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SOLVABILITY OF THE MIXED TYPE INTERACTION
PROBLEM OF ACOUSTIC WAVES AND GENERALIZED
THERMO-ELECTRO-MAGNETO-ELASTIC STRUCTURES

Abstract. In this paper, we consider a three-dimensional model of fluid-solid interaction when a generalized thermo-electro-magneto-elastic body, occupying a bounded region Ω^+ , is embedded in an inviscid fluid occupying an unbounded domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In this case, we have a six-dimensional Green-Lindsay model of generalized thermo-electro-magneto-elastic field (displacement vector with three components, electric potential, magnetic potential, and temperature distribution function) in the domain Ω^+ , while in the unbounded domain Ω^- we have a scalar acoustic pressure field. The physical kinematic and dynamic relations are described mathematically by appropriate boundary and transmission conditions. Using the potential method and the theory of pseudodifferential equations, we prove the existence and uniqueness theorems for the corresponding interaction problems in appropriate Sobolev–Slobodetskii spaces.

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რეზიუმე. ნაშრომში განხილულია სითხისა და სხეულის ურთიერთქმედების სამგანზომილებიანი მოდელი, სადაც განზოგადებული თერმო-ელექტრო-მაგნეტო-დრეკადი სხეული, რომელიც იკავებს სასრულ არეს Ω^+ , მოთავსებულია სითხეში, რომელიც იკავებს უსასრულო არეს $\Omega^- = \mathbb{R}^3 \setminus \Omega^+$. ამ შემთხვევაში, Ω^+ არეში გვაქვს განზოგადებული თერმო-ელექტრო-მაგნეტო-დრეკადი ველის ექვსგანზომილებიანი გრინ-ლინდსეის მოდელი (რომელიც მოიცავს გადაადგილების ვექტორს სამი კომპონენტით, ელექტრულ პოტენციალს, მაგნიტურ პოტენციალსა და ტემპერატურას), ხოლო უსასრულო Ω^- არეში გვაქვს აკუსტიკური წნევის სკალარული ველი. ფიზიკური კინემატიკური და დინამიკური ურთიერთქმედებები მათემატიკურად აღწერილია შესაბამისი სასაზღვრო და ტრანსმისიის პირობებით. პოტენციალთა მეთოდისა და ფსევდოდირექციულ განტოლებათა თეორიის გამოყენებით დამტკიცებულია შესაბამისი ურთიერთქმედების ამოცანების ამონახსნის არსებობისა და ერთადერთობის თეორემები სობოლევ-სლობოდსკის შესაბამის სივრცეებში.

1 Formulation of the problems

1.1 Introduction

The surge of interest in multiferroic materials over the past 15 years has been driven by their fascinating physical properties and huge potential for technological applications. Multiferroics belong to a newer class of thermo-electro-magneto-elastic materials in which ferromagnetic and ferroelectric properties occur simultaneously. Consequently, mathematical modeling related to the multiferroic complex composite structures and the corresponding fluid-solid interaction problems have become very important from the theoretical and practical points of view. Mathematically, this type of interaction problems is described by non-standard boundary-transmission problems for physical fields of different dimensions acting in adjacent domains. This type of interaction problems involving physical fields of different dimensions arises in mathematical models of electro-magneto transducers, sensors, actuators, energy harvesters, servomechanisms, phased array microphones, ultrasound equipment, inkjet droplet actuators, sonar transducers, bioimaging, immunochemistry, and acousto-biotherapeutics (see, e.g., [39, 41, 45] and the references therein).

In this paper, we analyze a three-dimensional model of fluid-solid interaction, when a thermo-electro-magneto-elastic body, occupying a bounded region Ω^+ , is embedded in an inviscid fluid occupying an unbounded domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In the solid region, we consider the generalized Green–Lindsay thermo-electro-magneto-elastic model. An essential feature of this model is that heat propagation has a finite speed, in contrast to the classical model. In the case under consideration, in the solid region we have a six-dimensional thermo-electro-magneto-elastic field (displacement vector with three components, electric potential, magnetic potential, and temperature distribution function), while in the fluid region, we have a scalar acoustic pressure field. The physical kinematic and dynamic relations are described mathematically by appropriate boundary and transmission conditions. We consider the boundary-transmission problems for the so-called pseudo-oscillation equations, which are obtained from the corresponding dynamical equations by the Laplace transform. With the help of the potential method and the theory of pseudodifferential equations, we prove the uniqueness and existence theorems for the corresponding boundary-transmission problems in appropriate Sobolev–Slobodetskii spaces. We derive explicit representation formulas of solutions in the form of single layer potentials.

Many papers are devoted to similar interaction problems, when in the solid region simpler mathematical models are considered. The Dirichlet-type, Neumann-type and mixed type steady-state oscillation interaction problems for acoustic waves and piezoelectric structures are studied in [11, 12, 15, 16] and [17].

Similar interaction problems with the classical model of elasticity have been investigated by a number of authors. An exhaustive information concerning theoretical and numerical results, for the case when the both interacting media are isotropic, can be found in the references [4–7, 20, 24–26, 30, 31, 34, 36–38].

2 Basic equations and operators, statement of problems, and uniqueness theorems

2.1 Generalized thermo-electro-magneto elastic field

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with C^∞ -smooth boundary $S = \partial\Omega^+$ unless otherwise stated, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. Throughout the paper, $n = (n_1, n_2, n_3)$ denotes the outward unit normal to S .

The basic linear system of pseudo-oscillation equations for the thermo-electro-magneto-elasticity theory under the Green–Lindsay model is obtained from the corresponding dynamical equations by applying the Laplace transform and, in matrix form, reads as follows (see [3, 21, 43] and the references therein) $A(\partial, \tau)U(x, \tau) = \Phi(x, \tau)$, where $U = (u, \varphi, \psi, \theta)^\top = (u_1, \dots, u_6)^\top$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\varphi = u_4$ is the electric potential, $\psi = u_5$ is the magnetic potential, $\theta = u_6$ is the temperature distribution, $\Phi = (\Phi_1, \dots, \Phi_6)^\top$ is the given vector-function, and $A(\partial, \tau)$ is the matrix

differential operator

$$A(\partial_x, \tau) = [A_{pq}(\partial_x, \tau)]_{6 \times 6} \\ := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \rho_1\tau^2\delta_{rk}]_{3 \times 3} & [e_{lrj}\partial_j\partial_l]_{3 \times 1} & [q_{lrj}\partial_j\partial_l]_{3 \times 1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3 \times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1 \times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1 \times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1 \times 3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l - \tau^2 h_0 - \tau d_0 \end{bmatrix}_{6 \times 6},$$

where $\tau = \sigma + i\omega$ is a complex parameter with $\sigma > \sigma_0 \geq 0$ and $\omega \in \mathbb{R}$, δ_{jk} is the Kronecker symbol, and the Einstein summation convention is used: summation over repeated indices is understood to be from 1 to 3, unless otherwise stated. Here and in what follows, we employ the following notation for the material parameters: ρ_1 is the mass density, c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the electromagnetic coupling coefficients, p_j , m_j , and λ_{rj} are coupling coefficients connecting dissimilar fields, η_{jk} are the heat conductivity coefficients, ν_0 and h_0 are two relaxation times, a_0 and d_0 are constitutive coefficients.

The constants involved in the above equations satisfy the symmetry conditions:

$$\begin{aligned} c_{rjkl} &= c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \\ \varkappa_{kj} &= \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad a_{kj} = a_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad r, j, k, l = 1, 2, 3. \end{aligned} \quad (2.1)$$

Some authors require more extended symmetry conditions for the piezoelectric and piezomagnetic constants: $e_{klj} = e_{kjl} = e_{ljk}$, $q_{klj} = q_{kjl} = q_{ljk}$ (see, e.g., [2, 3, 32, 33]). However, in our further analysis, we will require only the symmetry properties described in (2.1). From physical considerations it follows that (see, e.g., [3, 21, 32, 40, 43]):

$$\begin{aligned} c_{rjkl}\xi_{rj}\xi_{kl} &\geq \delta_0\xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq \delta_1|\xi|^2, \quad \mu_{kj}\xi_k\xi_j \geq \delta_2|\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq \delta_3|\xi|^2, \\ \text{for all } \xi_{kj} &= \xi_{jk} \in \mathbb{R} \text{ and for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \end{aligned} \quad (2.2)$$

$$\nu_0 > 0, \quad h_0 > 0, \quad d_0\nu_0 - h_0 > 0, \quad (2.3)$$

where δ_0 , δ_1 , δ_2 , and δ_3 are positive constants depending on the material parameters.

