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PSEUDO ELLIPSOID SPECTRUM IN A RIGHT QUATERNIONIC HILBERT SPACE

Abstract. In this research paper, we introduce the concept of pseudo ellipsoid spectrum in a right quaternionic Hilbert space and display some properties of this concept. Furthermore, we give a characterization of the Weyl pseudo ellipsoid spectrum in Hilbert space.

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1 Introduction

Several topics in mathematics and physics have been addressed to replicate the classical operator theory in Banach and Hilbert spaces. When quantum mechanics was first formulated around the turn of the century, it gave a huge boost to the development of operator theory, with one of the most notable achievements being the spectral theorem for unbounded normal operators on a Hilbert space. In the context of the application of spectral theory to quantum theories, mathematical physicists have concentrated on functional calculus in quaternionic Hilbert spaces. In reality, the standard technique for complex Hilbert spaces, which starts with continuous functional calculus and ends with the measurable functional calculus, has been largely ignored, while the measurable functional calculus has received virtually immediate attention. An overarching issue in the past was the lack of a clear definition of the spectrum of an operator on quaternionic Hilbert spaces. Only a few years ago, such a concept was introduced in the broader context of operators on quaternionic Banach spaces. In 1936, when Birkhoff and V. Neumann demonstrated that quantum physics can only be expressed in real, complex, and quaternionic numbers, the precise demonstration of the spectral theorem for quaternionic linear operators has been an open topic. The biggest stumbling block was the lack of clarity in the specification of a linear operator's quaternionic spectrum. Only in 2006 F. Colombo and I. Sabadini established the concept of S-spectrum, allowing quaternionic operator theory to be fully developed (see [6]). It took several more years to prove the spectral theorem based on the S-spectrum, but in 2016 D. Alpay, F. Colombo and D. P. Kimsey (see [1,5]), offered a complete demonstration of this fundamental theorem for both bounded and unbounded operators. Moreover, the S-functional calculus and the spectral theory on the S-spectrum in general started to take shape no sooner than 2006. The book's introduction (see [4]) describes how hypercomplex analysis methods were utilized to establish the proper notion of the quaternionic spectrum whose existence was suggested by quaternionic quantum physics. Furthermore, J. H. Wilkinson established the concept of pseudospectra in the complex case in 1986, when he defined it for an arbitrary matrix norm produced by a vector norm. In [3], A. Ammar, A. Jeribi and N. Lazrag investigated the concept of a bounded right quaternionic linear operator's pseudo-S-spectrum. Moreover, in [2], A. Ammar, A. Jeribi and S. Fakhfakh introduced the ellipsoid spectrum and demonstrated its various features. The purpose of this paper is to investigate the concept of pseudo ellipsoid spectrum in a right quaternionic Hilbert space and to highlight some conclusions.

The rest of this work is organized as follows. We review some preliminary results and basic concepts about right quaternionic linear operators in Section 2. In Section 3, we introduce the concept of a pseudo ellipsoid spectrum in a right quaternionic Hilbert space and look at some of its features. In the last section, we characterize the Weyl pseudo ellipsoid spectrum in quaternionic Hilbert space.

2 Preliminary and auxiliary results

In this section, we summarize some basic notions about the algebra of quaternions, quaternionic Hilbert space and operators. The space of quaternions $\mathbb{H}$ is the four-dimensional real algebra including three the so-called imaginary elements which we indicate by i, j, k; by definition they satisfy: $ij = -ji = k$, $ki = -ik = j$, $jk = -kj = i$, $ii = jj = kk = -1$, where the elements 1, i, j, k are assumed to form a real vector basis of $\mathbb{H}$, so that any element $q \in \mathbb{H}$ takes the form

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

where q_0, q_1, q_2, and q_3 belong to $\mathbb{R}$. Finally, we denote by $\mathcal{S}$ the sphere of unite imaginary quaternions:

$$\mathcal{S} = \Big\{x_1 i + x_2 j + x_3 k \in \mathbb{H} : \ x_1^2 + x_2^2 + x_3^2 = 1 \ \text{ for all } \ x_1, x_2, x_3 \in \mathbb{K}\Big\}.$$

Definition 2.1. Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $q \in \mathbb{H}$, the inner product

$$\langle \cdot\,, \cdot \rangle_{V_{\mathbb{H}}^R} : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \to \mathbb{H}$$

satisfies the following properties:

(i) $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$.

