, l^\varepsilon \in \mathbb{R}_+^* :\ \ \frac{\partial u^\varepsilon}{\partial n} + l^\varepsilon u^\varepsilon = 0 \ \text{ on }]0, L[\times]0, T[\,, \\
u^\varepsilon(x,0) = 0,\quad \frac{\partial^{\alpha-1} u^\varepsilon(x,0)}{\partial t^{\alpha-1}} = 0,\ \ \forall x \in Q^\varepsilon,
\end{aligned}\right\}$$

and the weak formulation

$$\left.\begin{aligned}
\text{Find } u^\varepsilon \in K^\varepsilon \text{ where } \frac{\partial^{\alpha-1} u^\varepsilon}{\partial t^{\alpha-1}} \in K^\varepsilon,\ \ \forall t \in]0, T[\text{ such that} \\
\Big(\frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}, \varphi\Big) + a_{\mu^\varepsilon}\Big(u^\varepsilon, \varphi\Big) + l^\varepsilon \int_0^L u^\varepsilon \varphi \, dx_1 = (f^\varepsilon, \varphi),\ \ \forall \varphi \in K^\varepsilon, \\
u^\varepsilon(x,0) = 0,\quad \frac{\partial^{\alpha-1} u^\varepsilon(x,0)}{\partial t^{\alpha-1}} = 0,
\end{aligned}\right\} \tag{4.1}$$

with

$$K^\varepsilon = \{v \in H^1(Q^\varepsilon) :\ \ v = 0 \ \text{ on } \partial Q_1^\varepsilon \cup \partial Q_2^\varepsilon\},$$

$$a_{\mu^\varepsilon}(u^\varepsilon, \varphi) = \sum_{i=1}^{2} \int_{Q^\varepsilon} \mu^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx_1 \, dz.$$

4.1 The problem in a fixed domain and some estimates

For the asymptotic analysis of problem (4.1), we use the approach which consists in transposing the problem initially posed in the domain, which depends on a small parameter ε, into an equivalent problem posed in the fixed domain, which is independent of ε. For this end, we use the technique of scaling (see, e.g., ([12, 13]).

Introducing the change of variables $z = \frac{x_2}{\varepsilon}$, we get the fixed domain

$$Q = \{(x_1, z) \in \mathbb{R}^2,\ \ 0 < x_1 < L,\ \ 0 < z < h(x_1)\}.$$

Now, we define on Q the following functions:

$$\left.\begin{aligned}
u^\varepsilon(x_1, x_2, t) &= \widehat{u}^\varepsilon(x_1, z, t), \\
\varepsilon^2 f^\varepsilon(x_1, x_2, t) &= \widehat{f}(x_1, z, t), \\
\varepsilon l^\varepsilon &= \widehat{l}, \\
\mu^\varepsilon(x_1, x_2) &= \widehat{\mu}(x_1, z),
\end{aligned}\right\}$$

with $\widehat{f}(\cdot)$, $\widehat{l}$, and $\widehat{\mu}(\cdot)$ not depending on ε.

Multiplying (4.1) by ε and passing to the fixed domain Ω, we get the following variational problem:

$$\left.\begin{aligned} &\text{Find } \widehat{u}^\varepsilon \in K^\varepsilon \text{ where } \frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}} \in K, \ \forall\, t\in]0,T[\text{ such that}\\ &\int_Q \varepsilon^2 \frac{\partial^\alpha \widehat{u}^\varepsilon}{\partial t^\alpha}\varphi\, dx_1\, dz + \varepsilon^2 \int_Q \widehat{\mu}(x_1,z)\frac{\partial \widehat{u}^\varepsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1}\, dx_1\, dz\\ &+\int_Q \widehat{\mu}(x_1,z)\frac{\partial \widehat{u}^\varepsilon}{\partial z}\frac{\partial\varphi}{\partial z}\, dx_1\, dz + \widehat{l}\int_0^L \widehat{u}^\varepsilon \varphi\, dx_1 = \int_Q \widehat{f}\varphi\, dx_1\, dz, \ \ \forall\, \varphi\in K,\\ &\widehat{u}^\varepsilon(x_1,z,0)=0, \quad \tfrac{\partial^{\alpha-1}\widehat{u}^\varepsilon(x_1,z,0)}{\partial t^{\alpha-1}} = 0, \end{aligned}\right\} \tag{4.2}$$