Due to the symmetry conditions (2.1), using (2.2), we easily derive

$$\begin{aligned} c_{rjkl}\zeta_{rj}\bar{\zeta}_{kl} &\geq \delta_0\zeta_{kl}\bar{\zeta}_{kl}, \quad \varkappa_{kj}\zeta_k\bar{\zeta}_j \geq \delta_1|\zeta|^2, \quad \mu_{kj}\zeta_k\bar{\zeta}_j \geq \delta_2|\zeta|^2, \quad \eta_{kj}\zeta_k\bar{\zeta}_j \geq \delta_3|\zeta|^2, \\ \text{for all } \zeta_{kj} &= \zeta_{jk} \in \mathbb{C} \text{ and for all } \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3. \end{aligned} \quad (2.4)$$

A more careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insures that the following 8×8 matrix

$$M = [M_{kj}]_{8 \times 8} := \begin{bmatrix} [\varkappa_{jl}]_{3 \times 3} & [a_{jl}]_{3 \times 3} & [p_j]_{3 \times 1} & [\nu_0 p_j]_{3 \times 1} \\ [a_{jl}]_{3 \times 3} & [\mu_{jl}]_{3 \times 3} & [m_j]_{3 \times 1} & [\nu_0 m_j]_{3 \times 1} \\ [p_j]_{1 \times 3} & [m_j]_{1 \times 3} & d_0 & h_0 \\ [\nu_0 p_j]_{1 \times 3} & [\nu_0 m_j]_{1 \times 3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8 \times 8} \quad (2.5)$$

is positive definite. Note that the positive definiteness of M remains valid if the parameters p_j and m_j in (2.5) are replaced by the opposite ones, $-p_j$ and $-m_j$. Moreover, it follows that the matrices

$$\Lambda^{(1)} := \begin{bmatrix} [\varkappa_{kj}]_{3 \times 3} & [a_{kj}]_{3 \times 3} \\ [a_{kj}]_{3 \times 3} & [\mu_{kj}]_{3 \times 3} \end{bmatrix}_{6 \times 6}, \quad \Lambda^{(2)} := \begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2 \times 2} \quad (2.6)$$

are also positive definite, i.e.,

$$\begin{aligned} \varkappa_{kj}\zeta'_k\bar{\zeta}'_j + a_{kj}(\zeta'_k\bar{\zeta}'_j + \bar{\zeta}'_k\zeta'_j) + \mu_{kj}\zeta''_k\bar{\zeta}''_j &\geq \kappa_1(|\zeta'|^2 + |\zeta''|^2) \quad \forall \zeta', \zeta'' \in \mathbb{C}^3, \\ d_0|z_1|^2 + h_0(z_1\bar{z}_2 + \bar{z}_1 z_2) + \nu_0 h_0|z_2|^2 &\geq \kappa_2(|z_1|^2 + |z_2|^2) \quad \forall z_1, z_2 \in \mathbb{C}, \end{aligned}$$

with some positive constants κ_1 and κ_2 depending on the material parameters involved in (2.6).

The principal homogeneous symbol matrix of the operator $A(\partial_x, \tau)$ is

$$A^{(0)}(-i\xi) = -A^{(0)}(\xi) = \begin{bmatrix} [-c_{rjkl}\xi_j\xi_l]_{3\times 3} & [-e_{lrj}\xi_j\xi_l]_{3\times 1} & [-q_{lrj}\xi_j\xi_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}\xi_j\xi_l]_{1\times 3} & -\varkappa_{jl}\xi_j\xi_l & -a_{jl}\xi_j\xi_l & 0 \\ [q_{jkl}\xi_j\xi_l]_{1\times 3} & -a_{jl}\xi_j\xi_l & -\mu_{jl}\xi_j\xi_l & 0 \\ [0]_{1\times 3} & 0 & 0 & -\eta_{jl}\xi_j\xi_l \end{bmatrix}_{6\times 6}.$$

From the symmetry conditions (2.1), inequalities (2.2), and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.6), it follows that there is a positive constant C_0 , depending only on the material parameters, such that

$$\operatorname{Re}(-A^{(0)}(-i\xi)\zeta \cdot \zeta) = \operatorname{Re}\left(\sum_{k,j=1}^6 A_{kj}^{(0)}(\xi)\zeta_j\bar{\zeta}_k\right) \geq C_0|\xi|^2|\zeta|^2 \text{ for all } \xi \in \mathbb{R}^3 \text{ and for all } \zeta \in \mathbb{C}^6.$$

Therefore, $-A(\partial_x, \tau)$ is a non-selfadjoint strongly elliptic differential operator. The overbar denotes complex conjugation and the central dot denotes the scalar product in the respective complex-valued vector space.

Further, we introduce the generalized stress operator $\mathcal{T}(\partial_x, n, \tau)$ associated with the pseudo-oscillation operator $A(\partial_x, \tau)$:

$$\begin{aligned} \mathcal{T} &= \mathcal{T}(\partial_x, n, \tau) = [\mathcal{T}_{pq}(\partial_x, n, \tau)]_{6\times 6} \\ &:= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -(1+\nu_0\tau)p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -(1+\nu_0\tau)m_jn_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}. \end{aligned}$$

Evidently, for a smooth six-dimensional vector $U := (u, \varphi, \psi, \vartheta)^\top$, we have

$$\mathcal{T}(\partial_x, n, \partial_t)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -T_0^{-1}q_jn_j)^\top. \quad (2.7)$$

Recall that $E = (E_1, E_2, E_3)^\top = -\nabla\varphi$ and $H = (H_1, H_2, H_3)^\top = -\nabla\psi$ are the electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials, respectively, and ϑ denotes the temperature change relative to the reference temperature T_0 , $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density, and the corresponding constitutive equations read as follows:

$$\begin{aligned} \sigma_{rj}(x, \tau) &= c_{rjkl}\varepsilon_{kl}(x, \tau) + e_{lrj}\partial_l\varphi(x, \tau) + q_{lrj}\partial_l\psi(x, \tau) - (1+\nu_0\tau)\lambda_{rj}\vartheta(x, \tau), \\ D_j(x, \tau) &= e_{jkl}\varepsilon_{kl}(x, \tau) - \varkappa_{jl}\partial_l\varphi(x, \tau) - a_{jl}\partial_l\psi(x, \tau) + (1+\nu_0\tau)p_j\vartheta(x, \tau), \\ B_j(x, \tau) &= q_{jkl}\varepsilon_{kl}(x, \tau) - a_{jl}\partial_l\varphi(x, \tau) - \mu_{jl}\partial_l\psi(x, \tau) + (1+\nu_0\tau)m_j\vartheta(x, \tau), \\ q_j(x, \tau) &= -T_0\eta_{jl}\partial_l\vartheta(x, \tau). \end{aligned}$$

The components of the vector $\mathcal{T}U$ given by (2.7) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the fourth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector.

In Green's formulas there also appears the boundary operator $\mathcal{P}(\partial_x, n, \tau)$ associated with the adjoint

differential operator $A^*(\partial_x, \tau) := [\overline{A(-\partial_x, \tau)}]^\top = A^\top(-\partial_x, \bar{\tau})$,

$$\begin{aligned} \mathcal{P} &= \mathcal{P}(\partial_x, n, \tau) = [\mathcal{P}_{pq}(\partial_x, n, \tau)]_{6 \times 6} \\ &= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3 \times 3} & [-e_{lrj}n_j\partial_l]_{3 \times 1} & [-q_{lrj}n_j\partial_l]_{3 \times 1} & [\bar{\tau}\lambda_{rj}n_j]_{3 \times 1} \\ [e_{jkl}n_j\partial_l]_{1 \times 3} & \kappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -\bar{\tau}p_jn_j \\ [q_{jkl}n_j\partial_l]_{1 \times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -\bar{\tau}m_jn_j \\ [0]_{1 \times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6 \times 6}. \end{aligned} \quad (2.8)$$

2.2 Scalar acoustic wave field

We assume that the unbounded exterior domain Ω^- is filled by a homogeneous, isotropic, inviscid fluid medium with constant density ρ_2 . Further, let the propagation of an acoustic pressure wave in Ω^- be described by a complex-valued scalar function (scalar field) w being a solution of the homogeneous Helmholtz equation (cf. [18])

$$\Delta w(x, \tau) - \rho_2 \tau^2 w(x, \tau) = 0, \quad (2.9)$$

satisfying the following asymptotic condition at infinity:

$$w(x) = O(|x|^m) \text{ as } |x| \rightarrow \infty \quad (2.10)$$

with some natural number $m \in \mathbb{N}$. Note that if $\operatorname{Re} \tau = \sigma > \sigma_0 \geq 0$, then, due to relation (2.10), it follows that

$$D^\alpha w(x) = O(|x|^{-n}) \text{ as } |x| \rightarrow \infty \quad (2.11)$$

for all multi-indices α and $n \in \mathbb{N}$. Actually, w decays exponentially at infinity.

2.3 Formulation of interaction problems

We denote by $H^s(\Omega^\pm)$ and $H^s(S)$ the well-known L_2 -based Bessel potential spaces. The symbols $\{\cdot\}_S^+$ and $\{\cdot\}_S^-$ denote one-sided limits (traces) on S from Ω^+ and Ω^- , respectively. We omit the subscript S whenever no confusion can arise.