(ii) $\|\phi\|^2 = \langle \phi, \phi \rangle > 0$ unless $\phi = 0$, a real norm.

(iii) $\langle \phi, \omega + \psi \rangle = \langle \phi, \omega \rangle + \langle \phi, \psi \rangle$.

(iv) $\langle \phi, \psi q \rangle = \langle \phi, \psi \rangle q$.

(v) $\langle \phi q, \psi \rangle = \overline{q} \langle \phi, \psi \rangle$, where $\overline{q}$ stands for the quaternionic conjugate.

Definition 2.2. A mapping $T : \mathcal{D}(T) \subseteq V_{\mathbb{H}}^R \to V_{\mathbb{H}}^R$, where $\mathcal{D}(T)$ stands for the domain of T, is said to be right $\mathbb{H}$-linear operator if

$$T(\alpha\phi + \beta\psi) = \alpha(T\phi) + \beta(T\psi) \text{ for } \phi, \psi \in \mathcal{D}(T) \text{ and } \alpha, \beta \in \mathbb{H}.$$

The set of all right linear operators from $V_{\mathbb{H}}^R$ to $V_{\mathbb{H}}^R$ is bounded by $\mathcal{L}(V_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and the identity linear operator on $V_{\mathbb{H}}^R$ is denoted by $I_{V_{\mathbb{H}}^R}$. For a given $T \in \mathcal{L}(V_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$, the range and the kernel is defined by

$$\operatorname{ran}(T) = \left\{ \psi \in V_{\mathbb{H}}^R : \ T\phi = \psi \text{ for } \phi \in \mathcal{D}(T) \right\},$$
$$\ker(T) = \left\{ \phi \in \mathcal{D}(T) : \ T\phi = 0 \right\}.$$

We call an operator $T \in \mathcal{L}(V_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ bounded if there exists $k \geq 0$ such that

$$\|T\phi\|_{V_{\mathbb{H}}^R} \leq k \|\phi\|_{V_{\mathbb{H}}^R} \text{ for all } \phi \in \mathcal{D}(T).$$

The set of all bounded right linear operators from $V_{\mathbb{H}}^R$ to $V_{\mathbb{H}}^R$ is denoted by $\mathcal{B}(V_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$. Let $(V_{\mathbb{H}}^R)'$ be the topological dual space of $(V_{\mathbb{H}}^R)$ consisting of all continuous right $\mathbb{H}$-linear functions on $(V_{\mathbb{H}}^R)$.

Definition 2.3. Let $V_{\mathbb{H}}^R$ be right quaternionic Hilbert space. A bounded operator $K : V_{\mathbb{H}}^R \to V_{\mathbb{H}}^R$ is compact if K maps bounded sets into precompact sets. We denote the set of all compact operators in $V_{\mathbb{H}}^R$ by $\mathcal{K}(V_{\mathbb{H}}^R)$.

Lemma 2.1.

(i) [8, *Theorem 3.2*] *Let* $T : \mathcal{D}(T) \subseteq V_{\mathbb{H}}^R \to V_{\mathbb{H}}^R$ *be a right linear operator. If* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ *is surjective, then* T *is open. In particular, if* T *is bijective, then* $T^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R)$.

(ii) [8, *Proposition 3.6*] *Let* $V_{\mathbb{H}}^R$ *be a right quaternionic Hilbert space. If* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ *such that* $\|T\| < 1$, *then the right linear operator* $I_{V_{\mathbb{H}}^R} - T$ *is invertible and the inverse is given by*

$$(I_{V_{\mathbb{H}}^R} - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Lemma 2.2. *Let* $V_{\mathbb{H}}^R$ *be a quaternionic Hilbert space and let* $u \in V_{\mathbb{H}}^R$. *If* $f(u) = 0$ *for every* $f \in (V_{\mathbb{H}}^R)'$, *then* $u = 0$.

