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MORE REFINEMENT ON THE HERMITE–HADAMARD INEQUALITY

Abstract. This study aims to further refine the Bullen inequality, the dual Bullen inequality, and the Hermite–Hadamard inequality. Subsequently, we present a specific case involving newly derived second Bullen-type inequalities. We improve these inequalities by applying a convex non-decreasing function GZ and a convex non-increasing function DF .

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1 Introduction

The Hermite–Hadamard inequality, which is considered to be a fundamental concept in the field of mathematical analysis, particularly in relation to convex functions, has attracted considerable attention since its first introduction (see [1, 4, 8, 9]). Within the context of a convex function, this inequality establishes a fundamental connection between the mean values of the function across an interval and the values of the function at its limit points.

A multitude of extensions, improvements, and variants—collectively referred to as Hermite–Hadamard-type inequalities—have been developed due to its numerous implications, which extend far beyond its original formulation. These developments clearly demonstrate that convex analysis will continue to play an important role in addressing challenging mathematical problems. The Hermite–Hadamard inequality [3] is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}, \quad (1.1)$$

where $f : I \rightarrow \mathbb{R}$ is a convex function and $a, b \in I \subseteq \mathbb{R}$ with $a < b$. Midpoint and trapezoid inequalities provide boundaries on the left and right sides of inequality (1.1), respectively.

1. The midpoint estimate

$$\frac{1}{b-a} \int_a^b f(t) dt \approx f\left(\frac{a+b}{2}\right)$$

for functions characterized by the convexity of the absolute values of their first derivatives is presented in [7] as follows:

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

2. The trapezoid estimate

$$\frac{1}{b-a} \int_a^b f(t) dt \approx \frac{f(a) + f(b)}{2}$$

has been established in [2] as

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

An improvement to the Hermite–Hadamard inequality, called Bullen-type inequality, was provided by Tseng et al. [10] in the form

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] \leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (1.2)$$

The third inequality is known in the literature as the Bullen inequality, which is an improvement to the trapezoid inequality. In 2016, the Bullen estimate was articulated as follows [5]:

$$\left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{16} [|f'(a)| + |f'(b)|].$$

We refer to the second inequality in (1.2) as the dual Bullen inequality, which is a refinement to the midpoint inequality; this inequality is a subject of our investigation.

The following significant inequalities are required for the main result.

Lemma 1.1 ([6]). *Let $a \leq y_1 \leq z_1 \leq z_2 \leq y_2 \leq b$. If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function and $z_1 + z_2 = y_1 + y_2$, then*

$$f(z_1) + f(z_2) \leq f(y_1) + f(y_2).$$

Lemma 1.2 ([11]). *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function and $A, B, C, D \in [a, b]$ and $A \leq C \leq B$ and $A \leq D \leq B$ with $A + B = C + D$, then*

$$f(C) + f(D) \leq f(A) + f(B). \quad (1.3)$$

2 Basic lemmas

Lemma 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function, $t \in [0, 1]$ and for $i = 1, 2, 3, 4$, let $G_{t,i} : [0, \frac{b-a}{8}] \rightarrow \mathbb{R}$ be defined by*

$$\begin{aligned} G_{t,1}(x) &= \left[f\left(\frac{7a+b}{8} - tx\right) + f\left(\frac{7a+b}{8} + tx\right) \right], \\ G_{t,2}(x) &= \left[f\left(\frac{5a+3b}{8} - tx\right) + f\left(\frac{5a+3b}{8} + tx\right) \right], \\ G_{t,3}(x) &= \left[f\left(\frac{3a+5b}{8} - tx\right) + f\left(\frac{3a+5b}{8} + tx\right) \right], \\ G_{t,4}(x) &= \left[f\left(\frac{a+7b}{8} - tx\right) + f\left(\frac{a+7b}{8} + tx\right) \right]. \end{aligned}$$

Then $G_{t,i}$ ($i = 1, 2, 3, 4$) is a convex non-decreasing function on $[0, \frac{b-a}{8}]$.