Assume that the domain $\bar{\Omega}^+$ is occupied by an anisotropic homogeneous material with the above-described generalized thermo-electro-magneto-elastic properties and is immersed in an inviscid fluid occupying the exterior domain Ω^- .

The fluid-solid interaction in the case under consideration is described by the boundary-transmission problems for the equations of the generalized thermo-electro-magneto-elasticity theory and the Helmholtz equation. Throughout the paper, we assume that $\tau = \sigma + i\omega$, $\sigma > \sigma_0 \geq 0$, $\omega \in \mathbb{R}$, unless otherwise stated.

Let the boundary $S = \partial\Omega^+ = \partial\Omega^- \in C^\infty$ be divided into two disjoint parts S_D and S_N , i.e. $S = \bar{S}_D \cup \bar{S}_N$, $S_D \cap S_N = \emptyset$ and $l_m := \partial S_D = \partial S_N \in C^\infty$.

Interaction Mixed Type Problem $(IM)_\tau$: Find a vector-function $U = (u, u_4, u_5, u_6)^\top \in [H^1(\Omega^+)]^6$ and scalar function $w \in H^1(\Omega^-)$ satisfying the differential equations

$$\begin{aligned} A(\partial_x, \tau)U &= 0 \text{ in } \Omega^+, \\ \Delta w - \rho_2 \tau^2 w &= 0 \text{ in } \Omega^-, \end{aligned} \quad (2.12)$$

the transmission conditions

$$\{u \cdot n\}^+ = -(\rho_2 \tau^2)^{-1} \{\partial_n w\}^- + g_0 \text{ on } S, \quad (2.13)$$

$$\{[\mathcal{T}U]_j\}^+ = -\{w\}^- n_j + f_j \text{ on } S, \quad j = 1, 2, 3, \quad (2.14)$$

and the mixed type boundary conditions

$$\{u_r\}^+ = f_r^{(D)} \text{ on } S_D, \quad r = 4, 5, 6, \quad (2.15)$$

$$\{[\mathcal{T}U]_r\}^+ = f_r^{(N)} \text{ on } S_N, \quad r = 4, 5, 6, \quad (2.16)$$

where $g_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_r^{(D)} \in H^{1/2}(S_D)$, $f_r^{(N)} \in H^{-1/2}(S_N)$, $r = 4, 5, 6$. In this interaction problem, we require that the scalar pressure function w satisfy the decay condition (2.11).

2.4 Uniqueness theorems

Theorem 2.1. *Let $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geq 0$ and $\omega \in \mathbb{R}$. The homogeneous problem $(IM)_\tau$ has only a trivial solution.*

Proof. Let a pair (U, w) be a solution to the homogeneous problem $(IM)_\tau$.

Let us write Green's first formula for the Helmholtz equation (2.12) in the domain Ω^- :

$$\int_{\Omega^-} (\Delta - \rho_2 \tau^2) w \bar{w} dx + \rho_2 \tau^2 \int_{\Omega^-} |w|^2 dx + \int_{\Omega^-} |\nabla w|^2 dx = -\langle \{\partial_n w\}^-, \{w\}^- \rangle_S. \quad (2.17)$$

Here and in what follows, the symbol $\langle \cdot, \cdot \rangle_S$ denotes the duality between the mutually adjoint function spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$, which extends the usual L_2 scalar product:

$$\langle f, g \rangle_S = \int_S f \bar{g} dS \quad \text{for } f, g \in L_2(S).$$

Now, we write Green's formula for the operator $A(\partial_x, \tau)$ in the domain Ω^+ (see Subsection 2.7.3 in [9]):

$$\begin{aligned} & \int_{\Omega^+} \left[[A(\partial_x, \tau)U]_j \bar{u}_j + [\overline{A(\partial_x, \tau)U}]_4 u_4 + [\overline{A(\partial_x, \tau)U}]_5 u_5 + \frac{1+\nu_0\tau}{\bar{\tau}} [\overline{A(\partial_x, \tau)U}]_6 u_6 + \mathcal{E}(U, \bar{U}) \right] dx \\ &= \langle \{\mathcal{T}U\}_j^+, \{u_j\}^+ \rangle_S + \langle \{\overline{\mathcal{T}U}\}_4^+, \{u_4\}^+ \rangle_S + \langle \{\overline{\mathcal{T}U}\}_5^+, \{u_5\}^+ \rangle_S + \frac{1+\nu_0\tau}{\bar{\tau}} \langle \{\overline{\mathcal{T}U}\}_6^+, \{u_6\}^+ \rangle_S, \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} \mathcal{E}(U, \bar{U}) &:= c_{rjkl} \partial_l u_k \overline{\partial_j u_r} + \rho_1 \tau^2 |u|^2 + \varkappa_{jl} \partial_l \varphi \overline{\partial_j \varphi} + a_{jl} (\partial_l \psi \overline{\partial_j \varphi} + \partial_j \varphi \overline{\partial_l \psi}) \\ &\quad + \mu_{jl} \partial_l \psi \overline{\partial_j \psi} - 2 \operatorname{Re} [p_l (1 + \nu_0 \tau) \vartheta \overline{\partial_l \varphi}] - 2 \operatorname{Re} [m_l (1 + \nu_0 \tau) \vartheta \overline{\partial_l \psi}] \\ &\quad + (1 + \nu_0 \tau) (h_0 \bar{\tau} + d_0) |\vartheta|^2 + \frac{1 + \nu_0 \tau}{\bar{\tau}} \eta_{jl} \partial_l \vartheta \overline{\partial_j \vartheta}. \end{aligned}$$

Since (U, w) is a solution of the homogeneous problem $(IM)_\tau$, from (2.18) we obtain

$$\int_{\Omega^+} \mathcal{E}(U, \bar{U}) dx = \langle \{\mathcal{T}U\}_j^+, \{u_j\}^+ \rangle_S = -\langle \{w\}^- n_j, \{u_j\}^+ \rangle_S = (\rho_2 \bar{\tau}^2)^{-1} \langle \{w\}^-, \{\partial_n w\}^- \rangle_S,$$

i.e.,

$$\langle \{w\}^-, \{\partial_n w\}^- \rangle_S = \rho_2 \bar{\tau}^2 \int_{\Omega^+} \mathcal{E}(U, \bar{U}) dx. \quad (2.19)$$

Substituting (2.19) into (2.17), we get

$$\int_{\Omega^-} |w|^2 dx + \frac{\tau^2}{\rho_2 |\tau|^4} \int_{\Omega^-} |\nabla w|^2 dx + \int_{\Omega^+} \mathcal{E}(U, \bar{U}) dx = 0. \quad (2.20)$$

Separating the imaginary part of (2.20), we obtain

$$\frac{2\sigma\omega}{\rho_2 |\tau|^4} \int_{\Omega^-} |\nabla w|^2 dx + \omega \int_{\Omega^+} \left[2\rho_1 \sigma^2 |u|^2 + (\nu_0 d_0 - h_0) |u_6|^2 + \frac{1 + 2\sigma\nu_0}{|\tau|^2} \eta_{jl} \partial_l u_6 \overline{\partial_j u_6} \right] dx = 0. \quad (2.21)$$

If $\omega \neq 0$, from (2.21) we conclude that $w = 0$ in Ω^- .

Keeping in mind that for $\omega = 0$ the inequality $\int_{\Omega^+} \mathcal{E}(U, \bar{U}) dx \geq 0$ holds (see the proof of Theorem 2.25 in Buchukuri et al. [9]), and separating the real part of (2.20), we deduce that $w = 0$ in Ω^- . Thus, $w = 0$ in Ω^- for arbitrary $\omega \in \mathbb{R}$. Therefore, from (2.19) we get $\int_{\Omega^+} \mathcal{E}(U, \bar{U}) dx = 0$ for $\sigma > \sigma_0 \geq 0$ and $\omega \in \mathbb{R}$. Due to relations (2.4) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.6), we find that

$$\begin{aligned} c_{ijk} \partial_i u_j \overline{\partial_l u_k} &\geq 0, \quad \eta_{jl} \partial_l \vartheta \overline{\partial_j \vartheta} \geq 0, \\ \left[\kappa_{jl} \partial_l \varphi \overline{\partial_j \varphi} + a_{jl} (\partial_l \psi \overline{\partial_j \varphi} + \partial_j \varphi \overline{\partial_l \psi}) + \mu_{jl} \partial_l \psi \overline{\partial_j \psi} \right] &\geq \lambda_0 (|\nabla \varphi|^2 + |\nabla \psi|^2), \end{aligned} \quad (2.22)$$

where λ_0 is a positive constant.

