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THE GLOBAL REPRESENTATION FORMULAS OF SOLUTIONS FOR THE NONLINEAR CONTROLLED FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH SEVERAL DELAYS AND THE CONTINUOUS INITIAL CONDITION

Dedicated to the memory of Academician Revaz Gamkrelidze

Abstract. In the paper, the global analytic representation formulas of solutions are proved for the nonlinear controlled functional-differential equations with several delays in the phase coordinates and controls. In the formulas, the effects of perturbations of the initial moment, the initial function, the control function, delays parameters contained in the phase coordinates, as well as the effect of the continuous initial condition are revealed. The representation formula of the solution is used when investigating optimization problems, finding an approximate solution of the perturbed functional-differential equation and carrying out a sensitivity analysis of mathematical models.

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რეზიუმე. ნაშრომში დამტკიცებულია ამონახსნების გლობალური ანალიზური წარმოდგენის ფორმულები არაწრფივი შეშფოთებული სამართი ფუნქციონალურ-დიფერენციალური განტოლებებისთვის მრავალი დაგვიანებით ფაზურ კოორდინატებსა და მართვებში. ფორმულებში გამოვლენილია საწყისი მომენტისა და ფაზურ კოორდინატებში შემავალი დაგვიანების პარამეტრების შეშფოთების ეფექტები, აგრეთვე უწყვეტი საწყისი პირობის ეფექტი. ამონახსნის წარმოდგენის ფორმულა გამოიყენება ოპტიმიზაციის ამოცანების გამოკვლევაში, შეშფოთებული განტოლების მიახლოებითი ამონახსნის მოძებნაში, მათემატიკური მოდელების სენსიტიურ ანალიზში.

1 Introduction

In the present paper, the functional-differential equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau_{10}),\dots,x(t-\tau_{s0}),u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_\nu)\big),\ \ t\in[t_{00},t_1] \tag{1.1}$$

with the continuous initial condition

$$x(t)=\varphi_0(t),\ \ t\le t_{00} \tag{1.2}$$

is considered. Condition (1.2) is called the continuous initial condition since $x(t_{00})=\varphi_0(t_{00})$ is always satisfied. Let $x_0(t)$ be a solution of the original Cauchy problem (1.1), (1.2) and let $x(t)$ be a solution of the perturbed (with respect to the initial moment t_{00}, delays τ_{i0}, $i=1,\dots,s$; the initial function $\varphi_0(t)$ and the control function $u_0(t)$) problem. The analytic relation between the solutions $x_0(t)$ and $x(t)$ (the representation formula of solution $x(t)$) is proved on the interval $[t_0^*,t_1]$, where the value t_0^* depends on which side the variation occurs at the initial moment t_{00}. In the formulas the effects of perturbations of the initial moment, the initial function, the control function, delays parameters contained in the phase coordinates, as well as the effect of the continuous initial condition are revealed. We note that the representation formula of solution is used in the investigation of optimization problems [1, 3–13, 15–17, 20, 23], in finding of an approximate solution of the perturbed functional-differential equation (see (2.11)) and to carry out a sensitivity analysis of mathematical models. The representation formula of the solution for the ordinary differential equation was first proved by **Revaz Gamkrelidze** in [5]. The representation formulas of solutions for various classes of functional-differential equations with perturbations defending of a parameter $\varepsilon>0$ are given in [18–21, 23–25]. The novelty here is that the main formula is proved without a parameter ε and taking into account perturbation of the initial moment. The representation formulas of solutions without the parameter $\varepsilon>0$ and perturbation of the initial moment are proved in [14, 22]. Finally, we note that the representation formulas of solutions for a functional-differential equation with one delay, with the continuous initial condition and taking into account perturbation of the initial moment were proved in [2].

The paper is organized as follows. In Section 2, the main theorems are formulated and some comments are given. In Section 3, the auxiliary theorems are given. In Sections 4 and 5, the main theorems are proved.

2 Formulation of main results

Let $\mathbb{R}^n$ be an n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, T means transpose. Suppose that $O\subset\mathbb{R}^n, U\subset\mathbb{R}^r$ are open, convex, bounded sets. Let $t_{10}<t_{20}<t_1, \tau_{2i}>\tau_{1i}>0$, $i=1,\dots,s$, $\theta_k>0$, $k=1,\dots,\nu$ be the given numbers with

$$t_{20}+\tau_2<t_1,\ \ \text{where}\ \ \tau_2=\max\big\{\tau_{21},\dots,\tau_{2s}\big\}.$$

Let an n-dimensional function $f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_\nu)$ be continuous on the set $I\times O^{1+s}\times U^{1+\nu}$, where $I=[t_{10},t_1]$, and continuously differentiable with respect to x, $x_1,\dots,x_s$, u, $u_1,\dots,u_\nu$. It is clear that for the compact sets $K_0\subset O$ and $U_0\subset U$, there exists a number $M_0=M_0(K_0,U_0)>0$ such that

$$\big|f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_\nu)\big|+|f_x(\,\cdot\,)|+\sum_{i=1}^{s}|f_{x_i}(\,\cdot\,)|+|f_u(\,\cdot\,)|+\sum_{i=1}^{\nu}|f_{u_i}(\,\cdot\,)|\le M_0, \tag{2.1}$$
$$\forall\,(t,x,x_1,\dots,x_s,u,u_1,\dots,u_\nu)\in I\times K_0^{1+s}\times U_0^{1+\nu}.$$