with

$$K = \{v\in H^1(Q):\ \ v=0 \text{ on } \Gamma_1\cup\Gamma_2\}.$$

Theorem 4.1. *Assume that* $f^\varepsilon, \frac{\partial^{\alpha-1}f^\varepsilon}{\partial t^{\alpha-1}}, \frac{\partial^\alpha f^\varepsilon}{\partial t^\alpha} \in L^2(0,T;L^2(Q^\varepsilon))$. *Then there exists a constant* c *independent of* ε *such that*

$$\Big\|\varepsilon\frac{\partial}{\partial x_1}\Big(\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}}\Big)\Big\|^2_{L^2(Q)} + \Big\|\frac{\partial}{\partial z}\Big(\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}}\Big)\Big\|^2_{L^2(Q)} + \Big\|\varepsilon\frac{\partial^\alpha \widehat{u}^\varepsilon}{\partial t^\alpha}\Big\|^2_{L^2(0,T;L^2(Q))} \le c, \tag{4.3}$$

$$\Big\|\varepsilon\frac{\partial \widehat{u}^\varepsilon}{\partial x_1}\Big\|^2_{L^2(0,T;L^2(Q))} + \Big\|\frac{\partial \widehat{u}^\varepsilon}{\partial z}\Big\|^2_{L^2(0,T;L^2(Q))} + \Big\|\varepsilon\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q)} \le c. \tag{4.4}$$

Proof. Let u^ε be the solution of the problem (4.1). We derive equation (4.1) with respect to t of order $\alpha-1$ and choose $\varphi = \frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}$. Then we have

$$\Big(\frac{\partial^{2\alpha-1}u^\varepsilon}{\partial t^{2\alpha-1}}, \frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}\Big) + a_{\mu^\varepsilon}\Big(\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}, \frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}\Big) + l^\varepsilon\int_0^L \frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}\, dx_1 = \Big(\frac{\partial^{\alpha-1}f^\varepsilon}{\partial t^{\alpha-1}}, \frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}\Big).$$

Using inequality (3.14) and Lemma 2.1, we get

$$\begin{aligned}\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\Big\|\frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)} + \frac{d}{dt}\Big[a_{\mu^\varepsilon}\Big(\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}, \frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big) + l^\varepsilon\Big\|\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(]0,L[)}\Big]&\\ \le \Big(\frac{\partial^{\alpha-1}f^\varepsilon}{\partial t^{\alpha-1}}, \frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha}\Big).&\end{aligned}$$

Integrating over $]0,t[$, we deduce

$$\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\int_0^t \Big\|\frac{\partial^\alpha u^\varepsilon(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)} ds + \mu_*\|\nabla\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\|^2_{L^2(Q^\varepsilon)^2} \le \int_0^t\int_{Q^\varepsilon} \frac{\partial^{\alpha-1}f^\varepsilon(s)}{\partial t^{\alpha-1}}\frac{\partial^\alpha u^\varepsilon(s)}{\partial t^\alpha}\, dx\, ds. \tag{4.5}$$

On the other hand, using integration by parts, the Cauchy–Schwarz inequality, the Poincaré inequality [11]

$$\|v\|_{L^2(Q^\varepsilon)} \le \varepsilon\overline{h}\|\nabla v\|_{L^2(Q^\varepsilon)^2}, \ \ \forall\, v\in K^\varepsilon,$$