Proof. If f is convex and g is linear, then the composition $(f \circ g)$ is convex. The sum of two convex functions is convex; hence, $G_{t,i}$ are convex on the interval $[0, \frac{b-a}{8}]$. Next, letting $t \in [0, 1]$ and $x_1, x_2 \in [0, \frac{b-a}{8}]$, where $x_1 < x_2$, we have

$$\frac{7a+b}{8} - tx_2 \leq \frac{7a+b}{8} - tx_1 \leq \frac{7a+b}{8} + tx_1 \leq \frac{7a+b}{8} + tx_2.$$

Given that f is a convex function, the application of Lemma 1.1 yields

$$f\left(\frac{7a+b}{8} - tx_1\right) + f\left(\frac{7a+b}{8} + tx_1\right) \leq f\left(\frac{7a+b}{8} - tx_2\right) + f\left(\frac{7a+b}{8} + tx_2\right),$$

then $G_{t,1}(x_1) \leq G_{t,1}(x_2)$, which indicates that $G_{t,1}$ is a non-decreasing function on the interval $[0, \frac{b-a}{8}]$. Analogously for the functions $G_{t,2}$, $G_{t,3}$, and $G_{t,4}$. $\square$

Lemma 2.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function, $t \in [0, 1]$, and for $i = 1, 2, 3, 4$, let $H_{t,i} : [0, \frac{b-a}{4}] \rightarrow \mathbb{R}$ be defined by*

$$\begin{aligned} H_{t,1}(x) &= \left[f(tx+a) + f\left(\frac{3a+b}{4} - tx\right) \right], \\ H_{t,2}(x) &= \left[f\left(tx + \frac{3a+b}{4}\right) + f\left(\frac{a+b}{2} - tx\right) \right], \\ H_{t,3}(x) &= \left[f\left(tx + \frac{a+b}{2}\right) + f\left(\frac{a+3b}{4} - tx\right) \right], \\ H_{t,4}(x) &= \left[f\left(tx + \frac{a+3b}{4}\right) + f(b-tx) \right]. \end{aligned}$$

Then $H_{t,i}$ ($i = 1, 2, 3, 4$) is a convex, non-increasing function on $[0, \frac{b-a}{4}]$.

Proof. If f is convex and g is linear, then the composition $(f \circ g)$ is convex. The sum of two convex functions is convex; hence, $H_{t,i}$ are convex on the interval $[0, \frac{b-a}{4}]$. Next, letting $t \in [0, 1]$ and $x, x_1, x_2 \in [0, \frac{b-a}{4}]$, we have:

1. $a \leq tx + a \leq \frac{3a+b}{4}$ and $a \leq \frac{3a+b}{4} - tx \leq \frac{3a+b}{4}$, applying Lemma 1.2 yields that for all $x \in [0, \frac{b-a}{4}]$,

$$H_{t,1}(x) = \left[f(tx + a) + f\left(\frac{3a+b}{4} - tx\right) \right] \leq \left[f(a) + f\left(\frac{3a+b}{4}\right) \right] = H_{t,1}(0).$$

Taking $0 < x_1 < x_2 < \frac{b-a}{4}$ results in

$$\frac{H_{t,1}(x_2) - H_{t,1}(x_1)}{x_2 - x_1} \leq \frac{H_{t,1}(x_2) - H_{t,1}(0)}{x_2} \leq 0,$$

thus $H_{t,1}(x_2) \leq H_{t,1}(x_1)$. This shows that $H_{t,1}$ is a non-increasing function on $[0, \frac{b-a}{4}]$. Similarly for all $x \in [0, \frac{b-a}{4}]$.

2. $\frac{3a+b}{4} \leq tx + \frac{3a+b}{4} \leq \frac{a+b}{2}$ and $\frac{3a+b}{4} \leq \frac{a+b}{2} - tx \leq \frac{a+b}{2}$, applying Lemma 1.2 yields

$$H_{t,2}(x) = \left[f\left(tx + \frac{3a+b}{4}\right) + f\left(\frac{a+b}{2} - tx\right) \right] \leq \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+b}{2}\right) \right] = H_{t,2}(0).$$

Putting $0 < x_1 < x_2 < \frac{b-a}{4}$, we find that

$$\frac{H_{t,2}(x_2) - H_{t,2}(x_1)}{x_2 - x_1} \leq \frac{H_{t,2}(x_2) - H_{t,2}(0)}{x_2} \leq 0.$$

This shows that $H_{t,2}$ is a non-increasing function on $[0, \frac{b-a}{4}]$.

