Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2025, 1–14

Hamza Benachour, Lakehal Belarbi, Mohamed Belkhelfa

WEIGHTED FLAT TRANSLATION SURFACES IN MINKOWSKI 3-SPACE WITH DENSITY

Abstract. In this work we classified the weighted flat translation surfaces in Minkowski 3-space with radial density $\Psi = e^{\phi} = e^{-a(x^2+y^2+z^2)+c}$.

2020 Mathematics Subject Classification. 49Q20, 53C22.

Key words and phrases. Manifolds with density, Flat Surfaces, homogeneous space, Lorentzian metric.

1 Introduction

During the recent years, there has been a rapidly growing interest in the geometry of surfaces. A manifold with density is a Riemannian manifold $\mathcal{M}^n$ with positive density function e^{φ} used to weight volume and hyperarea (and sometimes lower-dimensional area and length). In terms of underlying Riemannian volume dV_0 and area dA_0, the new weighted volume and area are given by

$$\begin{aligned} dV &= e^{\varphi} \cdot dV_0, \\ dA &= e^{\varphi} \cdot dA_0. \end{aligned}$$

One of the first examples of a manifold with density appeared in the realm of probability and statistics – Euclidean space with the Gaussian density $e^{-\pi|x|}$ (see [19] for a detailed exposition in the context of isoperimetric problems).

For reasons coming from the study of diffusion processes, Bakry and Émery [1] defined a generalization of the Ricci tensor of Riemannian manifold $\mathcal{M}^n$ with density e^{φ} (or the ∞-Bakry–Émery–Ricci tensor) by

$$\mathrm{Ric}_{\varphi}^{\infty} = \mathrm{Ric} - \mathrm{Hess}\,\varphi.$$

where Ric denotes the Ricci curvature of $\mathcal{M}^n$ and $\mathrm{Hess}\,\varphi$ the Hessian of φ.

According to Perelman in [18, 1.3, p. 6], in a Riemannian manifold $\mathcal{M}^n$ with density e^{φ}, in order for the Lichnerovicz formula to hold, the corresponding φ-scalar curvature is given by

$$S_{\varphi}^{\infty} = S - 2\Delta\varphi - |\nabla\varphi|^2,$$

where S denotes the scalar curvature of $\mathcal{M}^n$. Note that this is different from taking the trace of $\mathrm{Ric}_{\varphi}^{\infty}$, which is $S - \Delta\varphi$.

Following Gromov [12, p. 213], the natural generalization of the mean curvature of hypersurfaces on a manifold with density e^{φ} is given by

$$H_{\varphi} = H - \frac{1}{n-1}\frac{d\varphi}{d\mathbf{N}}\,, \tag{1.1}$$

where H is the Riemannian mean curvature and $\mathbf{N}$ is the unit normal vector field of hypersurface. For a 2-dimensional smooth manifold with density e^{φ}, Corwin et al. [10, p. 6] define a generalized Gauss curvature

$$K_{\varphi} = K - \Delta\varphi$$

and obtain a generalization of the Gauss-Bonnet formula for a smooth disc $\mathbf{D}$:

$$\int\limits_{\mathbf{D}} \mathbf{G}_{\varphi} + \int\limits_{\partial\mathbf{D}} \kappa_{\varphi} = 2\pi,$$

where κ_{φ} is the inward one-dimensional generalized mean curvature as in (1.1) and the integrals are with respect to the unweighted Riemannian area and arclength [16, p. 181].

Bayle [2] derived the first and second variation formulae for the weighted volume functional (see also [16, 19]). From the first variation formula, it can be shown that an immersed submanifold $\mathcal{N}^{n-1}$ in $\mathcal{M}^n$ is minimal if and only if the generalized mean curvature H_{φ} vanishes ($H_{\varphi} = 0$).

Doan The Hieu and Nguyen Minh Hoang [13] classified ruled minimal surfaces in $\mathbb{R}^3$ with density $\Psi = e^z$. In [21], weighted minimal translation surfaces in Minkowski 3-space are classified.

In [5], the second and third authors previously wrote the equations of minimal surfaces in $\mathbb{R}^3$ with linear density $\Psi = e^{\varphi}$ (in the case $\varphi(x,y,z) = x$, $\varphi(x,y,z) = y$ and $\varphi(x,y,z) = z$), and characterized some solutions of the equation of minimal graphs in $\mathbb{R}^3$ with linear density $\Psi = e^{\varphi}$.

In [4], the second and third authors studied the φ-Laplace–Beltrami operator of a nonparametric surface in $\mathbb{R}^3$ with density and proved that

$$\Delta_{\varphi}X = 2H_{\varphi}\cdot\mathbf{N} + \nabla\varphi = 2H\mathbf{N} + (\nabla\varphi)^T,$$

where X is the vector position of a nonparametric surface $z = f(x^1, x^2)$ in $\mathbb{R}^3$ with density $\Psi = e^{\varphi}$, and $(\nabla\varphi)^T$ is the component tangent of $\nabla\varphi$.