and the Young inequality, we obtain

$$\begin{aligned}\int_0^t\int_{Q^\varepsilon}\frac{\partial^{\alpha-1}f^\varepsilon(s)}{\partial t^{\alpha-1}}\frac{\partial^\alpha u^\varepsilon(s)}{\partial t^\alpha}\, dx\, ds &= \int_{Q^\varepsilon}\frac{\partial^{\alpha-1}f^\varepsilon(t)}{\partial t^{\alpha-1}}\frac{\partial^{\alpha-1}u^\varepsilon(t)}{\partial t^{\alpha-1}}\, dx - \int_0^t\int_{Q^\varepsilon}\frac{\partial^\alpha f^\varepsilon(s)}{\partial t^\alpha}\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\, dx\, ds\\ &\le \frac{2\varepsilon^2\overline{h}^2}{\mu_*}\Big\|\frac{\partial^{\alpha-1}f^\varepsilon(t)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)} + \frac{\mu_*}{2}\Big\|\nabla\frac{\partial^{\alpha-1}u^\varepsilon(t)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}\\ &\quad + \frac{2\varepsilon^2\overline{h}^2}{\mu_*}\int_0^t\Big\|\frac{\partial^\alpha f^\varepsilon(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)} ds + \frac{\mu_*}{2}\int_0^t\Big\|\nabla\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2} ds. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we get

$$\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\int_0^t \Big\|\frac{\partial^\alpha u^\varepsilon(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)}\,ds+\frac{\mu_*}{2}\Big\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}$$
$$\le \frac{2\varepsilon^2\overline{h}^2}{\mu_*}\Big\|\frac{\partial^{\alpha-1}f^\varepsilon(t)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)}+\frac{2\varepsilon^2\overline{h}^2}{\mu_*}\int_0^t\Big\|\frac{\partial^\alpha f^\varepsilon(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)}\,ds$$
$$+\frac{\mu_*}{2}\int_0^t\Big\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}\,ds. \tag{4.7}$$

As

$$\varepsilon^2\|f^\varepsilon\|^2_{L^2(Q^\varepsilon)}=\varepsilon^{-1}\|\widehat{f}\|^2_{L^2(Q)},$$

multiplying inequality (4.7) by ε, we obtain

$$\frac{g_{2-\alpha}(T)}{\Gamma(3-2\alpha)}\int_0^t \varepsilon\Big\|\frac{\partial^\alpha u^\varepsilon(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)}\,ds+\frac{\mu_*}{2}\,\varepsilon\Big\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}$$
$$\le\frac{\mu_*}{2}\int_0^t\varepsilon\Big\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}\,ds+A, \tag{4.8}$$

where

$$A=\frac{2\overline{h}^2}{\mu_*}\Big\|\frac{\partial^{\alpha-1}\widehat{f}(t)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q)}+\frac{2\overline{h}^2}{\mu_*}\Big\|\frac{\partial^\alpha\widehat{f}}{\partial t^\alpha}\Big\|^2_{L^2(0,T;L^2(Q))},$$

is a constant independent of ε.

Now, using Gronwall's lemma in inequality (4.8), we get

$$\int_0^t\varepsilon\Big\|\frac{\partial^\alpha u^\varepsilon(s)}{\partial t^\alpha}\Big\|^2_{L^2(Q^\varepsilon)}\,ds+\frac{\mu_*}{2}\,\varepsilon\Big\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}\le c, \tag{4.9}$$

where c is a positive constant independent of ε.

From (4.9) we deduce (4.3).

To show estimate (4.4), we choose $\varphi=\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}$ in (4.1), we find

$$\Big(\frac{\partial^\alpha u^\varepsilon}{\partial t^\alpha},\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big)+a_\mu\Big(u^\varepsilon,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big)+l^\varepsilon\int_0^L u^\varepsilon\,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\,dx_1=\Big(f^\varepsilon,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big).$$

Using Lemma 2.1, we have

$$\frac12\frac{d}{dt}\Big\|\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)}+a_\mu\Big(u^\varepsilon,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big)+l^\varepsilon\int_0^L u^\varepsilon\,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\,dx_1=\int_{Q^\varepsilon} f\,\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\,dx.$$

Integrating over $]0,t[$ and using inequality (5.21) from [8], we deduce

$$\Big\|\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)}+g_{2-\alpha}(T)\int_0^t\Big[\mu_*\|\nabla u^\varepsilon(s)\|^2_{L^2(Q^\varepsilon)^2}+l^\varepsilon\|u^\varepsilon(s)\|^2_{L^2(]0,L[)}\Big]\,ds$$
$$\le 2\int_0^t\int_{Q^\varepsilon} f(s)\,\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\,dx\,ds.$$