3. $\frac{a+b}{2} \leq tx + \frac{a+b}{2} \leq \frac{a+3b}{4}$ and $\frac{a+b}{2} \leq \frac{a+3b}{4} - tx \leq \frac{a+3b}{4}$, applying Lemma 1.2, we get

$$H_{t,3}(x) = \left[f\left(tx + \frac{a+b}{2}\right) + f\left(\frac{a+3b}{4} - tx\right) \right] \leq \left[f\left(\frac{a+b}{2}\right) + f\left(\frac{a+3b}{4}\right) \right] = H_{t,3}(0).$$

For $0 < x_1 < x_2 < \frac{b-a}{4}$, we arrive at

$$\frac{H_{t,3}(x_2) - H_{t,3}(x_1)}{x_2 - x_1} \leq \frac{H_{t,3}(x_2) - H_{t,3}(0)}{x_2} \leq 0,$$

which shows that $H_{t,3}$ is a non-increasing function on $[0, \frac{b-a}{4}]$.

4. $\frac{a+3b}{4} \leq tx + \frac{a+3b}{4} \leq b$ and $\frac{a+3b}{4} \leq b - tx \leq b$, applying Lemma 1.2 yields

$$H_{t,4}(x) = \left[f\left(tx + \frac{a+3b}{4}\right) + f(b - tx) \right] \leq \left[f\left(\frac{a+3b}{4}\right) + f(b) \right] = H_{t,4}(0).$$

For $0 < x_1 < x_2 < \frac{b-a}{4}$, we obtain

$$\frac{H_{t,4}(x_2) - H_{t,4}(x_1)}{x_2 - x_1} \leq \frac{H_{t,4}(x_2) - H_{t,4}(0)}{x_2} \leq 0.$$

This shows that $H_{t,4}$ is a non-increasing function on $[0, \frac{b-a}{4}]$. □

3 Main results

Theorem 3.1. Let $0 \leq a < b < \infty$ and $f : [a, b] \rightarrow \mathbb{R}$ be a convex mapping. Consider $GZ : [0, 1] \rightarrow \mathbb{R}$ as a function defined as follows:

$$GZ(t) = \frac{1}{b-a} \left\{ \int_a^{\frac{3a+b}{4}} f\left(tx + (1-t)\frac{7a+b}{8}\right) dx + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{5a+3b}{8}\right) dx \right. \\ \left. + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f\left(tx + (1-t)\frac{3a+5b}{8}\right) dx + \int_{\frac{a+3b}{4}}^b f\left(tx + (1-t)\frac{a+7b}{8}\right) dx \right\}.$$

The function GZ is convex and non-decreasing on the interval $[0, 1]$.

Proof.

1. Let $t_1, t_2 \in [0, 1]$ and $\alpha + \beta = 1$. We have

$$GZ(\alpha t_1 + \beta t_2) = \frac{1}{b-a} \left\{ \int_a^{\frac{3a+b}{4}} f\left([\alpha t_1 + \beta t_2]x + (1 - [\alpha t_1 + \beta t_2])\frac{7a+b}{8}\right) dx \right. \\ + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f\left([\alpha t_1 + \beta t_2]x + (1 - [\alpha t_1 + \beta t_2])\frac{5a+3b}{8}\right) dx \\ + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f\left([\alpha t_1 + \beta t_2]x + (1 - [\alpha t_1 + \beta t_2])\frac{3a+5b}{8}\right) dx \\ \left. + \int_{\frac{a+3b}{4}}^b f\left([\alpha t_1 + \beta t_2]x + (1 - [\alpha t_1 + \beta t_2])\frac{a+7b}{8}\right) dx \right\}.$$

Since f is a convex function defined on the interval $[a, b]$, we conclude that

$$f\left([\alpha t_1 + \beta t_2]x + (1 - [\alpha t_1 + \beta t_2])\frac{7a+b}{8}\right) \\ = f\left[\alpha\left(t_1x + (1-t_1)\frac{7a+b}{8}\right) + \beta\left(t_2x + (1-t_2)\frac{7a+b}{8}\right)\right] \\ \leq \alpha f\left(t_1x + (1-t_1)\frac{7a+b}{8}\right) + \beta f\left(t_2x + (1-t_2)\frac{7a+b}{8}\right).$$

Similarly, the expression $\frac{7a+b}{8}$ can be transformed into $\frac{5a+3b}{8}$, $\frac{3a+5b}{8}$, and $\frac{a+7b}{8}$ in other cases. As a result, $GZ(\alpha t_1 + \beta t_2) \leq \alpha GZ(t_1) + \beta GZ(t_2)$. Based on this, the function GZ is convex along the interval $[0, 1]$.