Definition 2.4. Let $V_{\mathbb{H}}^R$ be a right quaternionic Hilbert space and let $T \in \mathcal{D}(T) \to V_{\mathbb{H}}^R$ be a right linear operator with a dense domain. The adjoint $T^* : \mathcal{D}(T^*) \to V_{\mathbb{H}}^R$ of T is the unique right operator with the following properties:

$$\mathcal{D}(T^*) = \left\{ \psi \in V_{\mathbb{H}}^R : \ \exists \, \lambda \text{ such that } \langle \psi, T\phi \rangle = \langle \lambda, \phi \rangle \right\}$$

and

$$\langle \psi, T\phi \rangle = \langle T^*\psi, \phi \rangle \text{ for all } \phi \in \mathcal{D}(T), \ \ \psi \in \mathcal{D}(T^*).$$

Remark 2.1. It is proved in [7, Remark 2.16] that:

(i) If $T \in \mathcal{B}(V_{\mathbb{H}}^R)$, then $T^* \in \mathcal{B}(V_{\mathbb{H}}^R)$, and $\|T\| = \|T^*\|$.

(ii) Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$, then we have that T is bijective and $T^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R)$ if and only if T^* is bijective and $(T^*)^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R)$. Therefore we get $(T^*)^{-1} = (T^{-1})^*$.

Definition 2.5. Let $\mathbb{H}$ be a right quaternionic Hilbert space and T be a right linear operator on $\mathbb{H}$. For $q \in \mathbb{H}$, the associated operator $\delta_q(T)$ is defined by

$$\delta_q(T) = T^2 - 2T\,\mathrm{Re}(q) + I|q|^2.$$

The ellipsoid resolvent set of T is the set $\rho^E(T) \subset \mathbb{H}$ consisting of all quaternions q satisfying all the following conditions:

(i) $\ker(I - \delta_q(T)) = \{0\}$.

(ii) $\mathrm{ran}(I - \delta_q(T))$ is dense in $\mathbb{H}$.

(iii) $(I - \delta_q(T))^{-1} : ran(I - \delta_q(T)) \to \mathcal{D}(T^2)$ is bounded.

The ellipsoid spectrum σ^E of T is defined by setting $\sigma^E(T) = \mathbb{H} \setminus \rho^E(T)$. The ellipsoid approximate point spectrum of T is defined as

$$\sigma^E_{ap}(T) = \Big\{ q \in \mathbb{H} : \ \exists\,(\phi_n) \text{ such that } \|\phi_n\| = 1 \text{ and } \lim \|(I - \delta_q(T))\phi_n\| = 0 \Big\}.$$

Proposition 2.1. *If $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $q \in \mathbb{H}$, then the following statements are equivalent:*

(i) $q \notin \sigma^E_{ap}(T)$.

(ii) $\ker(I - \delta_q(T)) = \{0\}$ *and* $\mathrm{ran}(I - \delta_q(T))$ *is closed.*

Proof. The proof may be achieved in a similar way as [8, Proposition 5.3]. □

3 Main results

The goal of this section is to introduce and investigate the concept of pseudo ellipsoid spectrum of a bounded right quaternionic linear operator in $V_{\mathbb{H}}^R$. Moreover, we characterize the Weyl pseudo ellipsoid spectrum in a quaternionic Hilbert space.

Definition 3.1. Let $T : V_{\mathbb{H}}^R \to V_{\mathbb{H}}^R$ be a bounded right linear operator and $\varepsilon > 0$. The pseudo ellipsoid spectrum is defined by

$$\sigma^E_\varepsilon(T) = \sigma^E(T) \cup \Big\{ q \in \mathbb{H} : \ \big\|(I - \delta_q(T))^{-1}\big\| > \frac{1}{\varepsilon} \Big\}.$$

Proposition 3.1. *Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $\varepsilon > 0$.*

(i) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma^E(T) \subset \sigma^E_{\varepsilon_1+\alpha}(T) \subset \sigma^E_{\varepsilon_2+\alpha}(T)$ for all $\alpha > 0$.*

(ii) $\sigma^E(T) = \bigcap_{\varepsilon>0} \sigma^E_\varepsilon(T)$.

Proof.