By Poincaré's inequality, we obtain

$$\Big\|\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)} + \mu_* g_{2-\alpha}(T)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(Q^\varepsilon)^2}\,ds$$
$$\le 2(\varepsilon h)^2\int_0^t \|f^\varepsilon(s)\|^2_{L^2(Q^\varepsilon)}\,ds + 2\int_0^t \Big\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)^2}\,ds.$$

From (4.9), we note that $2\varepsilon\|\nabla\,\frac{\partial^{\alpha-1}u^\varepsilon(s)}{\partial t^{\alpha-1}}\|^2_{L^2(0,T;L^2(Q^\varepsilon))^2} \le c$ (where c is a constant independent of ε). Thus, by multiplying the inequality above by ε, we get

$$\varepsilon\Big\|\frac{\partial^{\alpha-1}u^\varepsilon}{\partial t^{\alpha-1}}\Big\|^2_{L^2(Q^\varepsilon)} + \mu_* g_{2-\alpha}(T)\int_0^t \varepsilon\|\nabla u^\varepsilon(s)\|^2_{L^2(Q^\varepsilon)^2}\,ds \le \overline{h}\|\widehat{f}\|_{L^2(0,T;L^2(Q))} + c.$$

This concludes the proof of the theorem. □

4.2 Passage to the limit as $\varepsilon \to 0$ and limit problem

The objective of this subsection is to obtain the theorem of convergence and the limit problem when $\varepsilon \to 0$. To this end, we introduce the Banach space V_z defined as follows:

$$V_z = \Big\{v \in L^2(Q):\ \frac{\partial v}{\partial z} \in L^2(Q),\ \ v = 0 \text{ on } Q_1\Big\},$$

with the norm

$$\|v\|_{V_z} = \Big(\|v\|^2_{L^2(Q)} + \Big\|\frac{\partial v}{\partial z}\Big\|^2_{L^2(Q)}\Big)^{\frac12}.$$

Theorem 4.2. *Under the hypotheses of Theorem 4.1, there exists $u^* \in L^2(0,T;V_z)$ such that*

$$\left.\begin{aligned} \widehat{u}^\varepsilon &\rightharpoonup u^* \\ \frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}} &\rightharpoonup \frac{\partial^{\alpha-1}u^*}{\partial t^{\alpha-1}} \end{aligned}\right\}\ \text{weakly in } L^2(0,T;V_z), \tag{4.10}$$

$$\left.\begin{aligned} \varepsilon\,\frac{\partial\widehat{u}^\varepsilon}{\partial x_1} \rightharpoonup 0, \quad \varepsilon\,\frac{\partial}{\partial z}\Big(\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}}\Big) \rightharpoonup 0, \\ \varepsilon\,\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}} \rightharpoonup 0, \quad \varepsilon^2\,\frac{\partial^{\alpha}\widehat{u}^\varepsilon}{\partial t^{\alpha}} \rightharpoonup 0 \end{aligned}\right\}\ \text{weakly in } L^2(0,T;L^2(Q)), \tag{4.11}$$

and u^* is the unique (weak) solution of the limit problem

$$\begin{cases} -\dfrac{\partial}{\partial z}\Big[\widehat{\mu}(x_1,z)\,\dfrac{\partial u^*(x_1,z,t)}{\partial z}\Big] = \widehat{f}(x_1,z,t) & \text{a.e. in } Q\times]0,T[\,, \\ \widehat{\mu}(x_1,0)\,\dfrac{\partial u^*(x_1,0,t)}{\partial z} - \widehat{l}u^*(x_1,0,t) = 0 & \text{a.e. on }]0,L[\times]0,T[\,, \\ u^*(x_1,z,0) = 0. \end{cases}$$

Proof. By Theorem 4.1, there exists a constant c independent of ε such that

$$\Big\|\frac{\partial\widehat{u}^\varepsilon(s)}{\partial z}\Big\|_{L^2(0,T;L^2(Q))} \le c, \qquad \Big\|\frac{\partial}{\partial z}\Big(\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon(s)}{\partial t^{\alpha-1}}\Big)\Big\|_{L^2(0,T;L^2(Q))} \le c.$$