2 Preliminary

The space $\mathbb{R}^3_1$ is defined as the space that is the usual three-dimensional $\mathbb{R}$-vector space consisting of vectors $\{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$, but endowed with the inner product

$$\langle \xi, \zeta \rangle_{\mathbb{R}^3_1} = -\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3.$$

This space is called the Minkowski space or the Lorentz space. Tangent vectors are defined precisely as in the case of Euclidean space $\mathbb{R}^3$. A vector ξ is said to be:

- space-like if $\langle \xi, \xi \rangle_{\mathbb{R}^3_1} > 0$;
- time-like if $\langle \xi, \xi \rangle_{\mathbb{R}^3_1} < 0$;
- light-like or isotropic or a null vector if $\langle \xi, \xi \rangle_{\mathbb{R}^3_1} = 0$, but $\xi \neq 0$.

Definition 2.1 ([14]). A regular surface element is defined as an immersion $X : U \to \mathbb{R}^3_1$, exactly as in $\mathbb{R}^3$. A regular surface element $X : U \to \mathbb{R}^3_1$ is called:

- space-like, in case the first fundamental form is positive definite, and if and only if at every point $p = X(u)$, there is a time-like vector $\xi \neq 0$ which is perpendicular, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^3_1}$ in the Minkowski space, to the tangent plane of the surface at the point p;
- time-like, in case the first fundamental form is indefinite, and if and only if at every point $p = X(u)$, there is a space-like vector $\xi \neq 0$ which is perpendicular, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^3_1}$ in the Minkowski space, to the tangent plane of the surface at the point p;
- isotropic, in case the first fundamental form has rank 1, and if and only if at every point $p = X(u)$, there is a isotropic vector $\xi \neq 0$ which is perpendicular, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^3_1}$ in the Minkowski space, to the tangent plane of the surface at the point p.

Definition 2.2 ([11]). A translation surface in the Minkowski 3-space is a surface that is parametrized by either

- $X(s, t) = (s, t, f(s) + g(t))$ if L is timelike;
- $X(s, t) = (f(s) + g(t), s, t)$ if L is spacelike;
- $X(s, t) = (s + t, g(t), f(s) + t)$ if L is lightlike,

with the intersection L of the two planes that contain the curves that generate the surface.

Definition 2.3 ([16]). In an n-dimensional Riemannian manifold with density e^φ, the mean curvature H_φ of a hypersurface with unit normal $\mathbf{N}$ is given by

$$H_\varphi = H - \frac{1}{n-1}\frac{d\varphi}{d\mathbf{N}},$$

where H is the Riemannian mean curvature.

Definition 2.4. A surface Σ in a 3-dimentional Riemannian manifold with density e^φ is weighted minimal if and only if

$$H_\varphi = 0.$$

Example 2.1. The surface S in $\mathbb{R}^3$ with linear density e^x defined by the parametrization

$$X : (x, y) \longmapsto \Big(x, y, -\frac{a^2}{\sqrt{1+a^2}} \arcsin(\beta e^{-\frac{1+a^2}{a^2}x}) + ay + b + \gamma\Big), \text{ where } (x, y) \in \mathbb{R}^2, \ a, b, \beta \in \mathbb{R}^*,$$

is weighted minimal.

Definition 2.5 ([10])**.** The φ-Gauss curvature K_φ of a two-dimensional Riemannian manifold with density e^φ is given by

$$K_\varphi = K - \Delta\varphi,$$

where K is the Riemannian–Gauss curvature and $\Delta\varphi$ is the Laplace–Beltrami operator of the function φ.

Definition 2.6. A surface Σ in 3-dimentional Riemannian manifold with density e^φ is weighted flat if and only if

$$K_\varphi = 0.$$

Example 2.2. The pseudosphere is the surface of revolution obtained by rotating the tractrix about the z-axis, so it is parametrized by

$$X : \mathbb{R}^2 \to \mathbb{R}^3,$$
$$(u, v) \longmapsto \Big(\frac{\cos v}{\cosh(u)}, \frac{\sin v}{\cosh(u)}, u - \frac{\sinh(u)}{\cosh(u)}\Big),$$

where $u > 0$ and $v \in [0, 2\pi]$.

The pseudosphere in $\mathbb{R}^3$ with density $e^{-\frac{1}{6}\rho^2+c}$ is a weighted flat surface.

3 Weighted flat translation surfaces in Minkowski 3-space with density

In this section, we give classifications of all weighted flat translation surfaces in Minkowski space with radial density $e^{-a(x^2+y^2+z^2)+c}$, where $a > 0$ and $c \in \mathbb{R}$.