Using inequalities (2.22), positive definiteness of the matrix M defined by (2.5), and the inequality $\sigma(d_0 \nu_0 - h_0) > 0$ (see (2.3)), we find that (see proof of Theorem 2.25 in [9])

$$u = 0, \quad u_4 = \varphi = b_1, \quad u_5 = \psi = b_2, \quad u_6 = \vartheta = 0 \quad \text{in } \Omega^+,$$

i.e., $U = (0, 0, 0, b_1, b_2, 0)^\top$, where b_1 and b_2 are arbitrary complex constants. The homogeneous Dirichlet conditions on S_D then imply $b_1 = b_2 = 0$, i.e., $U = 0$ in Ω^- . $\square$

3 Layer potentials

3.1 Potentials associated with the Helmholtz equation

Let us introduce the single and double layer potentials

$$\begin{aligned} V_\tau(g)(x) &:= \int_S \gamma(x - y, \tau) g(y) d_y S, \\ W_\tau(f)(x) &:= \int_S \partial_{n(y)} \gamma(x - y, \tau) f(y) d_y S, \end{aligned}$$

where

$$\gamma(x, \tau) := -\frac{\exp(-\sqrt{\rho_2} \tau |x|)}{4\pi |x|}, \quad \operatorname{Re} \tau > 0,$$

is the fundamental solution of the Helmholtz equation (2.9). These potentials satisfy the decay condition (2.11) at infinity.

For these potentials the following theorems are valid (see [18, 35]).

Theorem 3.1. *Let $g \in H^{-1/2}(S)$, $f \in H^{1/2}(S)$. Then the following jump relations hold on the manifold S :*

$$\begin{aligned} \{V_\tau(g)\}^\pm &= \mathcal{H}_\tau(g), \quad \{\partial_n V_\tau(g)\}^\pm = \mp 2^{-1} g + \mathcal{K}_\tau(g), \\ \{W_\tau(f)\}^\pm &= \pm 2^{-1} f + \mathcal{N}_\tau(f), \quad \{\partial_n W_\tau(f)\}^+ = \{\partial_n W_\tau(f)\}^- =: \mathcal{L}_\tau(f), \end{aligned} \quad (3.1)$$

where $\mathcal{H}_\tau$, $\mathcal{N}_\tau$, and $\mathcal{K}_\tau$ are integral operators with weakly singular kernels:

$$\mathcal{H}_\tau(g)(z) := \int_S \gamma(z - y, \tau) g(y) d_y S, \quad z \in S, \quad (3.2)$$

$$\mathcal{N}_\tau(f)(z) := \int_S \partial_{n(y)} \gamma(z - y, \tau) f(y) d_y S, \quad z \in S, \quad (3.3)$$

$$\mathcal{K}_\tau(g)(z) := \int_S \partial_{n(z)} \gamma(z - y, \tau) g(y) d_y S, \quad z \in S, \quad (3.4)$$

while $\mathcal{L}_\tau$ is a singular integro-differential operator (a pseudodifferential operator of order 1).

Theorem 3.2. *The operator $\mathcal{M} := (-2^{-1}I_1 + \mathcal{K}_\tau)\mathcal{H}_\tau^{-1} : H^{1/2}(S) \rightarrow H^{-1/2}(S)$ is invertible. Here, I_1 is the scalar identity operator.*

The mapping properties of the above potentials and the boundary integral operators are described in Appendix (see Theorems 5.1, 5.2).

3.2 Fundamental solution and potentials associated with the pseudo-oscillation operator of generalized thermo-electro-magneto-elasticity theory

The full symbol of the pseudo-oscillation operator $A(\partial_x, \tau)$ is elliptic provided $\operatorname{Re} \tau \neq 0$, i.e. (see Chapter 3 in [9])

$$\det A(-i\xi, \tau) \neq 0 \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\},$$

where

$$A(-i\xi, \tau) = - \begin{bmatrix} [c_{rjkl}\xi_j\xi_l + \rho_1\tau^2\delta_{rk}]_{3 \times 3} & [e_{lrj}\xi_j\xi_l]_{3 \times 1} & [q_{lrj}\xi_j\xi_l]_{3 \times 1} & [-i(1 + \nu_0\tau)\lambda_{rj}\xi_j]_{3 \times 1} \\ [-e_{jkl}\xi_j\xi_l]_{1 \times 3} & \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & -i(1 + \nu_0\tau)p_j\xi_j \\ [-q_{jkl}\xi_j\xi_l]_{1 \times 3} & a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & -i(1 + \nu_0\tau)m_j\xi_j \\ [-i\tau\lambda_{kl}\xi_l]_{1 \times 3} & i\tau p_l\xi_l & i\tau m_l\xi_l & \eta_{jl}\xi_j\xi_l + \tau^2 h_0 + \tau d_0 \end{bmatrix}_{6 \times 6}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\xi, \tau)$ are locally integrable functions decaying at infinity as $O(|\xi|^{-2})$. Therefore, one can construct the fundamental matrix $\Gamma(x, \tau) = [\Gamma_{rk}(x, \tau)]_{6 \times 6}$ of the operator $A(\partial_x, \tau)$ by the distributional Fourier transform technique,

$$\Gamma(x, \tau) = F_{\xi \rightarrow x}^{-1} [A^{-1}(-i\xi, \tau)]. \quad (3.5)$$

The properties of the fundamental matrix $\Gamma(x, \tau)$ in the neighbourhood of the pole and at infinity are studied in [9, Chapter 3].

Let us introduce the single and double layer vector potentials associated with the pseudo-oscillation operator $A(\partial_x, \tau)$:

$$\begin{aligned} \mathbf{V}_\tau(h) &= \int_S \Gamma(x - y, \tau) h(y) dy, \\ \mathbf{W}_\tau(h) &= \int_S [\mathcal{P}(\partial_y, n(y), \bar{\tau}) \Gamma^\top(x - y, \tau)]^\top h(y) dy, \end{aligned}$$

where $h = (h_1, h_2, h_3, h_4, h_5, h_6)^\top$ is a density vector-function and $\mathcal{P}$ is defined in (2.8).

These pseudo-oscillation potentials have the following jump properties (see Theorem 4.4 in [9]).

Theorem 3.3. *Let $h^{(1)} \in [H^{s-1}(S)]^6$, $h^{(2)} \in [H^s(S)]^6$, $s > 0$. Then on S the following jump relations hold:*

$$\begin{aligned} \{\mathbf{V}_\tau(h^{(1)})(z)\}^\pm &= \int_S \Gamma(z - y, \tau) h^{(1)}(y) dy, \quad z \in S, \\ \{\mathbf{W}_\tau(h^{(2)})(z)\}^\pm &= \pm 2^{-1} h^{(2)}(z) + \int_S [\mathcal{P}(\partial_y, n(y), \bar{\tau}) \Gamma^\top(z - y, \tau)]^\top h^{(2)}(y) dy, \quad z \in S, \\ \{\mathcal{T}\mathbf{V}_\tau(h^{(1)})(z)\}^\pm &= \mp 2^{-1} h^{(1)}(z) + \int_S \mathcal{T}(\partial_z, n(z), \tau) \Gamma(z - y, \tau) h^{(1)}(y) dy, \quad z \in S, \\ \{\mathcal{T}\mathbf{W}_\tau(h^{(2)})(z)\}^+ &= \{\mathcal{T}\mathbf{W}_\tau(h^{(2)})(z)\}^-, \quad z \in S. \end{aligned}$$

Further, we introduce the boundary pseudodifferential operators

$$\begin{aligned}\mathbf{H}_\tau h(z) &= \int_S \Gamma(z-y, \tau) h(y) d_y S, \quad z \in S, \\ \mathbf{K}_\tau h(z) &= \int_S \mathcal{T}(\partial_z, n(z), \tau) \Gamma(z-y, \tau) h(y) d_y S, \quad z \in S, \\ \mathbf{N}_\tau h(z) &= \int_S [\mathcal{P}(\partial_y, n(y), \bar{\tau}) \Gamma^\top(z-y, \tau)]^\top h(y) d_y S, \quad z \in S, \\ \mathbf{L}_\tau h(z) &= \{\mathcal{T} \mathbf{W}_\tau h(z)\}^+ = \{\mathcal{T} \mathbf{W}_\tau(h)(z)\}^-, \quad z \in S.\end{aligned}$$

Note that $\mathbf{H}_\tau$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathbf{K}_\tau$ and $\mathbf{N}_\tau$ are singular integral operators (pseudodifferential operators of order 0), and $\mathbf{L}_\tau$ is a singular integro-differential operator (pseudodifferential operator of order 1). The mapping properties of these potentials and the corresponding boundary operators are described in Appendix (see Theorems 5.3, 5.4).

3.3 Some results for pseudodifferential equations on manifold with boundary

The Fredholm properties of strongly elliptic pseudodifferential operators on a compact manifold with boundary are studied in [1, 19, 22, 23, 42].