(i). Let $0 < \varepsilon_1 < \varepsilon_2$. It is clear that $0 < \varepsilon_1 + \alpha < \varepsilon_2 + \alpha$ for all $\alpha > 0$ and $\varepsilon > 0$. So, it suffices to show that

$$\Big\{ q \in \mathbb{H} : \ \big\|(I - \delta_q(T))^{-1}\big\| > \frac{1}{\varepsilon_1 + \alpha} \Big\} \subset \sigma^E_{\varepsilon_2+\alpha}(T).$$

Let $q \in \{q \in \mathbb{H} : \ \|(I - \delta_q(T))^{-1}\| > \frac{1}{\varepsilon_1+\alpha}\}$. Then

$$\big\|(I - \delta_q(T))^{-1}\big\| > \frac{1}{\varepsilon_1 + \alpha} > \frac{1}{\varepsilon_2 + \alpha}.$$

Thus
$$q \in \sigma^E_{\varepsilon_2+\alpha}(T).$$
So,
$$\sigma^E(T) \subset \sigma^E_{\varepsilon_1+\alpha}(T) \subset \sigma^E_{\varepsilon_2+\alpha}(T).$$

(ii). Let us assume that
$$\begin{aligned}\bigcap_{\varepsilon>0} \sigma^E_\varepsilon(T) &= \bigcap_{\varepsilon>0} \Big(\sigma^E(T) \cup \Big\{q \in \mathbb{H} : \ \big\|(I-\delta_q(T))^{-1}\big\| > \frac{1}{\varepsilon}\Big\}\Big)\\ &= \sigma^E(T) \cup \Big(\bigcap_{\varepsilon>0} \Big\{q \in \mathbb{H} : \ \big\|(I-\delta_q(T))^{-1}\big\| > \frac{1}{\varepsilon}\Big\}\Big).\end{aligned}$$
We show that
$$\bigcap_{\varepsilon>0} \Big\{q \in \mathbb{H} : \ \big\|(I-\delta_q(T))^{-1}\big\| > \frac{1}{\varepsilon}\Big\} \subset \sigma^E(T).$$
Let $q \in \bigcap\limits_{\varepsilon>0} \{q \in \mathbb{H} : \ \|(I-\delta_q(T))^{-1}\| > \frac{1}{\varepsilon}\}$, then $\|(I-\delta_q(T))^{-1}\| > \frac{1}{\varepsilon}$ for all $\varepsilon > 0$. If $\varepsilon \to 0^+$, we obtain $\|(I-\delta_q(T))^{-1}\| = +\infty$. Hence $q \in \sigma^E(T)$. □

Theorem 3.1. *Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$ and $\varepsilon > 0$. Then*
$$\sigma^E_\varepsilon(T) = \sigma^E_\varepsilon(T^*).$$

Proof. Since $T \in \mathcal{B}(V^R_{\mathbb{H}})$, Proposition 2.1 implies that
$$(I-\delta_q(T))^* = (I-\delta_q(T^*)) \ \text{ for every } \ q \in \mathbb{H}.$$
Hence, using Remark 2.1, we have
$$\big\|(I-\delta_q(T^*))^{-1}\big\| = \big\|(I-\delta_q(T^{-1}))^*\big\| = \big\|(I-\delta_q(T))^{-1}\big\|$$
and we conclude that
$$\sigma^E_\varepsilon(T) = \sigma^E_\varepsilon(T^*).$$
□

Proposition 3.2. *Let (T_n) be a sequence of bounded right linear operators on $V^R_{\mathbb{H}}$ which converges in the norm to the operator T such that $T_nT = TT_n$ for all $n \in \mathbb{N}$. Then $\delta_q(T_n)$ converges in the norm to the operator $\delta_q(T)$.*

Proof. Using the fact that $T_nT = TT_n$ for all $n \geq 0$, we infer that
$$\begin{aligned}\big\|\delta_q(T_n) - \delta_q(T)\big\| &= \big\|T_n^2 - T^2 - 2\delta(q)(T_n - T)\big\| = \Big\|(T_n - T)(T_n + T) - 2\operatorname{Re}(q)(T_n - T)\Big\|\\ &= \Big\|(T_n - T)\big[(T_n + T) - 2\operatorname{Re}(q)\big]\Big\| < \|T_n - T\| \,\big\|(T_n + T) - 2\operatorname{Re}(q)\big\| \to 0.\end{aligned}$$
Hence, $\delta_q(T_n)$ converges in the norm to the operator $\delta_q(T)$. □

Theorem 3.2. *Let (T_n) be a sequence of bounded right linear operators on $V^R_{\mathbb{H}}$ which converges in the norm to the operator such that $T_nT = TT_n$ for all $n \geq 1$. Then, to every pair $(\varepsilon_1, \varepsilon_2)$ of real numbers with $0 < \varepsilon_1 < \varepsilon_2$, we have*
$$\sigma^E_{\varepsilon_1}(T) \subset \sigma^E_{\varepsilon_2}(T_n) \subset \sigma^E_{\varepsilon_1+\varepsilon_2}(T) \ \text{ for all } \ n \geq N.$$