Using these estimates with the Poincaré inequality in the domain Q, we get

$$\|\widehat{u}^\varepsilon(s)\|_{L^2(0,T;V_z)} \le \overline{h}\Big\|\frac{\partial\widehat{u}^\varepsilon(s)}{\partial z}\Big\|^2_{L^2(0,T;L^2(Q))},$$
$$\Big\|\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon(s)}{\partial t^{\alpha-1}}\Big\|_{L^2(0,T;V_z)} \le \overline{h}\Big\|\frac{\partial}{\partial z}\Big(\frac{\partial^{\alpha-1}\widehat{u}^\varepsilon(s)}{\partial t^{\alpha-1}}\Big)\Big\|_{L^2(0,T;L^2(Q))}.$$

So, $(\widehat{u}^\varepsilon, \frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}})_\varepsilon$ is bounded in $L^2(0,T;V_z)^2$, which implies the existence of an element $(u^*, \frac{\partial^{\alpha-1}u^*}{\partial t^{\alpha-1}})$ in $L^2(0,T;V_z)^2$ such that $(\widehat{u}^\varepsilon, \frac{\partial^{\alpha-1}\widehat{u}^\varepsilon}{\partial t^{\alpha-1}})_\varepsilon$ converges weakly to $(u^*, \frac{\partial^{\alpha-1}u^*}{\partial t^{\alpha-1}})$ in $L^2(0,T;V_z)^2$. Thus we obtain (4.10). To find (4.11), we use (4.3), (4.4) and (4.10).

Now, by passage to the limit when ε tends to zero in the variational problem (4.2), and using the convergence results, we deduce

$$\int_Q \widehat{\mu}(x_1,z)\,\frac{\partial u^*}{\partial z}\,\frac{\partial \varphi}{\partial z}\,dx_1\,dz + \widehat{l}\int_0^L u^*\varphi\,dx_1 = \int_Q \widehat{f}\varphi\,dx_1\,dz,\quad \forall\,\varphi\in K. \tag{4.12}$$

Choosing $\varphi\in H_0^1(Q)$ and using Green's formula, we obtain

$$-\int_Q \frac{\partial}{\partial z}\Big[\widehat{\mu}(x_1,z)\,\frac{\partial u^*}{\partial z}\Big]\varphi\,dx_1\,dz = \int_Q \widehat{f}\varphi\,dx_1\,dz,$$

thus

$$-\frac{\partial}{\partial z}\left[\widehat{\mu}(x_1,z)\,\frac{\partial u^*}{\partial z}\right] = \widehat{f}\ \text{ in } H^{-1}(Q). \tag{4.13}$$

As $\widehat{f}\in L^2(0,T;L^2(Q))$, then (4.13) is valid a.e in $Q\times\,]0,T[$.

Going back to formula (4.12) and using Green's formula for $\varphi\in K$, we get

$$-\int_Q \frac{\partial}{\partial z}\left[\widehat{\mu}(x_1,z)\,\frac{\partial u^*}{\partial z}+\widehat{f}\right]\varphi\,dx_1\,dz - \int_0^L \widehat{\mu}(x_1,0)\,\frac{\partial u^*(x_1,0,t)}{\partial z}\,\varphi\,dx_1 + \widehat{l}\int_0^L u^*\varphi\,dx_1 = 0,\quad \forall\,\varphi\in K,$$

and using (4.13), we have

$$\widehat{\mu}(x_1,0)\,\frac{\partial u^*(x_1,0,t)}{\partial z} - \widehat{l}u^*(x_1,0,t) = 0 \ \text{ a.e on }]0,L[\,\times\,]0,T[\,.$$

To prove the uniqueness result, we suppose that there exist two solutions u^* and u^{**} of the variational problem (4.12). We have

$$\int_Q \widehat{\mu}(x_1,z)\,\frac{\partial u^*}{\partial z}\,\frac{\partial \varphi}{\partial z}\,dx_1\,dz + \widehat{l}\int_0^L u^*\varphi\,dx_1 = \int_Q \widehat{f}\varphi\,dx_1\,dz,\quad \forall\,\varphi\in K, \tag{4.14}$$

and

$$\int_Q \widehat{\mu}(x_1,z)\,\frac{\partial u^{**}}{\partial z}\,\frac{\partial \varphi}{\partial z}\,dx_1\,dz + \widehat{l}\int_0^L u^{**}\varphi\,dx_1 = \int_Q \widehat{f}\varphi\,dx_1\,dz,\quad \forall\,\varphi\in K. \tag{4.15}$$