The spaces H_p^s and $B_{p,p}^s$ (with $s \in \mathbb{R}$, $1 < p < \infty$) denote, respectively, the well-known Bessel potential and Besov function spaces.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathcal{M} \subset \mathcal{M}_0$, we denote by $\tilde{H}_p^s(\mathcal{M})$ and $\tilde{B}_{p,p}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M})$ and $B_{p,p}^s(\mathcal{M})$, respectively:

$$\begin{aligned}\tilde{H}_p^s(\mathcal{M}) &= \{g : g \in H_p^s(\mathcal{M}_0), \text{ supp } g \subset \overline{\mathcal{M}}\}, \\ \tilde{B}_{p,q}^s(\mathcal{M}) &= \{g : g \in B_{p,q}^s(\mathcal{M}_0), \text{ supp } g \subset \overline{\mathcal{M}}\},\end{aligned}$$

where $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ stand for the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively (see [44]).

Let $\Sigma \in C^\infty$ be a compact n -dimensional, nonselfintersecting manifold with the boundary $\partial\Sigma \in C^\infty$ and let $\mathbf{A}$ be a strongly elliptic scalar pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\Sigma}$, that is, there is a positive constant c_0 such that $\text{Re } \mathfrak{S}_{\mathbf{A}}(x, \xi) \geq c_0$ for $x \in \overline{\Sigma}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Here, we denote by $\mathfrak{S}_{\mathbf{A}}(x, \xi)$ the principal homogeneous symbol of the operator $\mathbf{A}$ in some local coordinate system ($x \in \overline{\mathbb{R}_+^n}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

The Fredholm properties of strongly elliptic scalar pseudodifferential operators on a compact manifold with boundary are investigated by the Wiener–Hopf factorization method. The factorization index of the principal homogeneous symbol of the operator $\mathbf{A}$ is calculated by the following formula (see [19]):

$$\begin{aligned}\kappa(x) &= \frac{\nu}{2} + \frac{1}{2\pi i} \ln \frac{\mathfrak{S}_{\mathbf{A}}(x, 0, \dots, 0, -1)}{\mathfrak{S}_{\mathbf{A}}(x, 0, \dots, 0, +1)} \\ &= \frac{\nu}{2} + \frac{1}{2\pi} (\arg \mathfrak{S}_{\mathbf{A}}(x, 0, \dots, 0, -1) - \arg \mathfrak{S}_{\mathbf{A}}(x, 0, \dots, 0, +1)) - \frac{i}{2\pi} \ln \left| \frac{\mathfrak{S}_{\mathbf{A}}(x, 0, \dots, 0, -1)}{\mathfrak{S}_{\mathbf{A}}(x, 0, \dots, 0, +1)} \right|,\end{aligned}$$

where

$$-\frac{\pi}{2} < \arg \mathfrak{S}_{\mathbf{A}}(x, 0, \pm 1) < \frac{\pi}{2}, \quad x \in \partial\Sigma.$$

Theorem 3.4. *Let $s \in \mathbb{R}$, $1 < p < \infty$, and let $\mathbf{A}$ be a strongly elliptic pseudodifferential scalar operator of order $\nu \in \mathbb{R}$. Then the operator*

$$\mathbf{A} : \tilde{H}_p^s(\Sigma) \rightarrow H_p^{s-\nu}(\Sigma) \quad (\tilde{B}_{p,p}^s(\Sigma) \rightarrow B_{p,p}^{s-\nu}(\Sigma)) \quad (3.6)$$

is Fredholm with index zero if

$$\frac{1}{p} - 1 + \sup_{x \in \partial\Sigma} \operatorname{Re} \kappa(x) < s < \frac{1}{p} + \inf_{x \in \partial\Sigma} \operatorname{Re} \kappa(x). \quad (3.7)$$

Moreover, the null space and the index of operator (3.6) are the same for all s and p satisfying inequality (3.7).

4 Existence of solutions to the interaction Problem $(IM)_\tau$

By Theorems 5.4 and 5.2 (see Appendix), the operators

$$\mathbf{H}_\tau : [H^s(S)]^6 \rightarrow [H^{s+1}(S)]^6 \text{ and } \mathcal{H}_\tau : H^s(S) \rightarrow H^{s+1}(S)$$

are invertible for all $s \in \mathbb{R}$ and we can look for a solution pair (U, w) of the problem $(IM)_\tau$ in the form of single layer potentials:

$$U = \mathbf{V}_\tau(\mathbf{H}_\tau^{-1}g) \text{ in } \Omega^+, \quad w = V_\tau(\mathcal{H}_\tau^{-1}h) \text{ in } \Omega^-,$$

where $g = (\tilde{g}, g_4, g_5, g_6)^\top \in [H^{1/2}(S)]^6$, $\tilde{g} = (g_1, g_2, g_3)^\top$, $h \in H^{1/2}(S)$ are unknown densities. Theorems 5.1, 5.3, and 5.4 imply the inclusion $U \in [H^1(\Omega^+)]^6$ and $w \in H^1(\Omega^-)$.

The transmission conditions (2.13), (2.14), the mixed type conditions (2.15), (2.16) and Theorems 3.1 and 3.3 lead to the following system of pseudodifferential equations with respect to the unknowns $\tilde{g}$, g_4 , g_5 , g_6 , and h :

$$\tilde{g} \cdot n + (\rho_2 \tau^2)^{-1} (2^{-1} I_1 + \mathcal{K}_\tau) \mathcal{H}_\tau^{-1} h = g_0 \text{ on } S, \quad (4.1)$$

$$[(-2^{-1} I_6 + \mathbf{K}_\tau) \mathbf{H}_\tau^{-1} g]_j + n_j h = f_j \text{ on } S, \quad j = 1, 2, 3, \quad (4.2)$$

$$r_{SD} g_r = f_r^{(D)} \text{ on } S_D, \quad r = 4, 5, 6, \quad (4.3)$$

$$r_{SN} [(-2^{-1} I_6 + \mathbf{K}_\tau) \mathbf{H}_\tau^{-1} g]_r = f_r^{(N)} \text{ on } S_N, \quad r = 4, 5, 6. \quad (4.4)$$

Here and in what follows, I_m stands for the $m \times m$ unit matrix.

The matrix operator generated by the left-hand side expressions in system (4.1)–(4.4) is given by

$$\mathcal{P}_{\tau, M} := \begin{pmatrix} [n]_{1 \times 3} & 0 & 0 & 0 & (\rho_2 \tau^2)^{-1} \mathcal{M} \\ [\mathcal{A}_\tau^{jk}]_{3 \times 3} & [\mathcal{A}_\tau^{j4}]_{3 \times 1} & [\mathcal{A}_\tau^{j5}]_{3 \times 1} & [\mathcal{A}_\tau^{j6}]_{3 \times 1} & [n]_{3 \times 1} \\ [0]_{1 \times 3} & r_{SD} I_1 & 0 & 0 & 0 \\ [0]_{1 \times 3} & 0 & r_{SD} I_1 & 0 & 0 \\ [0]_{1 \times 3} & 0 & 0 & r_{SD} I_1 & 0 \\ r_{SN} [\mathcal{A}_\tau^{4j}]_{1 \times 3} & r_{SN} [\mathcal{A}_\tau^{44}] & r_{SN} [\mathcal{A}_\tau^{45}] & r_{SN} [\mathcal{A}_\tau^{46}] & 0 \\ r_{SN} [\mathcal{A}_\tau^{5j}]_{1 \times 3} & r_{SN} [\mathcal{A}_\tau^{54}] & r_{SN} [\mathcal{A}_\tau^{55}] & r_{SN} [\mathcal{A}_\tau^{56}] & 0 \\ r_{SN} [\mathcal{A}_\tau^{6j}]_{1 \times 3} & r_{SN} [\mathcal{A}_\tau^{64}] & r_{SN} [\mathcal{A}_\tau^{65}] & r_{SN} [\mathcal{A}_\tau^{66}] & 0 \end{pmatrix}_{10 \times 7}, \quad j, k = 1, 2, 3,$$

where

$$\mathcal{A}_\tau := (-2^{-1} I_6 + \mathbf{K}_\tau) \mathbf{H}_\tau^{-1} = [\mathcal{A}_\tau^{jk}]_{6 \times 6}, \quad j, k = \overline{1, 6}, \quad \mathcal{M} = (-2^{-1} I_1 + \mathcal{K}_\tau) \mathcal{H}_\tau^{-1}$$

are the Steklov–Poincaré type operators. These operators are strongly elliptic pseudodifferential operators of order 1 (see [8, 10] for details).

By Theorems 5.2 and 5.4 (see Appendix), the operator $\mathcal{P}_{\tau, M}$ possesses the following mapping property:

$$\mathcal{P}_{\tau, M} : [H^s(S)]^7 \rightarrow [H^{s-1}(S)]^4 \times [H^s(S_D)]^3 \times [H^{s-1}(S_N)]^3, \quad s \in \mathbb{R}.$$

Next, we show that system (4.1)–(4.4) is uniquely solvable in the space $[H^{1/2}(S)]^7$.