Proof. Since $T_n \to T$ and $T_nT = TT_n$ for all $n \geq 1$, using Proposition 3.2, we can deduce that $\delta_q(T_n) \to \delta_q(T)$ as $n \to +\infty$. We infer that there exists $N \in \mathbb{N}$ such that

$$\left\|\delta_q(T_n) - \delta_q(T)\right\| < \varepsilon_2 - \varepsilon_1 \text{ for all } n \geq N.$$

Let $q \notin \sigma^E_{\varepsilon_2}(T_n)$, where n is fixed. Thus we have that $(I - \delta_q(T_n))$ is invertible and

$$\left\|(I - \delta_q(T_n))^{-1}\right\| \leq \frac{1}{\varepsilon_2}.$$

Hence,

$$\left\|(I - \delta_q(T_n))^{-1}(\delta_q(T) - \delta_q(T_n))\right\| \leq \left\|(I - \delta_q(T_n))^{-1}\right\| \left\|(\delta_q(T) - \delta_q(T_n))\right\| \leq \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} < 1 - \frac{\varepsilon_1}{\varepsilon_2}.$$

Then, using Lemma 2.1, we infer that $I + (I - \delta_q(T_n))^{-1}(\delta_q(T) - \delta_q(T_n))$ is invertible and

$$\Big(I + (I - \delta_q(T_n))^{-1}(\delta_q(T) - \delta_q(T_n))\Big)^{-1} = \sum_{k=0}^{\infty} \Big((I - \delta_q(T_n))^{-1}(\delta_q(T) - \delta_q(T_n))\Big)^k.$$

We can express $\delta_q(T)$ in the form

$$\delta_q(T) = (I - \delta_q(T_n))\Big[I + (I - \delta_q(T_n))(\delta_q(T) - \delta_q(T_n))\Big],$$

which is invertible and

$$\begin{aligned}\left\|(I - \delta_q(T))^{-1}\right\| &\leq \left\|I + (I - \delta_q(T_n))^{-1}(\delta_q(T) - \delta_q(T_n))^{-1}\right\| \left\|(I - \delta_q(T_n))^{-1}\right\| \\ &\leq \frac{(I - \delta_q(T_n))^{-1}}{1 - \|(I - \delta_q(T_n))^{-1}\| \, \|\delta_q(T) - \delta_q(T_n)\|} \leq \frac{1}{\varepsilon_2} \times \frac{1}{1 - (1 - \frac{\varepsilon_1}{\varepsilon_2})} \leq \frac{1}{\varepsilon_1}.\end{aligned}$$

This implies that $q \notin \sigma^E_{\varepsilon_1}$. As a result,

$$\sigma^E_{\varepsilon_1}(T) \subset \sigma^E_{\varepsilon_2}(T_n).$$

In order to show that the $\sigma^E_{\varepsilon_2}(T_n) \subset \sigma^E_{\varepsilon_1+\varepsilon_2}(T)$ for all $n \geq N$, it suffices to take

$$\left\|(I - \delta_q(T)) - (I - \delta_q(T_n))\right\| < \varepsilon_1. \qquad \square$$

Theorem 3.3. *Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$ and $\varepsilon > 0$. Then*

$$\sigma^E_\varepsilon(T) = \sigma^E(T) \cup \Big\{q \in \mathbb{H} : \ \exists\, \psi \in D(T^2) \text{ and } \|(I - \delta_q(T))\psi\| < \varepsilon\|\psi\|\Big\}.$$

Proof. Let $q \in \sigma^E_\varepsilon(T)$. This implies that $q \in \sigma^E(T)$ or $\|(I - \delta_q(T))^{-1}\| > \frac{1}{\varepsilon}$. Let $q \in \sigma^E_\varepsilon(T) \setminus \sigma^E(T)$. Then

$$\sup_{\psi \in V^R_{\mathbb{H}} \setminus \{0\}} \frac{\|(I - \delta_q(T))^{-1}\psi\|}{\|\psi\|} > \frac{1}{\varepsilon}.$$