2. Let $0 \leq t \leq 1$, then

$$I_1 = \int_a^{\frac{3a+b}{4}} f\left(tx + (1-t)\frac{7a+b}{8}\right) dx \\ = \int_a^{\frac{7a+b}{8}} f\left(tx + (1-t)\frac{7a+b}{8}\right) dx + \int_{\frac{7a+b}{8}}^{\frac{3a+b}{4}} f\left(tx + (1-t)\frac{7a+b}{8}\right) dx.$$

Taking $u = \frac{7a+b}{8} - x$ and $v = x - \frac{7a+b}{8}$, yields

$$I_1 = \int_0^{\frac{b-a}{8}} f\left(\frac{7a+b}{8} - tu\right) du + \int_0^{\frac{b-a}{8}} f\left(tv + \frac{7a+b}{8}\right) dv = \int_0^{\frac{b-a}{8}} \left[f\left(\frac{7a+b}{8} - tx\right) + f\left(\frac{7a+b}{8} + tx\right) \right] dx.$$

In a similar manner, we obtain

$$I_2 = \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{5a+3b}{8}\right) dx = \int_0^{\frac{b-a}{8}} \left[f\left(\frac{5a+3b}{8} - tx\right) + f\left(\frac{5a+3b}{8} + tx\right) \right] dx,$$

$$I_3 = \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f\left(tx + (1-t)\frac{3a+5b}{8}\right) dx = \int_0^{\frac{b-a}{8}} \left[f\left(\frac{3a+5b}{8} - tx\right) + f\left(\frac{3a+5b}{8} + tx\right) \right] dx,$$

and

$$I_4 = \int_{\frac{a+3b}{4}}^b f\left(tx + (1-t)\frac{a+7b}{8}\right) dx = \int_0^{\frac{b-a}{8}} \left[f\left(\frac{a+7b}{8} - tx\right) + f\left(\frac{a+7b}{8} + tx\right) \right] dx.$$

Therefore,

$$\begin{aligned} GZ(t) &= \frac{1}{b-a} \{I_1 + I_2 + I_3 + I_4\} \\ &= \frac{1}{b-a} \int_0^{\frac{b-a}{8}} \left\{ \left[f\left(\frac{7a+b}{8} - tx\right) + f\left(\frac{7a+b}{8} + tx\right) \right] \right. \\ &\quad + \left[f\left(\frac{5a+3b}{8} - tx\right) + f\left(\frac{5a+3b}{8} + tx\right) \right] \\ &\quad + \left[f\left(\frac{3a+5b}{8} - tx\right) + f\left(\frac{3a+5b}{8} + tx\right) \right] + \left. \left[f\left(\frac{a+7b}{8} - tx\right) + f\left(\frac{a+7b}{8} + tx\right) \right] \right\} dx \\ &= \frac{1}{b-a} \int_0^{\frac{b-a}{8}} \{G_{t,1}(x) + G_{t,2}(x) + G_{t,3}(x) + G_{t,4}(x)\} dx. \end{aligned}$$

According to Lemma 2.1, the functions $G_{t,1}$, $G_{t,2}$, $G_{t,3}$, and $G_{t,4}$ are non-decreasing on the interval $[0, \frac{b-a}{8}]$. Therefore, GZ is a non-decreasing function on the interval $[0, 1]$. $\square$

Remark 3.1. Since GZ is a non-decreasing function on $[0, 1]$, we get $GZ(0) \leq GZ(t) \leq GZ(1)$. That is,

$$\frac{1}{4} \left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] \leq GZ(t) \leq \frac{1}{b-a} \int_a^b f(x) dx. \quad (3.1)$$

Corollary 3.1. Under the hypotheses of Theorem 3.1, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ &\leq \frac{1}{4} \left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] \\ &\leq GZ(t) \leq \frac{1}{b-a} \int_a^b f(x) dx. \quad (3.2) \end{aligned}$$