Any extension of the Dirichlet data $f_r^{(D)} \in H^{1/2}(S_D)$, $r = 4, 5, 6$, to the entire boundary S has the form $G_0^{(r)} + g_0^{(r)} \in H^{1/2}(S)$, $r = 4, 5, 6$, where $G_0^{(r)} \in H^{1/2}(S)$ are some fixed extensions of $f_r^{(D)}$, respectively, while $g_0^{(r)} \in \tilde{H}^{1/2}(S_N)$.

We rewrite system (4.1)–(4.4) in the equivalent form with respect to $\tilde{g}, g_4, g_5, g_6, h, g_0^{(4)}, g_0^{(5)}, g_0^{(6)}$:

$$\tilde{g} \cdot n + (\rho_2 \tau^2)^{-1} \mathcal{M}(h) = f_0 \text{ on } S, \quad (4.5)$$

$$[\mathcal{A}_\tau(\tilde{g}, g_4, g_5, g_6)^\top]_j + n_j h = f_j \text{ on } S, \quad j = 1, 2, 3, \quad (4.6)$$

$$g_r - g_0^{(r)} = G_0^{(r)} \text{ on } S, \quad r = 4, 5, 6, \quad (4.7)$$

$$r_{S_N}[\mathcal{A}_\tau(\tilde{g}, g_4, g_5, g_6)^\top]_r = f_r^{(N)} \text{ on } S_N, \quad r = 4, 5, 6. \quad (4.8)$$

Remark 4.1. Systems (4.1)–(4.4) and (4.5)–(4.8) are equivalent in the following sense:

- (i) if $(\tilde{g}, g_4, g_5, g_6, h)$ is a solution of system (4.1)–(4.4), then $(\tilde{g}, g_4, g_5, g_6, h, g_0^{(4)}, g_0^{(5)}, g_0^{(6)})$, where $g_0^{(r)} = g_r - G_0^{(r)}$, $r = 4, 5, 6$, and $G_0^{(r)}$ are some fixed extensions of the functions $f_r^{(D)}$ from S_D to the entire boundary S , is a solution of system (4.5)–(4.8).
- (ii) If $(\tilde{g}, g_4, g_5, g_6, h, g_0^{(4)}, g_0^{(5)}, g_0^{(6)})$ is a solution of system (4.5)–(4.8), then $(\tilde{g}, g_4, g_5, g_6, h)$ is a solution of system (4.1)–(4.4).

From the results obtained in [13], system (4.5)–(4.8) is uniquely solvable with respect to $\tilde{g}, g_4, g_5, g_6, h$ for any extensions $G_0^{(r)} + g_0^{(r)}$ of the Dirichlet boundary data $f_r^{(D)}$, $r = 4, 5, 6$. We have to find the extensions of $f_r^{(D)}$ (i.e., the functions $g_0^{(r)}$) such the equations (4.8) are satisfied.

The operator corresponding to system (4.5)–(4.8) has the following form:

$$\mathcal{P}_{\tau, M}^{(1)} := \begin{pmatrix} [n]_{1 \times 3} & 0 & 0 & 0 & (\rho_2 \tau^2)^{-1} \mathcal{M} & 0 & 0 & 0 \\ [\mathcal{A}_\tau^{jk}]_{3 \times 3} & [\mathcal{A}_\tau^{j4}]_{3 \times 1} & [\mathcal{A}_\tau^{j5}]_{3 \times 1} & [\mathcal{A}_\tau^{j6}]_{3 \times 1} & [n]_{3 \times 1} & [0]_{3 \times 1} & [0]_{3 \times 1} & [0]_{3 \times 1} \\ [0]_{1 \times 3} & I_1 & 0 & 0 & 0 & -I_1 & 0 & 0 \\ [0]_{1 \times 3} & 0 & I_1 & 0 & 0 & 0 & -I_1 & 0 \\ [0]_{1 \times 3} & 0 & 0 & I_1 & 0 & 0 & 0 & -I_1 \\ r_{S_N}[\mathcal{A}_\tau^{4j}]_{1 \times 3} & r_{S_N}[\mathcal{A}_\tau^{44}] & r_{S_N}[\mathcal{A}_\tau^{45}] & r_{S_N}[\mathcal{A}_\tau^{46}] & 0 & 0 & 0 & 0 \\ r_{S_N}[\mathcal{A}_\tau^{5j}]_{1 \times 3} & r_{S_N}[\mathcal{A}_\tau^{54}] & r_{S_N}[\mathcal{A}_\tau^{55}] & r_{S_N}[\mathcal{A}_\tau^{56}] & 0 & 0 & 0 & 0 \\ r_{S_N}[\mathcal{A}_\tau^{6j}]_{1 \times 3} & r_{S_N}[\mathcal{A}_\tau^{64}] & r_{S_N}[\mathcal{A}_\tau^{65}] & r_{S_N}[\mathcal{A}_\tau^{66}] & 0 & 0 & 0 & 0 \end{pmatrix}_{10 \times 10}, \quad j, k = 1, 2, 3,$$

The operator $\mathcal{P}_{\tau, M}^{(1)}$ is bounded in the following spaces:

$$\mathcal{P}_{\tau, M}^{(1)} : [H^s(S)]^7 \times [\tilde{H}^s(S_N)]^3 \rightarrow [H^{s-1}(S)]^4 \times [H^s(S)]^3 \times [H^{s-1}(S_N)]^3, \quad s \in \mathbb{R}.$$

Let us consider the system generated by equations (4.5)–(4.8) with respect to the unknowns $\tilde{g}, g_4, g_5, g_6$, and h :

$$\tilde{g} \cdot n + (\rho_2 \tau^2)^{-1} \mathcal{M}(h) = f_0 \text{ on } S, \quad (4.9)$$

$$[\mathcal{A}_\tau(\tilde{g}, g_4, g_5, g_6)^\top]_j + n_j h = f_j \text{ on } S, \quad j = 1, 2, 3, \quad (4.10)$$

$$g_r = G_0^{(r)} + g_0^{(r)} \text{ on } S, \quad r = 4, 5, 6. \quad (4.11)$$

System (4.9)–(4.11) is uniquely solvable in the class $[H^{1/2}(S)]^7$ and the corresponding operator of this system

$$\mathcal{P}_{\tau, D} : [H^s(S)]^7 \rightarrow [H^{s-1}(S)]^4 \times [H^s(S)]^3$$

is invertible for all $s \in \mathbb{R}$ (cf. [13, 14]).

Define h from equation (4.9) (see Theorem 3.2) and substitute it in equation (4.10). Then we obtain a uniquely solvable system with respect to $\tilde{g}, g_4, g_5, g_6$:

$$\mathcal{B}_\tau(\tilde{g}, g_4, g_5, g_6)^\top = \Psi \text{ on } S, \quad (4.12)$$

where

$$\mathcal{B}_\tau = \begin{pmatrix} [C_\tau]_{3 \times 3} & [\mathcal{A}_\tau^{j4}]_{3 \times 1} & [\mathcal{A}_\tau^{j5}]_{3 \times 1} & [\mathcal{A}_\tau^{j6}]_{3 \times 1} \\ [0]_{1 \times 3} & I_1 & 0 & 0 \\ [0]_{1 \times 3} & 0 & I_1 & 0 \\ [0]_{1 \times 3} & 0 & 0 & I_1 \end{pmatrix}_{6 \times 6},$$

$$[C_\tau]_{3 \times 3} = [\mathcal{A}_\tau^{jk}]_{3 \times 3} - (\rho_2 \tau^2) [n_j \mathcal{M}^{-1} n_k]_{3 \times 3}, \quad j, k = 1, 2, 3,$$

$$\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in [H^{-1/2}(S)]^3 \times [H^{1/2}(S)]^3,$$

$$\Psi_j = f_j - (\rho_2 \tau^2) n_j \mathcal{M}^{-1} f_0, \quad j = 1, 2, 3, \quad \Psi_r = G_0^{(r)} + g_0^{(r)}, \quad r = 4, 5, 6.$$

Since the operator $\mathcal{B}_\tau : [H^{1/2}(S)]^6 \rightarrow [H^{-1/2}(S)]^3 \times [H^{1/2}(S)]^3$ is invertible, from equation (4.12) we deduce

$$(\tilde{g}, g_4, g_5, g_6)^\top = \mathcal{B}_\tau^{-1} \Psi \text{ on } S. \quad (4.13)$$