3.1 Weighted flat timelike translation surfaces in Minkowski 3-space with density

In this subsection, we study the weighted flat timelike translation surfaces Σ in Minkowski 3-space $\mathbb{R}^3_1$, which are parameterized by

$$X(s, t) = (s, t, f(s) + g(t)), \ \ (s, t) \in \mathbb{R}^2,$$

where f and g are the real functions $\mathcal{C}^2(\mathbb{R})$, and have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_s = (1, 0, f'(s))$$

and

$$e_2 := X_t = (0, 1, g'(t)).$$

The coefficients of the first fundamental form are

$$E = \langle e_1, e_1\rangle_{\mathbb{R}^3_1} = 1 - f'^2, \quad F = \langle e_1, e_2\rangle_{\mathbb{R}^3_1} = -f'g', \quad G = \langle e_2, e_2\rangle_{\mathbb{R}^3_1} = 1 - g'^2.$$

As a unit normal field, we can take

$$N = \frac{-1}{\sqrt{|-1 + f'^2 + g'^2|}}\,(f', g', 1).$$

The coefficients of the second fundamental form are

$$l = \langle X_{ss}, N\rangle_{\mathbb{R}^3_1} = \frac{f''}{\sqrt{-1 + f'^2 + g'^2}}\,,$$
$$m = \langle X_{st}, N\rangle_{\mathbb{R}^3_1} = 0,$$
$$n = \langle X_{tt}, N\rangle_{\mathbb{R}^3_1} = \frac{g''}{\sqrt{-1 + f'^2 + g'^2}}\,.$$

Let K be the Gauss curvature of Σ,

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f''g''}{(-1 + f'^2 + g'^2)^2}.$$

The weighted Gauss curvature of Σ

$$K_\varphi = K - \Delta\varphi,$$

where $\Delta\varphi$ is the Laplacian of the function φ in the Minkowski 3-space. We have $\varphi(x, y, z) = -a(x^2 + y^2 + z^2) + c$, thus the Laplacian of φ is

$$\Delta\varphi = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g_{ij}} \cdot (\nabla\varphi)^i \right) = -2a. \tag{3.1}$$

Then

$$K_\varphi = \frac{f''g''}{(-1 + f'^2 + g'^2)^2} + 2a = \frac{f''g'' + 2a(-1 + f'^2 + g'^2)^2}{(-1 + f'^2 + g'^2)^2}.$$

Thus Σ is a weighted flat timelike translation surface in the Minkowski 3-space with density e^φ if and only if

$$K_\varphi = 0,$$

that is, if and only if

$$f''g'' + 2a(-1 + f'^2 + g'^2)^2 = 0. \tag{3.2}$$

To classify weighted flat timelike translation surfaces, it is necessary to solve equation (3.2).

- $f' = \alpha \in [-1, 1]$.

  We replace $f(s) = \alpha s + \alpha_1$ in (3.2) and obtain

$$g'^2 = 1 - \alpha^2,$$

so, we have $g(t) = \pm\sqrt{1 - \alpha^2}\, t + \alpha_2$. In this case Σ is a timelike plane.

- $g' = \beta \in [-1, 1]$.

  We replace $g(t) = \beta t + \beta_1$ in (3.2) and obtain

$$f'^2 = 1 - \beta^2,$$

and so $f(s) = \pm\sqrt{1 - \beta^2}\, s + \beta_2$. In this case Σ is a timelike plane.

- f' and g' are not constant smooth functions.

  In this case, we take derivation of th equation (3.2) by s and t, respectively,

$$f'''g''' + 16af'f''g'g'' = 0. \tag{3.3}$$

  We can write equation (3.3) as

$$\frac{f'''}{f'f''} = -\frac{16ag'g''}{g'''} = \lambda, \tag{3.4}$$

where λ is a real constant. Solving equation (3.4) with respect to the variable s, the first integration gives

$$f'' = \frac{\lambda}{2} f'^2 + \beta, \tag{3.5}$$

where β is a real constant.