Using the definition of the upper bounded, we find that there exists $\psi \neq 0$ such that

$$\left\|(I - \delta_q(T))^{-1}\psi\right\| > \frac{\psi}{\varepsilon}.$$

Let $\xi = (I - \delta_q(T))^{-1}\psi$, then we have $\xi \in \mathcal{D}(T^2)$ and $(I - \delta_q(T))\xi = \psi$. Hence

$$\|(I - \delta_q(T))\xi\| < \varepsilon\|\xi\|.$$

Conversely, let $q \in \mathbb{H}$ such that there exists $\psi \in \mathcal{D}(T^2)$ and $\|(I - \delta_q(T))\psi\| < \varepsilon\|\psi\|$ or $q \in \sigma^E(T)$. Let us assume that $q \notin \sigma^E(T)$ and we put $\xi = (I - \delta_q(T))\psi$, this implies that $\psi = (I - \delta_q(T))^{-1}\xi$. So, we have

$$\|\xi\| < \varepsilon\left\|(I - \delta_q(T))^{-1}\xi\right\|.$$

Since $\xi \neq 0$, we can see that

$$\frac{\|(I-\delta_q(T))^{-1}\xi\|}{\|\xi\|} > \frac{1}{\varepsilon}\,.$$

So,

$$\|(I-\delta_q(T))^{-1}\| > \frac{1}{\varepsilon}\,.$$

Then $q \in \sigma_\varepsilon^E(T)$. □

Theorem 3.4. *Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon^E = \sigma^E(T) \cup \Big\{q \in \mathbb{H} : \ \exists\,(\xi_n) \in \mathcal{D}(T^2), \ \ \|\xi_n\| = 1 \ \ \textit{and} \ \lim_{n\to+\infty} \big\|(I-\delta_q(T))\xi_n\big\| < \varepsilon\Big\}.$$

Proof. Let $q \in \sigma_\varepsilon^E(T) \setminus \sigma^E(T)$. Then we have $q \in \rho^E(T)$ and $\|(I-\delta_q(T))^{-1}\| > \frac{1}{\varepsilon}$. First, we have

$$\big\|(I-\delta_q(T))^{-1}\big\| = \sup_{\|\xi\|=1} \big\|(I-\delta_q(T))^{-1}\xi\big\|$$

which implies that for every $n \in \mathbb{N}\setminus\{0\}$, there exists a sequence of vectors $\xi_n \in V_{\mathbb{H}}^R$ with $\|\xi_n\| = 1$ such that

$$\big\|(I-\delta_q(T))^{-1}\big\| - \frac{1}{n} \leq \big\|(I-\delta_q(T))^{-1}\xi_n\big\| \leq \big\|(I-\delta_q(T))^{-1}\big\|.$$

Then

$$\lim_{n\to+\infty} \big\|(I-\delta_q(T))^{-1}\xi_n\big\|$$
$$= \lim_{n\to+\infty}\left(\|(I-\delta_q(T))^{-1}\| - \frac{1}{n}\right) = \lim_{n\to+\infty}\big\|(I-\delta(T))^{-1}\big\| = \big\|(I-\delta_q(T))^{-1}\big\|.$$

Second, let us assume that $\theta_n = (I-\delta_q(T))^{-1}\xi_n / \|(I-\delta_q(T))^{-1}\xi_n\|$. Then (θ_n) is a sequence of $\mathcal{D}(T^2)$ with $\|\theta_n\| = 1$ and $(I-\delta_q(T))\theta_n = \|(I-\delta_q(T))^{-1}\xi_n\|^{-1}\xi_n$. Hence

$$\big\|(I-\delta_q(T))\theta_n\big\| = \big\|(I-\delta_q(T))^{-1}\xi_n\big\|^{-1}.$$

Thus, by using Theorem 3.3, we infer that

$$\lim_{n\to+\infty}\big\|(I-\delta_q(T))\theta_n)\big\| < \varepsilon.$$

Conversely, let $q \in \mathbb{H}$ such that there exists $\theta_n \in \mathcal{D}(T^2)$ with $\|\xi_n\| = 1$ and $\lim\limits_{n\to+\infty}\|(I-\delta_q(T))\xi_n\| < \varepsilon$ or $q \in \sigma^E(T)$.