After substituting (4.13) into equations (4.8), on the manifold S_N with boundary, we obtain the following scalar pseudodifferential equations with respect to the unknown functions $g_0^{(4)}$, $g_0^{(5)}$ and $g_0^{(6)}$:

$$\begin{aligned} r_{S_N} \mathbf{A}_\tau^{(1)} g_0^{(4)} &= F^{(1)} \text{ on } S_N, \\ r_{S_N} \mathbf{A}_\tau^{(2)} g_0^{(5)} &= F^{(2)} \text{ on } S_N, \\ r_{S_N} \mathbf{A}_\tau^{(3)} g_0^{(6)} &= F^{(2)} \text{ on } S_N, \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_\tau^{(1)} g_0^{(4)} &:= [\mathcal{A}_\tau \mathcal{B}_\tau^{-1} (0, 0, 0, g_0^{(4)}, 0, 0)^\top]_4, \\ F^{(1)} &:= f_1^{(N)} - r_{S_N} [\mathcal{A}_\tau \mathcal{B}_\tau^{-1} (\Psi_1, \Psi_2, \Psi_3, G_0^{(4)}, 0, 0)^\top]_4 \in H^{-1/2}(S_N), \\ \mathbf{A}_\tau^{(2)} g_0^{(5)} &:= [\mathcal{A}_\tau \mathcal{B}_\tau^{-1} (0, 0, 0, 0, g_0^{(5)}, 0)^\top]_5, \\ F^{(2)} &:= f_2^{(N)} - r_{S_N} [\mathcal{A}_\tau \mathcal{B}_\tau^{-1} (\Psi_1, \Psi_2, \Psi_3, 0, G_0^{(5)}, 0)^\top]_5 \in H^{-1/2}(S_N), \\ \mathbf{A}_\tau^{(3)} g_0^{(6)} &:= [\mathcal{A}_\tau \mathcal{B}_\tau^{-1} (0, 0, 0, 0, 0, g_0^{(6)})^\top]_6, \\ F^{(3)} &:= f_3^{(N)} - r_{S_N} [\mathcal{A}_\tau \mathcal{B}_\tau^{-1} (\Psi_1, \Psi_2, \Psi_3, 0, 0, G_0^{(6)})^\top]_6 \in H^{-1/2}(S_N). \end{aligned}$$

Thus, system (4.5)–(4.8) can be reduced to the following equivalent system with respect to the unknowns $\tilde{g}$, g_4 , g_5 , g_6 , h , $g_0^{(4)}$, $g_0^{(5)}$, $g_0^{(6)}$:

$$\mathcal{P}_{\tau, M}^{(2)}(\tilde{g}, g_4, g_5, g_6, h, g_0^{(4)}, g_0^{(5)}, g_0^{(6)})^\top = \Phi,$$

where

$$\begin{aligned} \mathcal{P}_{\tau, M}^{(2)} &:= \begin{pmatrix} [n]_{1 \times 3} & 0 & 0 & 0 & (\rho_2 \tau^2)^{-1} \mathcal{M} & 0 & 0 & 0 \\ [\mathcal{A}_\tau^{jk}]_{3 \times 3} & [\mathcal{A}_\tau^{j4}]_{3 \times 1} & [\mathcal{A}_\tau^{j5}]_{3 \times 1} & [\mathcal{A}_\tau^{j6}]_{3 \times 1} & [n]_{3 \times 1} & [0]_{3 \times 1} & [0]_{3 \times 1} & [0]_{3 \times 1} \\ [0]_{1 \times 3} & I_1 & 0 & 0 & 0 & -I_1 & 0 & 0 \\ [0]_{1 \times 3} & 0 & I_1 & 0 & 0 & 0 & -I_1 & 0 \\ [0]_{1 \times 3} & 0 & 0 & I_1 & 0 & 0 & 0 & -I_1 \\ [0]_{1 \times 3} & 0 & 0 & 0 & 0 & r_{S_N} \mathbf{A}_\tau^{(1)} & 0 & 0 \\ [0]_{1 \times 3} & 0 & 0 & 0 & 0 & 0 & r_{S_N} \mathbf{A}_\tau^{(2)} & 0 \\ [0]_{1 \times 3} & 0 & 0 & 0 & 0 & 0 & 0 & r_{S_N} \mathbf{A}_\tau^{(3)} \end{pmatrix}_{10 \times 10} \\ &= \begin{pmatrix} \mathcal{P}_{\tau, D} & \tilde{I}_{7 \times 1}^* & I_{7 \times 1}^* & \tilde{I}_{7 \times 1} \\ [0]_{1 \times 6} & r_{S_N} \mathbf{A}_\tau^{(1)} & 0 & 0 \\ [0]_{1 \times 6} & 0 & r_{S_N} \mathbf{A}_\tau^{(2)} & 0 \\ [0]_{1 \times 6} & 0 & 0 & r_{S_N} \mathbf{A}_\tau^{(3)} \end{pmatrix}_{10 \times 10}, \quad j, k = 1, 2, 3, \end{aligned}$$

with $I_{7 \times 1}^* = (0, 0, 0, 0, -I_1, 0, 0)^\top$, $\tilde{I}_{7 \times 1}^* = (0, 0, 0, 0, 0, -I_1, 0)^\top$, $\tilde{I}_{7 \times 1} = (0, 0, 0, 0, 0, 0, -I_1)^\top$ and $\Phi = (f_0, f_1, f_2, f_3, G_0^{(4)}, G_0^{(5)}, G_0^{(6)}, F^{(1)}, F^{(2)}, F^{(3)})^\top \in [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^3 \times [H^{-1/2}(S_N)]^3$.

Therefore, the operator

$$\mathcal{P}_{\tau, M}^{(2)} : [H^{1/2}(S)]^7 \times [\tilde{H}^{1/2}(S_N)]^3 \rightarrow [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^3 \times [H^{-1/2}(S_N)]^3$$

is Fredholm with index zero if the operators $r_{S_N} \mathbf{A}_\tau^{(j)} : \tilde{H}^{1/2}(S_N) \rightarrow H^{-1/2}(S_N)$, $j = 1, 2, 3$, are Fredholm with index zero.

The pseudodifferential operators $r_{S_N} \mathbf{A}_\tau^{(j)}$, $j = 1, 2, 3$, are strongly elliptic (cf. [13]). Therefore, it follows from Theorem 3.4 for $p = 2$ that the operators $r_{S_N} \mathbf{A}_\tau^{(j)} : \tilde{H}^s(S_N) \rightarrow H^{s-1}(S_N)$ are Fredholm with index zero if

$$-\frac{1}{2} + \sup_{x \in \partial S_N} \operatorname{Re} \kappa_j(x) < s < \frac{1}{2} + \inf_{x \in \partial S_N} \operatorname{Re} \kappa_j(x), \quad (4.14)$$

where $\kappa_j(x)$, $j = 1, 2$, are the factorization indices of the symbols $\mathfrak{S}_{\mathbf{A}_\tau^{(j)}}(x, \xi) = \mathfrak{S}_{\mathbf{A}_\tau^{(j)}}(\xi)$, $j = 1, 2, 3$, at the “frozen” point $x \in \partial S_N$, whose real part is calculated by the formula (see Subsection 3.3)

$$\begin{aligned} \operatorname{Re} \kappa_j(x) &= \frac{1}{2} + \frac{1}{2\pi} \arg \mathfrak{S}_{\mathbf{A}_\tau^{(j)}}(x, 0, -1) - \frac{1}{2\pi} \arg \mathfrak{S}_{\mathbf{A}_\tau^{(j)}}(x, 0, +1), \\ -\frac{\pi}{2} &< \arg \mathfrak{S}_{\mathbf{A}_\tau^{(j)}}(x, 0, \pm 1) < \frac{\pi}{2}, \quad j = 1, 2, \quad x \in \partial S_N. \end{aligned} \quad (4.15)$$

It is evident that $0 < \operatorname{Re} \kappa_j(x) < 1$, $j = 1, 2$, for $x \in \partial S_N$ and $s = \frac{1}{2}$ satisfies condition (4.14).

Whence we find that the operators

$$\mathcal{P}_{\tau, M}^{(1)}, \mathcal{P}_{\tau, M}^{(2)} : [H^s(S)]^7 \times [\tilde{H}^s(S_N)]^3 \rightarrow [H^{s-1}(S)]^4 \times [H^s(S)]^3 \times [H^{s-1}(S_N)]^3$$

are Fredholm with index zero for all s satisfying

$$-\frac{1}{2} + \sup_{x \in \partial S_N, j=1,2} \operatorname{Re} \kappa_j(x) < s < \frac{1}{2} + \inf_{x \in \partial S_N, j=1,2} \operatorname{Re} \kappa_j(x) \quad (4.16)$$

(see Theorem 3.4).

We now show that the operator $\mathcal{P}_{\tau, M}^{(1)}$ is injective.

Let $(\tilde{g}, g_4, g_5, g_6, h, g_0^{(4)}, g_0^{(5)}, g_0^{(6)})^\top \in [H^{1/2}(S)]^7 \times [\tilde{H}^{1/2}(S_N)]^3$ be some solutions of the homogeneous system

$$\mathcal{P}_{\tau, M}^{(1)}(\tilde{g}, g_4, g_5, g_6, h, g_0^{(4)}, g_0^{(5)}, g_0^{(6)})^\top = 0.$$

It is clear that $g_r = g_0^{(r)}$, $r = 4, 5, 6$, and the potentials

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \text{ in } \Omega^+, \quad w = V_\tau(\mathcal{H}_\tau^{-1} h) \text{ in } \Omega^-$$

solve the homogeneous problem $(IM)_\tau$.