◇ Now, if $\beta = 0$, the solutions of equation (3.5) are

$$f(s) = \frac{-2}{\lambda} \ln \left| \frac{-\lambda}{2} s + \alpha \right| + \alpha_1. \tag{3.6}$$

Replacing the function f given in (3.6) into equation (3.2) gives

$$\begin{aligned}
&\frac{a\lambda^4}{8}(g'^2-1)^2s^4-\frac{a\lambda^3\alpha_1}{4}(g'^2-1)^2s^3\\
&\qquad+\Big[\frac{a\lambda^2\alpha_1^2}{2}(g'^2-1)^2+\frac{\lambda^2}{4}\Big(\frac{\lambda}{2}g''+4a(g'^2-1)\Big)\Big]s^2\\
&\qquad\qquad-\Big[a\lambda\alpha_1^3(g'^2-1)^2+\lambda\alpha_1^2\Big(\frac{\lambda}{2}g''+4a(g'^2-1)\Big)\Big]s\\
&\qquad\qquad\qquad+\Big[2a\alpha_1^4(g'^2-1)^2+\alpha_1^2\Big(\frac{\lambda}{2}g''+4a(g'^2-1)\Big)+2a\Big]=0. \qquad (3.7)
\end{aligned}$$

Equation (3.7) is a polynomial in the variable s, so the coefficients must vanish. It follows that the function g satisfies

$$\begin{cases}
g'^2-1=0,\\
\dfrac{a\lambda^2\alpha_1^2}{2}(g'^2-1)^2+\dfrac{\lambda^2}{4}\Big(\dfrac{\lambda}{2}g''+4a(g'^2-1)\Big)=0,\\
a\lambda\alpha_1^3(g'^2-1)^2+\lambda\alpha_1^2\Big(\dfrac{\lambda}{2}g''+4a(g'^2-1)\Big)=0,\\
2a\alpha_1^4(g'^2-1)^2+\alpha_1^2\Big(\dfrac{\lambda}{2}g''+4a(g'^2-1)\Big)+2a=0.
\end{cases}$$

Thus, $g'=\pm1$, and this is a contradiction.

◇ In the case $\beta\neq0$, we integrate equation (3.5) with respect to s and get

$$f(s)=\begin{cases}
\dfrac{-2}{\lambda}\ln\Big|\cos\Big(\dfrac{\lambda}{2}\sqrt{\dfrac{2\beta}{\lambda}}s+\beta_1\sqrt{\dfrac{2\beta}{\lambda}}\Big)\Big|+\beta_2, & \text{if } \dfrac{\beta}{\lambda}>o,\\
\sqrt{\dfrac{-2\beta}{\lambda}}s-\dfrac{2}{\lambda}\ln\Big|1-e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}s+\beta_1}\Big|+\beta_2, & \text{if } \dfrac{\beta}{\lambda}<0.
\end{cases} \qquad (3.8)$$

By replacing f in (3.2), we have:

– if $\frac{\beta}{\lambda}>0$,

$$\begin{aligned}
&\frac{8a\beta^2}{\lambda^2}\tan^4\Big(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s+\beta_1\sqrt{\frac{2\beta}{\lambda}}\Big)\\
&\qquad+\Big[\beta g''+\frac{4a\beta}{\lambda}(g'^2-1)\Big]\tan^2\Big(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s+\beta_1\sqrt{\frac{2\beta}{\lambda}}\Big)+\big[2a(g'^2-1)^2+g''\big]=0. \qquad (3.9)
\end{aligned}$$

Equation (3.9) is a polynomial of the function $\tan(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s+\beta_1\sqrt{\frac{2\beta}{\lambda}})$ and thus the coefficients must vanish. It follows that the function g satisfies

$$\begin{cases}
\dfrac{8a\beta^2}{\lambda^2}=0,\\
\beta g''+\dfrac{4a\beta}{\lambda}(g'^2-1)=0,\\
2a(g'^2-1)^2+g''=0.
\end{cases}$$

Thus, $\beta=0$, $g'=\pm1$, and this is a contradiction.

– if $\frac{\beta}{\lambda}<0$, we have

$$2a\Big[1-g'^2+\frac{2\beta}{\lambda}\Big]^2e^{4\lambda\sqrt{\frac{-2\beta}{\lambda}}s+4\beta_1}+\Big[-\beta g''-8a(1-g'^2)^2+\frac{32a\beta^2}{\lambda^2}\Big]e^{3\lambda\sqrt{\frac{-2\beta}{\lambda}}s+3\beta_1}$$

$$+\Big[2\beta g'' - \frac{-16a\beta}{\lambda}(1-g'^2) + 12a(1-g'^2)^2 + \frac{48a\beta^2}{\lambda^2}\Big]e^{2\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+2\beta_1}$$

$$+\Big[-\beta g'' - 8a(1-g'^2)^2 + \frac{32a\beta^2}{\lambda^2}\Big]e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+\beta_1} + 2a\Big[1-g'^2+\frac{2\beta}{\lambda}\Big]^2 = 0. \quad (3.10)$$

Equation (3.10) is a polynomial of the function $e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+\beta_1}$ and thus the coefficients must vanish. It follows that the function g satisfies

$$\begin{cases} 1-g'^2+\dfrac{2\beta}{\lambda}=0, \\ -\beta g'' - 8a(1-g'^2)^2 + \dfrac{32a\beta^2}{\lambda^2} = 0, \\ 2\beta g'' - \dfrac{-16a\beta}{\lambda}(1-g'^2) + 12a(1-g'^2)^2 + \dfrac{48a\beta^2}{\lambda^2} = 0. \end{cases}$$

Hence $\beta = 0$, $g' = \pm 1$, and this is a contradiction.