Proof. As a result of f being convex function, we have the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left[\frac{3a+b}{4} + \frac{a+3b}{4}\right]\right) \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right], \\ f\left(\frac{3a+b}{4}\right) &= f\left(\frac{1}{2}\left[\frac{7a+b}{8} + \frac{5a+3b}{8}\right]\right) \leq \frac{1}{2}\left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right)\right], \\ f\left(\frac{a+3b}{4}\right) &= f\left(\frac{1}{2}\left[\frac{3a+5b}{8} + \frac{a+7b}{8}\right]\right) \leq \frac{1}{2}\left[f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right]. \end{aligned} \quad (3.3)$$

Hence,

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{4}\left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right]. \quad (3.4)$$

The needed inequalities (3.2) are obtained by combining inequalities (3.3), (3.4) and (3.1). $\square$

Remark 3.2. Utilizing Corollary 3.1, we are able to establish the second dual Bullen inequality in the following manner:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{4}\left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right] \leq \frac{1}{b-a} \int_a^b f(x) dx. \end{aligned} \quad (3.5)$$

Inequality (3.5) provides a refinement of the dual Bullen inequality.

Theorem 3.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function, and suppose that the function $DF : [0, 1] \rightarrow \mathbb{R}$ is defined as follows:

$$\begin{aligned} DF(t) &= \frac{1}{2(b-a)} \left\{ \int_a^{\frac{3a+b}{4}} \left(f(tx + (1-t)a) + f\left(tx + (1-t)\frac{3a+b}{4}\right) \right) dx \right. \\ &\quad + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left(f\left(tx + (1-t)\frac{3a+b}{4}\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) \right) dx \\ &\quad + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left(f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(tx + (1-t)\frac{a+3b}{4}\right) \right) dx \\ &\quad \left. + \int_{\frac{a+3b}{4}}^b \left(f\left(tx + (1-t)\frac{a+3b}{4}\right) + f(tx + (1-t)b) \right) dx \right\}. \end{aligned}$$

The function DF is convex and non-increasing on the interval $[0, 1]$.

Proof.

1. Let $t_1, t_2 \in [0, 1]$ and $\alpha + \beta = 1$. Since f is a convex function defined on the interval $[a, b]$, we conclude that for $\Delta \in \{a, \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4}, b\}$,

$$\begin{aligned} &f\left([\alpha t_1 + \beta t_2]x + (1 - [\alpha t_1 + \beta t_2])\Delta\right) \\ &= f\left[\alpha(t_1x + (1-t_1)\Delta) + \beta(t_2x + (1-t_2)\Delta)\right] \\ &\leq \alpha f(t_1x + (1-t_1)\Delta) + \beta f(t_2x + (1-t_2)\Delta), \end{aligned}$$

consequently, $DF(\alpha t_1 + \beta t_2) \leq \alpha DF(t_1) + \beta DF(t_2)$. This implies that the function DF is convex along the interval $[0, 1]$.

2. Let $t \in [0, 1]$. Taking $u = x - a$ and $v = \frac{3a+b}{4} - x$, we have

$$\begin{aligned}
 J_1 &= \int_a^{\frac{3a+b}{4}} \left(f(tx + (1-t)a) + f\left(tx + (1-t)\frac{3a+b}{4}\right) \right) dx \\
 &= \int_a^{\frac{3a+b}{4}} f(tx + (1-t)a) dx + \int_a^{\frac{3a+b}{4}} f\left(tx + (1-t)\frac{3a+b}{4}\right) dx \\
 &= \int_0^{\frac{b-a}{4}} f(tu + a) du + \int_0^{\frac{b-a}{4}} f\left(\frac{3a+b}{4} - tv\right) dv \\
 &= \int_0^{\frac{b-a}{4}} \left[f(tx + a) + f\left(\frac{3a+b}{4} - tx\right) \right] dx = \int_0^{\frac{b-a}{4}} H_{t,1}(x) dx.
 \end{aligned}$$