Let us assume that $q \notin \sigma^E(T)$ and we set

$$\theta_n = \frac{(I-\delta_q(T)\xi_n}{\|(I-\delta_q(T))\xi_n\|}\,.$$

Then (θ_n) is a sequence of unit vectors in $V_{\mathbb{H}}^R$. Hence

$$\big\|(I-\delta_q(T))^{-1}\theta_n\big\| = \big\|(I-\delta_q(T))\xi_n\big\|^{-1}.$$

Since

$$\big\|(I-\delta_q(T))^{-1}\theta_n\big\| \leq \big\|(I-\delta_q(T))^{-1}\big\| \ \text{ for } \ \|\theta\| = 1,$$

we get

$$\lim_{n\to+\infty}\big\|(I-\delta_q(T))^{-1}\theta_n\big\| \leq \lim_{n\to+\infty}\big\|(I-\delta_q(T))^{-1}\big\|.$$

Therefore,

$$\big\|(I-\delta_q)(T))^{-1}\big\| \geq \lim \big\|(I-\delta_q(T)\xi_n\big\|^{-1} \geq \Big(\lim_{n\to+\infty}\big\|(I-\delta_q(T)\xi_n\big\|\Big)^{-1} \geq \frac{1}{\varepsilon}\,.$$

So, we can conclude that $q \in \sigma_\varepsilon^E(T)$. □

Theorem 3.5. *Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $\varepsilon > 0$. If $q \in \sigma_\varepsilon^E(T)$, then there exists a right bounded operator D such that $\|D\| < \varepsilon$ and $q \in \sigma^E(T + D)$.*

Proof. Let us assume that $\|(I - \delta_q(T))^{-1}\| > \frac{1}{\varepsilon}$. Since

$$\left\|(I - \delta_q(T))^{-1}\right\| = \sup_{\phi \neq 0} \frac{\|(I - \delta_q(T))^{-1}\phi\|}{\|\phi\|},$$

there exists a nonzero vector $\phi \in V_{\mathbb{H}}^R$ such that

$$\left\|(I - \delta_q(T))^{-1}\phi\right\| > \frac{\|\phi\|}{\varepsilon}. \tag{3.1}$$

We suppose that $\psi = (I - \delta_q(T))^{-1}\phi$. Since $\phi \neq 0$, we get $0 \neq \psi \in \mathcal{D}(T^2)$ and $(I - \delta_q(T))\psi = \phi$. Thus, using equality 3.1, we get

$$\left\|(I - \delta_q(T))\psi\right\| < \varepsilon\|\psi\|.$$

Put $\alpha = \xi/\|\xi\|$, then α is a unit vector in $V_{\mathbb{H}}^R$ and $\|(I - \delta_q(T)\alpha\| < \varepsilon$. Since $\alpha \neq 0$, by using Lemma 2.2 we have $f \in (V_{\mathbb{H}}^R)'$ such that $f(\alpha) \neq 0$. We can conclude that $\|f\| > 0$. Thus we define the right linear operator

$$D : V_{\mathbb{H}}^R \to V_{\mathbb{H}}^R, \ \ t = (I - \delta_q(T))\, \alpha\, \frac{f(t)}{\|f\|}.$$

First, we have

$$\|Dt\| = \left\|(I - \delta_q(T))\, t\, \frac{f(t)}{\|f\|}\right\| < \varepsilon\, \frac{|f(t)|}{\|f\|} < \varepsilon\,\|t\|,$$

which implies that $\|D\| < \varepsilon$. Second, we have

$$\begin{aligned}(I - \delta_q)(T + D)\alpha &= I - (T + D)^2\alpha + 2\operatorname{Re}(q)(T + D)\alpha - |q|^2\alpha \\ &= (I - \delta_q(T))\alpha + TD\alpha + DT\alpha + D^2\alpha - 2\operatorname{Re}(q)D\alpha.\end{aligned}$$

Hence

$$\begin{aligned}\left\|(I - \delta_q(T + D))\alpha\right\| &\leq \|(I - \delta_q(T))\alpha\| + \left\|(T - 2\operatorname{Re}(q))D\alpha\right\| + \|D(T\alpha)\| + \|D^2\alpha\| \\ &\leq \|(I - \delta_q(T))\alpha\| + \big(\|T\| + 2|\operatorname{Re}(q)|\big)\|D\|\|\alpha\| + \|D\|\big(\|T\| + \|D\|\big)\|\alpha\|. \\ &< \big(1 + 2\big(\|T\| + |\operatorname{Re}(q)|\big) + \varepsilon\big)\varepsilon\|\phi\|.\end{aligned}$$