Thus we have the following

Theorem 3.1. *Let Σ be a timelike translation surface in the Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ parameterized by*

$$X(s,t) = (s, t, f(s)+g(t)), \;\; (s,t)\in\mathbb{R}^2.$$

Then Σ is weighted flat timelike translation surface in the Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ if and only if

- $X(s,t) = \big(s, t, \alpha s \pm \sqrt{1-\alpha^2}\,t + \alpha_1\big)$, $\alpha\in[-1,1]$, $\alpha_1\in\mathbb{R}$,

or

- $X(s,t) = \big(s, t, \beta t \pm \sqrt{1-\beta^2}\,s + \beta_1\big)$, $\beta\in[-1,1]$, $\beta_1\in\mathbb{R}$.

3.2 Weighted flat spacelike translation surfaces in Minkowski 3-space with density

In this subsection, we study the weighted flat spacelike translation surfaces Σ in the Minkowski 3-space $\mathbb{R}^3_1$ which are parameterized by

$$X(s,t) = (f(s)+g(t), s, t), \;\; (s,t)\in\mathbb{R}^2,$$

where f and g are real functions from $\mathcal{C}^2(\mathbb{R})$, and have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_s = (f'(s), 1, 0)$$

and

$$e_2 := X_t = (g'(t), 0, 1).$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1\rangle_{\mathbb{R}^3_1} = 1+f'^2, \quad F = \langle e_1, e_2\rangle_{\mathbb{R}^3_1} = f'g', \quad G = \langle e_2, e_2\rangle_{\mathbb{R}^3_1} = -1+g'^2.$$

As a unit normal field, we can take

$$N = \frac{1}{\sqrt{|1+f'^2-g'^2|}}(1, -f', -g').$$

The coefficients of the second fundamental form are:

$$l = \langle X_{ss}, N\rangle_{\mathbb{R}^3_1} = \frac{f''}{\sqrt{1+f'^2-g'^2}}\,,$$
$$m = \langle X_{st}, N\rangle_{\mathbb{R}^3_1} = 0,$$
$$n = \langle X_{tt}, N\rangle_{\mathbb{R}^3_1} = \frac{g''}{\sqrt{1+f'^2-g'^2}}\,.$$

Let K be the Gauss curvature of Σ,

$$K = \frac{ln-m^2}{EG-F^2} = \frac{f''g''}{(1+f'^2-g'^2)^2}\,. \tag{3.11}$$

According to (3.1) and (3.11), the weighted Gaussian curvature of Σ is given by

$$K_\varphi = K - \Delta\varphi = \frac{f''g''}{(1+f'^2-g'^2)^2} + 2a = \frac{f''g''+2a(1+f'^2-g'^2)^2}{(1+f'^2-g'^2)^2}\,.$$

Thus Σ is a weighted flat spacelike translation surface in the Minkowski 3-space with density e^φ if and only if

$$K_\varphi = 0,$$

that is, if and only if

$$f''g''+2a(1+f'^2-g'^2)^2 = 0. \tag{3.12}$$

To classify weighted flat spacelike translation surfaces, it is necessary to solve equation (3.12).

- $f' = \alpha \in \mathbb{R}$.

We replace $f(s) = \alpha s + \alpha_1$ in (3.12) and obtain

$$g'^2 = 1-\alpha^2,$$

so $g(t) = \pm\sqrt{1+\alpha^2}\,t + \alpha_2$. In this case Σ is a spacelike plane.

- $g' = \beta \in]-\infty,-1[\cup]1,+\infty[$.

We replace $g(t) = \beta t + \beta_1$ in (3.12) and obtain

$$f'^2 = -1+\beta^2,$$

so $f(s) = \pm\sqrt{-1+\beta^2}\,s + \beta_2$. In this case Σ is a timelike plane.

- f' and g' are not constants smooth functions.

In this case, we take the derivation of equation (3.12) by s and t, respectively,

$$f'''g''' - 16af'f''g'g'' = 0. \tag{3.13}$$

We can write equation (3.13) as

$$\frac{f'''}{f'f''} = \frac{16ag'g''}{g'''} = \lambda, \tag{3.14}$$

where λ is a real constant. Solving equation (3.14) with respect to the variable s, according to (3.5) and (3.14), the function f is given by (3.8).