It follows from the uniqueness result for the problem $(IM)_\tau$ (see Theorem 2.1) that $U = 0$ in Ω^+ and $w = 0$ in Ω^- . Then $\{U\}^+ = (\tilde{g}, g_0^{(4)}, g_0^{(5)}, g_0^{(6)})^\top = 0$ on S and $\{\partial_n w\}^- = \mathcal{M}h = 0$ on S . From the invertibility of the operator $\mathcal{M}$ (see Theorem 3.2), we obtain that $h = 0$ on S .

Hence we obtain that system (4.5)–(4.8) is uniquely solvable. It means that the corresponding operator of system (4.5)–(4.8)

$$\mathcal{P}_{\tau, M}^{(1)} : [H^{1/2}(S)]^7 \times [\tilde{H}^{1/2}(S_N)]^3 \rightarrow [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^3 \times [H^{-1/2}(S_N)]^3$$

is invertible.

The null space and the index of the operator $\mathcal{P}_{\tau, M}^{(1)}$ are the same for all s satisfying (4.16).

Hence it follows that the operator

$$\mathcal{P}_{\tau, M}^{(1)} : [H^s(S)]^7 \times [\tilde{H}^s(S_N)]^3 \rightarrow [H^{s-1}(S)]^4 \times [H^s(S)]^3 \times [H^{s-1}(S_N)]^3$$

is invertible for all s satisfying (4.16).

Note that the solutions $\tilde{g}$, g_4 , g_5 , g_6 , h of system (4.5)–(4.8) do not depend on the extensions $G_0^{(r)} + g_0^{(r)}$ of the functions $f_r^{(D)}$, $r = 4, 5, 6$, respectively, whereas $g_0^{(r)}$ do. However, the sums $G_0^{(r)} + g_0^{(r)}$ are defined uniquely for $r = 4, 5, 6$.

Due to Remark 4.1, we conclude that the operator

$$\mathcal{P}_{\tau, M} : [H^s(S)]^7 \rightarrow [H^{s-1}(S)]^4 \times [H^s(S_D)]^3 \times [H^{s-1}(S_N)]^3$$

is invertible for all s satisfying (4.16).

Thus the following existence theorem holds for the problem $(IM)_\tau$.

Theorem 4.1. *Let $\tau = \sigma + i\omega$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_r^{(D)} \in H^{1/2}(S_D)$ and $f_r^{(N)} \in H^{-1/2}(S_N)$, $r = 4, 5, 6$. Then problem $(IM)_\tau$ has a unique solution (U, w) , $U \in [H^1(\Omega^+)]^6$, $w \in H^1(\Omega^-)$, which is represented by the potentials*

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \text{ in } \Omega^+, \quad w = V_\tau(\mathcal{H}_\tau^{-1} h) \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^6$ and $h \in H^{1/2}(S)$ are defined from the uniquely solvable system (4.1)–(4.4). If the conditions $f_0 \in H^{s-1}(S)$, $f_j \in H^{s-1}(S)$, $j = 1, 2, 3$, $f_r^{(D)} \in H^s(S_D)$, $f_r^{(N)} \in H^{s-1}(S_N)$, $r = 4, 5, 6$, hold for the data in (2.13)–(2.16) and

$$\frac{1}{2} < s < \frac{1}{2} + \inf_{x \in \partial S_N, j=1,2} \operatorname{Re} \kappa_j(x), \quad (4.17)$$

then the solution (U, w) of the mixed type problem $(IM)_\tau$ exists, is unique, and $U \in [H^{s+1/2}(\Omega^+)]^6$, $w \in H^{s+1/2}(\Omega^-)$.

Moreover, if the conditions $f_0 \in H^s(S)$, $f_j \in H^{s-1}(S)$, $j = 1, 2, 3$, $f_r^{(D)} \in H^s(S_D)$, $f_r^{(N)} \in H^{s-1}(S_N)$, $r = 4, 5, 6$, hold for the data in (2.13)–(2.16) and (4.17) is satisfied, then the solution (U, w) of the mixed type problem $(IM)_\tau$ exists, is unique, and $U \in [H^{s+1/2}(\Omega^+)]^6$, $w \in H^{s+3/2}(\Omega^-)$.

Remark 4.2. In the last statement of Theorem 4.1, the smoothness of w follows from the representation of h (see (4.1)),

$$h = -\rho_2 \tau^2 \mathcal{M}^{-1}(\tilde{g} \cdot n) + \rho_2 \tau^2 \mathcal{M}^{-1}(f_0) \in H^{s+1}(S) \text{ on } S,$$

and the mapping properties of potentials W_τ and V_τ (see Appendix, Theorem 5.1), where $f_0 \in H^s(S)$, $\tilde{g} \in [H^s(S)]^3$ and s satisfies (4.17).

Note that the invertibility of the operator $\mathcal{P}_{\tau, M}^{(1)}$ implies that of the operator $\mathcal{P}_{\tau, M}^{(2)}$. Therefore, the following Remark holds.

Remark 4.3. The operators $r_{S_N} \mathbf{A}_\tau^{(j)} : \tilde{H}^s(S_N) \rightarrow H^{s-1}(S_N)$, $j = 1, 2, 3$, are invertible for all s satisfying (4.14).

5 Appendix: Mapping Properties of Potentials

For the reader's convenience, we collect here several results describing the properties of the layer potentials. We retain the notation used in the main text of the paper.

For the potentials associated with the Helmholtz equation, the following theorems hold (see [18, 35]).

Theorem 5.1. *Let $s \in \mathbb{R}$, $S \in C^\infty$. Then the single and double layer scalar potentials can be extended to the following continuous operators:*

$$\begin{aligned} V_\tau : H^s(S) &\rightarrow H^{s+3/2}(\Omega^+), \quad V_\tau : H^s(S) \rightarrow H^{s+3/2}(\Omega^-), \\ W_\tau : H^s(S) &\rightarrow H^{s+1/2}(\Omega^+), \quad W_\tau : H^s(S) \rightarrow H^{s+1/2}(\Omega^-). \end{aligned}$$

Theorem 5.2. *Let $s \in \mathbb{R}$, $S \in C^\infty$. Then the operators (see (3.1)–(3.4))*

$$\begin{aligned} \mathcal{H}_\tau : H^s(S) &\rightarrow H^{s+1}(S), \quad \pm 2^{-1}I_1 + \mathcal{N}_\tau : H^s(S) \rightarrow H^s(S), \\ \mathcal{L}_\tau : H^{s+1}(S) &\rightarrow H^s(S), \quad \pm 2^{-1}I_1 + \mathcal{K}_\tau : H^s(S) \rightarrow H^s(S) \end{aligned}$$

are continuous and invertible for $\operatorname{Re} \tau > 0$.

For the vector potentials associated with the pseudo-oscillation operators $A(\partial, \tau)$ and $A^*(\partial, \tau)$, the following theorems hold (see [9, 10]).

Theorem 5.3. *Let $s \in \mathbb{R}$, $S \in C^\infty$. Then the vector potentials $\mathbf{V}_\tau$ and $\mathbf{W}_\tau$ are continuous in the following spaces*

$$\begin{aligned} \mathbf{V}_\tau : [H^s(S)]^6 &\rightarrow [H^{s+3/2}(\Omega^+)]^6, \quad \mathbf{V}_\tau : [H^s(S)]^6 \rightarrow [H_{loc}^{s+3/2}(\Omega^-)]^6, \\ \mathbf{W}_\tau : [H^s(S)]^6 &\rightarrow [H^{s+1/2}(\Omega^+)]^6, \quad \mathbf{W}_\tau : [H^s(S)]^6 \rightarrow [H_{loc}^{s+1/2}(\Omega^-)]^6. \end{aligned}$$

Theorem 5.4. *Let $s \in \mathbb{R}$, $S \in C^\infty$. Then the operators*

$$\mathbf{H}_\tau : [H^s(S)]^6 \rightarrow [H^{s+1}(S)]^6, \quad \mathbf{K}_\tau, \mathbf{N}_\tau : [H^s(S)]^6 \rightarrow [H^s(S)]^6, \quad \mathbf{L}_\tau : [H^s(S)]^6 \rightarrow [H^{s-1}(S)]^6$$

are bounded. The operators $\mathbf{H}_\tau$ and $\mathbf{L}_\tau$ are strongly elliptic pseudodifferential operators of order -1 and 1 , respectively, while the operators $\pm 2^{-1}I_6 + \mathbf{K}_\tau$ and $\pm 2^{-1}I_6 + \mathbf{N}_\tau$ are elliptic pseudodifferential operators of order 0 .

Moreover, the operators $\mathbf{H}_\tau$, $2^{-1}I_6 + \mathbf{K}_\tau$, and $2^{-1}I_6 + \mathbf{N}_\tau$ are invertible, whereas the operators $\mathbf{L}_\tau$, $-2^{-1}I_6 + \mathbf{N}_\tau$, and $-2^{-1}I_6 + \mathbf{K}_\tau$ are the Fredholm operators with zero index.

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