We take $\varphi = u^{**}-u^*$ in (4.14), then $\varphi = u^*-u^{**}$ in (4.15) and by summing the two equations, we obtain

$$\int_Q \widehat{\mu}(x_1,z)\Big|\frac{\partial u^*}{\partial z}-\frac{\partial u^{**}}{\partial z}\Big|^2\,dx_1\,dz + \widehat{l}\int_0^L |u^*-u^{**}|^2\,dx_1 = 0.$$

Then, by the Poincaré inequality, we conclude

$$\|u^*-u^{**}\|_{L^2(0,T;V_z)} = 0. \qquad \square$$

5 Conclusion

Although several studies, including those by M. Muslim [22], B. Ahmad [2,3] and others (see [1,9]), have explored various aspects of fractional differential equations, including the existence and uniqueness of solutions under different conditions, this paper addresses the unique topic of the asymptotic behavior of solutions in thin domains. The application of the Faedo–Galerkin method to demonstrate the existence and uniqueness of weak solutions and to provide some estimates for these solutions represents a notable advancement in the mathematical apparatus available for addressing FPDEs.

Compared to existing literature, this work offers a detailed examination of the behavior of solutions, where the thickness of the domain tends to zero, a topic not extensively covered in previous studies. For instance, while Dilmi et al. [10–12] focus on the asymptotic behavior of solutions in thin domains using the classic Sobolev space, this study addresses a gap in the literature by providing insights into the behavior of FPDEs in thin domains, where the difficulty here lies in how to estimate the solution and its gradient when studying the asymptotic behavior and obtaining our limit problem, a crucial aspect for applications in material science and related fields. This makes the work a valuable reference for future research.

In conclusion, this work not only advances the theoretical understanding of fractional partial differential equations (FPDEs), but also bridges a critical gap in the literature regarding their behavior in thin domains. These results are likely findings establish a foundation for further exploration of FPDEs, with applications in various scientific and engineering fields. The insights obtained from this research could guide upcoming studies focused on solving more intricate fractional differential equations and exploring their applications.

References

[1] M. Abbaszadeh and M. Dehghan, Numerical and analytical investigations for neutral delay fractional damped diffusion-wave equation based on the stabilized interpolating element free Galerkin (IEFG) method. *Appl. Numer. Math.* **145** (2019), 488–506.

[2] B. Ahmad, M. S. Alhothuali, H. H. Alsulami, M. Kirane and S. Timoshin, On a time fractional reaction diffusion equation. *Appl. Math. Comput.* **257** (2015), 199–204.

[3] B. Ahmad, A. Alsaedi, M. Berbiche and M. Kirane, Global existence and blow-up of solutions for a system of fractional wave equations. *Taiwanese J. Math.* **26** (2022), no. 1, 103–135.

[4] U. Ali, A. Iqbal, M. Sohail, F. A. Abdullah and Z. Khan, Compact implicit difference approximation for time-fractional diffusion-wave equation. *Alexandria Engineering Journal* **61** (2022), 4119–4126.

[5] A. A. Alikhanov, A priori estimates for solutions of boundary value problems for equations of fractional order. (Russian) *Differ. Uravn.* **46** (2010), no. 5, 658–664; translation in *Differ. Equ.* **46** (2010), no. 5, 660–666.

[6] R. Almeida and D. F. M. Torres, Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives. *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011), no. 3, 1490–1500.

[7] N. Anakira, Z. Chebana, T.-E. Oussaeif, I. M. Batiha and A. Ouannas, A study of a weak solution of a diffusion problem for a temporal fractional differential equation. *Nonlinear Funct. Anal. Appl.* **27** (2022), no. 3, 679–689.

[8] E. G. Bajlekova, Fractional evolution equations in Banach spaces. Dissertation, Technische Universiteit Eindhoven, Eindhoven, 2001; *Eindhoven University of Technology, Eindhoven*, 2001.