By replacing f in equation (3.12), we have:

– if $\frac{\beta}{\lambda} > 0$,

$$\frac{8a\beta^2}{\lambda^2}\tan^4\Big(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}\,s+\beta_1\sqrt{\frac{2\beta}{\lambda}}\Big) + \Big[\beta g'' + \frac{4a\beta}{\lambda}(1-g'^2)\Big]\tan^2\Big(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}\,s+\beta_1\sqrt{\frac{2\beta}{\lambda}}\Big) + \big[2a(1-g'^2)^2+g''\big] = 0. \quad (3.15)$$

Equation (3.15) is a polynomial of the function $\tan(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}\,s+\beta_1\sqrt{\frac{2\beta}{\lambda}})$ and thus the coefficients must vanish. It follows that the function g satisfies

$$\begin{cases} \dfrac{8a\beta^2}{\lambda^2} = 0, \\ \beta g'' + \dfrac{4a\beta}{\lambda}(1-g'^2) = 0, \\ 2a(1-g'^2)^2 + g'' = 0. \end{cases}$$

Hence $\beta = 0$, $g' = \pm 1$, and this is a contradiction.

– if $\frac{\beta}{\lambda} < 0$, we have

$$2a\Big[-1+g'^2+\frac{2\beta}{\lambda}\Big]^2 e^{4\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+4\beta_1} + \Big[-\beta g'' - 8a(-1+g'^2)^2 + \frac{32a\beta^2}{\lambda^2}\Big]e^{3\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+3\beta_1} + \Big[2\beta g'' - \frac{-16a\beta}{\lambda}(1-g'^2) + 12a(-1+g'^2)^2 + \frac{48a\beta^2}{\lambda^2}\Big]e^{2\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+2\beta_1} + \Big[-\beta g'' - 8a(1-g'^2)^2 + \frac{32a\beta^2}{\lambda^2}\Big]e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+\beta_1} + 2a\Big[-1+g'^2+\frac{2\beta}{\lambda}\Big]^2 = 0. \quad (3.16)$$

Equation (3.16) is a polynomial of the function $e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}\,s+\beta_1}$ and thus the coefficients must vanish. It follows that the function g satisfies

$$\begin{cases} -1+g'^2+\frac{2\beta}{\lambda} = 0, \\ -\beta g'' - 8a(-1+g'^2)^2 + \dfrac{32a\beta^2}{\lambda^2} = 0, \\ 2\beta g'' - \dfrac{-16a\beta}{\lambda}(-1+g'^2) + 12a(1-g'^2)^2 + \dfrac{48a\beta^2}{\lambda^2} = 0. \end{cases}$$

Hence $\beta = 0$, $g' = \pm 1$, and this is a contradiction.

Thus we have the following

Theorem 3.2. *Let Σ be a spacelike translation surface in the Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ parameterized by*

$$X(s,t) = (f(s)+g(t), s, t), \ \ (s,t) \in \mathbb{R}^2,$$

Then Σ is weighted flat timelike translation surface in the Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ if and only if

- $X(s,t) = \big(\alpha s \pm \sqrt{1-\alpha^2}\,t + \alpha_1, s, t\big)$, $\alpha, \alpha_1 \in \mathbb{R}$,

or

- $X(s,t) = \big(\beta t \pm \sqrt{1-\beta^2}\,s + \beta_1, s, t\big)$, $\beta \in]-\infty,-1[\cup]1,+\infty[$, $\beta_1 \in \mathbb{R}$.

3.3 Weighted flat lightlike translation surfaces in Minkowski 3-space with density

In this subsection, we study the weighted flat lightlike translation surfaces Σ in the Minkowski 3-space $\mathbb{R}^3_1$ which are parameterized by

$$X(s,t) = (s+t, g(t), f(s)+t), \quad (s,t) \in \mathbb{R}^2,$$

where f and g are real functions from $\mathcal{C}^2(\mathbb{R})$, and have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_s = (1, 0, f'(s))$$

and

$$e_2 := X_t = (1, g'(t), 1).$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1 \rangle_{\mathbb{R}^3_1} = 1 - f'^2, \quad F = \langle e_1, e_2 \rangle_{\mathbb{R}^3_1} = 1 - f', \quad G = \langle e_2, e_2 \rangle_{\mathbb{R}^3_1} = g'^2.$$

As a unit normal field, we can take

$$N = \frac{1}{\sqrt{|f'^2 g'^2 + (f'-1)^2 - g'^2|}} (-f'g', f'-1, -g').$$

The coefficients of the second fundamental form are:

$$\begin{aligned}
l &= \langle X_{ss}, N \rangle_{\mathbb{R}^3_1} = \frac{f''g''}{\sqrt{|f'^2 g'^2 + (f'-1)^2 - g'^2|}},\\
m &= \langle X_{st}, N \rangle_{\mathbb{R}^3_1} = 0,\\
n &= \langle X_{tt}, N \rangle_{\mathbb{R}^3_1} = \frac{g''(f'-1)}{\sqrt{|f'^2 g'^2 + (f'-1)^2 - g'^2|}}.
\end{aligned}$$