Choosing $u = x - \frac{3a+b}{4}$ and $v = \frac{a+b}{2} - x$, we result in

$$\begin{aligned}
 J_2 &= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left(f\left(tx + (1-t)\frac{3a+b}{4}\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) \right) dx \\
 &= \int_0^{\frac{b-a}{4}} f\left(tu + \frac{3a+b}{4}\right) du + \int_0^{\frac{b-a}{4}} f\left(\frac{a+b}{2} - tv\right) dv \\
 &= \int_0^{\frac{b-a}{4}} \left[f\left(tx + \frac{3a+b}{4}\right) + f\left(\frac{a+b}{2} - tx\right) \right] dx = \int_0^{\frac{b-a}{4}} H_{t,2}(x) dx.
 \end{aligned}$$

Furthermore, setting $u = x - \frac{a+b}{2}$ and $v = \frac{a+3b}{4} - x$, we obtain

$$\begin{aligned}
 J_3 &= \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left(f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(tx + (1-t)\frac{a+3b}{4}\right) \right) dx \\
 &= \int_0^{\frac{b-a}{4}} \left[f\left(tx + \frac{a+b}{2}\right) + f\left(\frac{a+3b}{4} - tx\right) \right] dx = \int_0^{\frac{b-a}{4}} H_{t,3}(x) dx,
 \end{aligned}$$

and by setting $u = x - \frac{a+3b}{4}$ and $v = b - x$, we deduce

$$\begin{aligned}
 J_4 &= \int_{\frac{a+3b}{4}}^b \left(f\left(tx + (1-t)\frac{a+3b}{4}\right) + f(tx + (1-t)b) \right) dx \\
 &= \int_0^{\frac{b-a}{4}} \left[f\left(tx + \frac{a+3b}{4}\right) + f(b - tx) \right] dx = \int_0^{\frac{b-a}{4}} H_{t,4}(x) dx.
 \end{aligned}$$

As a result,

$$DF(t) = \frac{1}{2(b-a)} \{J_1 + J_2 + J_3 + J_4\} = \frac{1}{2(b-a)} \int_0^{\frac{b-a}{4}} \{H_{t,1}(x) + H_{t,2}(x) + H_{t,3}(x) + H_{t,4}(x)\} dx.$$

Based on Lemma 2.2, the functions $H_{t,1}$, $H_{t,2}$, $H_{t,3}$, and $H_{t,4}$ are non-increasing on the interval $[0, \frac{b-a}{4}]$. Consequently, DF is a non-increasing function on the interval $[0, 1]$. $\square$

Remark 3.3. Due to the fact that DF is a non-increasing function on the interval $[0, 1]$, we obtain $DF(1) \leq DF(t) \leq DF(0)$, which is equivalent to

$$\frac{1}{b-a} \int_a^b f(x) dx \leq DF(t) \leq \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right]. \quad (3.6)$$

Corollary 3.2. Under the hypotheses of Theorem 3.2, we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq DF(t) \\ &\leq \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \\ &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \quad (3.7)$$

Proof. Since f is a convex function, we can establish the following inequalities:

$$\begin{aligned} f\left(\frac{3a+b}{4}\right) &= f\left(\frac{1}{2} \left[a + \frac{a+b}{2} \right] \right) \leq \frac{1}{2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right], \\ f\left(\frac{a+3b}{4}\right) &= f\left(\frac{1}{2} \left[\frac{a+b}{2} + b \right] \right) \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + f(b) \right]. \end{aligned}$$

Then

$$\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \leq \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right].$$

As a result

$$\frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right]. \quad (3.8)$$

The required inequalities (3.7) are derived by combining inequalities (3.6) and (3.8). $\square$

Remark 3.4. Employing Corollary 3.2, we present the second Bullen inequality as follows:

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \\ &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \quad (3.9)$$

Inequality (3.9) offers an improvement of the Bullen inequality.

3.1 Second Bullen-type inequalities

Remark 3.5. The following new inequality, which is referred to as second Bullen-type inequality, can be derived from inequalities (3.5) and (3.9) in the following manner:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ &\leq \frac{1}{4} \left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right], \end{aligned} \quad (3.10)$$

where f is a convex mapping in the interval $[a, b]$. Inequality (3.10) is an improvement to the Bullen-type inequalities (1.2).

We now present a new refinement of the second Bullen-type inequalities by using the results established in Corollaries 3.1 and 3.2.

Corollary 3.3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex mapping. Then the following inequalities are valid:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ &\leq \frac{1}{4} \left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] \leq GZ(t) \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt \leq DF(t) \leq \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \\ &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

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