[9] A. Chadha, D. Bahuguna and D. N. Pandey, Faedo-Galerkin approximate solutions for nonlocal fractional differential equation of Sobolev type. *Fract. Differ. Calc.* **8** (2018), no. 2, 205–222.

[10] V. Daftardar-Gejji and S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method. *Comput. Math. Appl.* **59** (2010), no. 5, 1801–1809.

[11] M. Dilmi, Asymptotic analysis for coupled parabolic problem with Dirichlet-Fourier boundary conditions in a thin domain. *J. Math. Appl.* **46** (2023), 163–185.

[12] M. Dilmi, M. Dilmi and H. Benseridi, Asymptotic behavior for the elasticity system with a nonlinear dissipative term. *Rend. Istit. Mat. Univ. Trieste* **51** (2019), 41–60.

[13] M. Dilmi, M. Dilmi and H. Benseridi, A 3D-2D asymptotic analysis of viscoelastic problem with nonlinear dissipative and source terms. *Math. Methods Appl. Sci.* **42** (2019), no. 18, 6505–6521.

[14] M. Dilmi, M. Dilmi and H. Benseridi, Variational formulation and asymptotic analysis of viscoelastic problem with Riemann–Liouville fractional derivatives. *Math. Methods Appl. Sci.* **44** (2021), no. 3, 2294–2313.

[15] R. Hilfer, *Applications of Fractional Calculus in Physics*. World Sci. Publishing, River Edge, NJ, 2000.

[16] J. Huang, J. Zhang, S. Arshad and Y. Tang, A numerical method for two-dimensional multi-term time-space fractional nonlinear diffusion-wave equations. *Appl. Numer. Math.* **159** (2021), 159–173.

[17] H. Jafari, Iterative Methods for solving system of fractional differential equations. *Ph.D. Thesis, Pune University*, 2006.

[18] H. Jafari and S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method. *Commun. Nonlinear Sci. Numer. Simul.* **14** (2009), no. 5, 1962–1969.

[19] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[20] A. N. Kochubei, Fractional-hyperbolic equations and systems. Cauchy problem. *Handbook of fractional calculus with applications. Vol. 2*, 197–222, *De Gruyter, Berlin*, 2019.

[21] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* (French) Dunod, Paris; Gauthier-Villars, Paris, 1969.

[22] M. Muslim, Faedo-Galerkin approximations to fractional integro-differential equation of order $\alpha \in (1, 2]$ with deviated argument. *Dyn. Partial Differ. Equ.* **13** (2016), no. 4, 351–368.

[23] Z. M. Odibat and Sh. Momani, Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlinear Sci. Numer. Simul.* **7** (2006), no. 1, 27–34.

[24] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications.* Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[25] M. Ruzhansky, N. Tokmagambetov and B. T. Torebek, Bitsadze–Samarskii type problem for the integro-differential diffusion-wave equation on the Heisenberg group. *Integral Transforms Spec. Funct.* **31** (2020), no. 1, 1–9.

[26] A. M. Spasic and M. P. Lazarevic, Electroviscoelasticity of liquid/liquid interfaces: fractional-order model. *Journal of Colloid and Interface Science* **282** (2005), no. 1, 223–230.

[27] K. Van Bockstal, Existence of a unique weak solution to a non-autonomous time-fractional diffusion equation with space-dependent variable order. *Adv. Difference Equ.* **2021**, Paper no. 314, 43 pp.

[28] Y. Zhou and L. Peng, Weak solutions of the time-fractional Navier–Stokes equations and optimal control. *Comput. Math. Appl.* **73** (2017), no. 6, 1016–1027.

(Received 2.11.2023; revised 22.07.2024; accepted 17.10.2024)

Authors' addresses:

Mohamed Dilmi

Department of Mathematics, University of Blida 1, Po. Box 270 Route de Soumaa, LAMDA-RO Laboratory Blida Algeria

E-mail: mohamed77dilmi@gmail.com

Mourad Dilmi

Laboratory for Applied Mathematics, Department of Mathematics, Setif I-University, 19000, Algeria

E-mail: mourad.dilmi@univ-setif.dz