Let K be the Gauss curvature of Σ,

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f''g''g'(f'-1)}{(f'^2 g'^2 + (f'-1)^2 - g'^2)^2}. \tag{3.17}$$

According to (3.1) and (3.17), the weighted Gaussian curvature of Σ is given by

$$\begin{aligned}
K_\varphi &= K - \Delta\varphi\\
&= \frac{f''g''g'(f'-1)}{(f'^2 g'^2 + (f'-1)^2 - g'^2)^2} + 2a = \frac{f''g''g'(f'-1) + 2a((f'-1) + g'^2(f'^2-1))^2}{(f'^2 g'^2 + (f'-1)^2 - g'^2)^2}.
\end{aligned}$$

Since the surface is non-degenerate, $f' \neq 1$ for all s.

Thus Σ is a weighted flat lightlike translation surface in the Minkowski 3-space with density e^φ if and only if

$$K_\varphi = 0,$$

that is, if and only if

$$f''g''g'(f'-1) + 2a\big((f'-1)^2 + g'^2(f'^2-1)\big)^2 = 0. \tag{3.18}$$

To classify weighted flat lightlike translation surfaces, it is necessary to solve equation (3.18).

- If $f' = \alpha \in]-1, 1[$, it is a trivial solution of (3.18), $f(s) = \alpha s + \alpha_1$, $g(t) = \pm\sqrt{\frac{1-\alpha}{1+\alpha}}\, t + \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, in this case the surface is lightlike space.
- If $g' = \beta \in \mathbb{R}$, it is a trivial solution (3.18), $g(t) = \beta t + \beta_1$, $f(s) = \frac{1-\beta^2}{1+\beta^2} s + \beta_2$, $\beta_1, \beta_2 \in \mathbb{R}$, in this case the surface is lightlike space.

- If f' is non-constant smooth function, we divide (3.18) by $(f'-1)(f'+1)^2$ and take derivatives with s and t, respectively. Then we obtain

$$\Big(\frac{f''}{(f'-1)(f'+1)^2}\Big)'(g''g')' + 4a\Big(\frac{f'-1}{f'+1}\Big)'(g'^2)' = 0.$$

Suppose $g' = 0$. From (3.18), $f' = 1$, a contradiction. Therefore, there exists $\lambda \in \mathbb{R}$ such that

$$-\frac{4a(\frac{f'-1}{f'+1})'}{(\frac{f''}{(f'-1)(f'+1)^2})'} = \frac{(g''g')}{(g'^2)'} = \lambda.$$

⋄ If $\lambda = 0$, then we have $g'^2 = 2\beta$ for some non-zero constant β.

From (3.18), $f' = \frac{1-2\beta}{1+2\beta}$, a contradiction.

⋄ If $\lambda \neq 0$, in this case we have

$$g''g' = \lambda g'^2 + \lambda_1, \quad \lambda_1 \in \mathbb{R}. \tag{3.19}$$

We can write equation (3.19) as

$$\frac{2g'g''}{g'^2 + \frac{\lambda_1}{\lambda}} = 2\lambda,$$

and its solution is given by

$$g'^2 = \kappa e^{2\lambda t} - \frac{\lambda_1}{\lambda}, \quad \kappa \in \mathbb{R}^{*,+}. \tag{3.20}$$

Substituting (3.20) into (3.18) and (3.19), the result is polynomial of $e^{2\lambda t}$ and thus the coefficients must vanish. It follows that f satisfies the following three differential equations:

$$\begin{cases} f'+1 = 0 = 0, \\ 2a(f'^2-1) - \dfrac{4a\lambda_1}{\lambda}(f'+1)^2 + \lambda f'' = 0, \\ (\lambda_1 + \lambda_1\lambda)f'' + (f'-1)^2 + \dfrac{2a\lambda_1^2}{\lambda^2}(f'+1)^2 + 2a\lambda_1(f'^2-1) = 0. \end{cases}$$

From this we conclude that $f' = 1$, again a contradiction.

Thus we have the following

Theorem 3.3. *Let Σ be a lightlike translation surface in the Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ parameterized by*

$$X(s,t) = (s+t, g(t), f(s)+t), \quad (s,t) \in \mathbb{R}^2.$$

Then Σ is a weighted flat lightlike translation surface in the Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ if and only if

- $X(s,t) = \Big(s+t, \pm\sqrt{\dfrac{1-\alpha}{1+\alpha}}\,t + \alpha_2, \alpha s + \alpha_1\Big)$, $\alpha \in]-1,1[$, $\alpha_1, \alpha_2 \in \mathbb{R}$,

or

- $X(s,t) = \Big(s+t, \beta t + \beta_1, \dfrac{1-\beta^2}{1+\beta^2}s + \beta_2 + t\Big)$, $\beta, \beta_1, \beta_2 \in \mathbb{R}$.