For each n, we have

$$\left\|(I - \delta_q)(T + D)\phi_n\right\| \leq \left[1 + 2\big(\|T\| + |\operatorname{Re}(q)|\big) + \frac{1}{n}\right]\frac{1}{n} \to 0 \ \text{ as n } \ \to +\infty.$$

Then $q \in \sigma_{ap}^E$. We can conclude from Proposition 2.1 that

$$\ker(I - \delta_q(T + D)) \neq 0 \ \text{ or } \ \operatorname{ran}(I - \delta_q(T + D)) \ \text{ is not closed.}$$

So, $(I - \delta_q(T + D))$ is not invertible, which gives $q \in \sigma^E(T + D)$. □

4 The Weyl pseudo ellipsoid spectrum

Definition 4.1. Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $\varepsilon > 0$. We define the Weyl pseudo ellipsoid spectrum of T by

$$\sigma_{w,\varepsilon}^E(T) = \bigcap_{K \in \mathcal{K}(V_{\mathbb{H}}^R)} \sigma_\varepsilon^E(T + K).$$

Proposition 4.1. *Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $\varepsilon > 0$.*

(i) $\sigma_w^E(T) = \bigcap_{\varepsilon>0} \sigma_{w,\varepsilon}^E(T)$.

(ii) *If* $0 < \varepsilon_1 < \varepsilon_2$, *then* $\sigma_w^E(T) \subset \sigma_{w,\varepsilon_1}^E(T) \subset \sigma_{w,\varepsilon_2}^E(T)$.

Proof. It follows from Proposition 3.1 that

$$\bigcap_{\varepsilon>0} \sigma_{w,\varepsilon}^E(T) = \bigcap_{\varepsilon>0} \Big(\bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \sigma_\varepsilon^E(T+K) \Big)$$

$$= \bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \Big(\bigcap_{\varepsilon>0} \sigma_\varepsilon^E(T+K) \Big) = \bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \sigma^E(T+K) = \sigma_w^E(T).$$

Let $0 < \varepsilon_1 < \varepsilon_2$. Using Proposition 3.1, we can deduce that

$$\sigma^E(T+K) \subset \sigma_{\varepsilon_1}^E(T+K) \subset \sigma_{\varepsilon_2}^E(T+K).$$

Thus we get

$$\bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \sigma^E(T+K) \subset \bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \sigma_{\varepsilon_1}^E(T+K) \subset \bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \sigma_{\varepsilon_2}^E(T+K)$$

and infer that

$$\sigma_w^E(T) \subset \sigma_{w,\varepsilon_1}^E(T) \subset \sigma_{w,\varepsilon_2}^E(T).$$

□

Theorem 4.1. *Let T_n be a sequence of bounded right linear operators on $V_{\mathbb{H}}^R$ which converges in the norm to the operator T such that $T_nT = TT_n$ for all $n \geq 0$. Then, for every pair $(\varepsilon_1, \varepsilon_2)$ of real numbers with $0 < \varepsilon_1 < \varepsilon_2$, there exists $N \in \mathbb{N}$ such that*

$$\sigma_{w,\varepsilon_1}^E(T) \subset \sigma_{w,\varepsilon_2}^E(T_n) \subset \sigma_{w,\varepsilon_1+\varepsilon_2}(T).$$

Proof. Let us assume that $q \in \sigma_{w,\varepsilon_2}^E(T_n)$. Then there exists $K \in \mathcal{K}(V_{\mathbb{H}}^R)$ such that $q \notin \sigma_{\varepsilon_2}^E(T_n+K)$. Since $T_n + K \to T + K$, using Proposition 3.2, we infer that $\delta_q(T_n+K) \to \delta_q(T+K)$ as $n \to +\infty$. So, by Theorem 3.2, we can conclude that $q \notin \sigma_{\varepsilon_1}^E(T+K)$. Then

$$q \notin \bigcap_{K\in\mathcal{K}(V_{\mathbb{H}}^R)} \sigma_{\varepsilon_1}^E(T+K) = \sigma_{w,\varepsilon_1}^E(T).$$

As a result, we get

$$\sigma_{w,\varepsilon_1}^E(T) \subset \sigma_{w,\varepsilon_2}^E(T_n).$$

In the same way, we get $\sigma_{w,\varepsilon_1}(T_n) \subset \sigma_{w,\varepsilon_1+\varepsilon_2}(T)$ for all $n > \mathbb{N}$. □

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