References

[1] D. Bakry and M. Émery, Diffusions hypercontractives. (French) [Hypercontractive diffusions] *Séminaire de probabilités, XIX, 1983/84*, 177–206, Lecture Notes in Math., 1123, *Springer, Berlin*, 1985.

[2] V. Bayle, *Propriétés de concavité du profil isopérimétrique et applications*. Mathématiques [math]. Université Joseph–Fourier – Grenoble I, 2004; `https://theses.hal.science/tel-00004317/`.

[3] L. Belarbi, Surfaces with constant extrinsically Gaussian curvature in the Heisenberg group. *Ann. Math. Inform.* **50** (2019), 5–17.

[4] L. Belarbi and M. Belkhelfa, Surfaces in $\mathbb{R}^3$ with density. *i-manager's Journal on Mathematics* **1** (2012), no. 1, 34–48.

[5] L. Belarbi and M. Belkhelfa, Variational problem in Euclidean space with density. *Geometric science of information*, 257–264, Lecture Notes in Comput. Sci., 8085, *Springer, Heidelberg*, 2013.

[6] L. Belarbi and M. Belkhelfa, Some results in Riemannian manifolds with density. *An. Univ. Oradea Fasc. Mat.* **22** (2015), no. 2, 81–86.

[7] L. Belarbi and M. Belkhelfa, On the minimal surfaces in Euclidean space with density. *Nonlinear Stud.* **22** (2015), no. 4, 739–749.

[8] L. Belarbi and M. Belkhelfa, First and second variation of arc length and energy in Riemannian manifold with density. *Adv. Nonlinear Var. Inequal.* **22** (2019), no. 1, 64–78.

[9] L. Belarbi and M. Belkhelfa, On the ruled minimal surfaces in Heisenberg 3-space with density. *J. Interdiscip. Math.* **23** (2020), no. 6, 1141–1155.

[10] I. Corwin, N. Hoffman, S. Hurder, V. Šešum and Y. Xu, Differential geometry of manifolds with density. *Undergrad. Math J.* **7** (2006), no. 1, 15 pp.

[11] W. Goemans, Surfaces in three-dimensional Euclidean and Minkowski space, in particular a study of Weingarten surfaces. *Katholieke Universiteit Leuven – Faculty of Science*, 2010.

[12] M. Gromov, Isoperimetry of waists and concentration of maps. *GAFA, Geom. Funct. Anal.* **13** (2003), 178–215.

[13] D. T. Hieu and N. M. Hoang, Ruled minimal surfaces in $\mathbb{R}^3$ with density e^z. *Pacific J. Math.* **243** (2009), no. 2, 277–285.

[14] W. Kühnel, *Differential Geometry. Curves-Surfaces-Manifolds*. Third edition [of MR1882174]. Translated from the 2013 German edition by Bruce Hunt, with corrections and additions by the author. Student Mathematical Library, 77. American Mathematical Society, Providence, RI, 2015.

[15] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)* **169** (2009), no. 3, 903–991.

[16] F. Morgan, *Geometric Measure Theory. A Beginner's Guide*. Fourth edition. Elsevier/Academic Press, Amsterdam, 2009.

[17] F. Morgan, Manifolds with density. *Notices Amer. Math. Soc.* **52** (2005), no. 8, 853–858.

[18] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. *Preprint* arXiv:math/0211159, 2002; `https://arxiv.org/abs/math/0211159`.

[19] C. Rosales, A. Cañete, V. Bayle and F. Morgan, On the isoperimetric problem in Euclidean space with density. *Calc. Var. Partial Differential Equations* **31** (2008), no. 1, 27–46.

[20] W. P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[21] D. W. Yoon, Weighted minimal translation surfaces in Minkowski 3-space with density. *Int. J. Geom. Methods Mod. Phys.* **14** (2017), no. 12, Article no. 1750178, 10 pp.

(Received 25.04.2024; accepted 28.06.2024)

Authors' addresses:

Hamza Benachour
Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P. 227, 27000 Mostaganem, Algeria
E-mail: `hamza.benachour@univ-mosta.dz`

Lakehal Belarbi
Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P.227,27000, Mostaganem, Algeria
E-mail: `lakehalbelarbi@gmail.com`

Mohamed Belkhelfa
Laboratoire de Physique Quantique de la Matière, et Modélisations Mathématiques (LPQ3M), Université de Mascara, B.P. 305, 29000, Route de Mamounia Mascara, Algérie
E-mail: `mohamed.belkhelfa@gmail.com`