Further, denote by Φ the set of continuously differentiable functions $\varphi(t)\in O$, $t\in[\widehat{\tau},t_{20}]$, where $\widehat{\tau}=t_{10}-\tau_2$. By Ω denote piecewise continuous functions $u(t)\in U, t\in I_\theta=[t_{10}-\theta,t_1]$, where $\theta=\max\{\theta_1,\dots,\theta_\nu\}$, with the set $\operatorname{cl}u(I_\theta)\subset U$.

To each element

$$w = (t_0, \tau_1, \dots, \tau_s, \varphi(t), u(t)) \in W = (t_{10}, t_{20}) \times (\tau_{11}, \tau_{21}) \times \cdots \times (\tau_{1s}, \tau_{2s}) \times \Phi \times \Omega,$$

we assign the controlled functional-differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_\nu)\big), \;\; t \in [t_0, t_1], \tag{2.2}$$

with the continuous initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0]. \tag{2.3}$$

Definition 2.1. Let $w \in W$. A function $x(t) = x(t; w) \in O$, $t \in I_1 = [\widehat{\tau}, t_1]$, is called a solution of equation (2.2) with condition (2.3), or a solution corresponding to the element w and defined on the interval I_1 if $x(t)$ satisfies condition (2.3), is absolutely continuous on the interval $[t_0, t_1]$, and satisfies equation (2.2) almost everywhere (a.e.) on $[t_0, t_1]$.

Let us introduce the notation

$$|w| = |t_0| + \sum_{i=1}^{s} |\tau_i| + \|\varphi\|_1 + \|u\|,$$

where

$$\|\varphi\|_1 = \sup\big\{|\varphi(t)| + |\dot{\varphi}(t)| : \; t \in [\widehat{\tau}, t_{20}]\big\}, \quad \|u\| = \sup\big\{|u(t)| : \; t \in I_\theta\big\}.$$

In addition,

$$W_\varepsilon(w_0) = \big\{w \in W : \; |w - w_0| \le \varepsilon\big\},$$

here, $\varepsilon > 0$ is a fixed number and $w_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t)) \in W$ is a fixed element. Furthermore,

$$\delta t_0 = t_0 - t_{00}, \;\; \delta\tau_i = \tau_i - \tau_{i0}, \;\; \delta\varphi(t) = \varphi(t) - \varphi_0(t), \;\; \delta u(t) = u(t) - u_0(t),$$

$$\delta w = w - w_0 = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta\varphi(t), \delta u(t)), \quad |\delta w| = |\delta t_0| + \sum_{i=1}^{s} |\delta\tau_i| + \|\delta\varphi\|_1 + \|\delta u\|$$

and

$$W - w_0 = \big\{\delta w = w - w_0 : \; w \in W\big\}.$$

Proposition 2.1. *Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element*

$$w_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t)) \in W$$

and defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $w = w_0 + \delta w \in W_{\varepsilon_1}(w_0)$ there corresponds the solution $x(t; w) := x(t; w_0 + \delta w)$ defined on the interval I_1 (see Theorem 3.1 in Section 3). In other words, for an arbitrary $\delta w \in W_{\varepsilon_1}(w_0) - w_0$, the perturbed problem (2.2), (2.3), where

$$t_0 = t_{00} + \delta t_0, \tau_i = \tau_{i0} + \delta\tau_i, \;\; i = 1, \dots, s, \quad \varphi(t) = \varphi_0(t) + \delta\varphi(t), \;\; u(t) = u_0(t) + \delta u(t),$$

has the unique solution $x(t; w_0 + \delta w)$, $t \in I_1$.

Theorem 2.1. *Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element*

$$w_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t)) \in W$$

and defined on the interval I_1. Then there exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that for arbitrary

$$\delta w \in \delta W_{\varepsilon_2}^{-} = \big\{\delta w \in W - w_0 : \; |\delta w| \le \varepsilon_2, \; \delta t_0 < 0\big\},$$

on the interval $[t_{00},t_1]$, *the following representation holds:*

$$x(t;w)=x(t;w_0+\delta w)=x_0(t)+\delta x(t;\delta w)+o(t;\delta w), \tag{2.4}$$

where

$$\delta x(t;\delta w)=Y(t_{00};t)\big(\dot\varphi_0(t_{00})-f_0^-\big)\delta t_0+\beta(t;\delta w) \tag{2.5}$$

and

$$\begin{aligned}\beta(t;\delta w)&=Y(t_{00};t)\delta\varphi(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi\\&-\sum_{i=1}^{s}\bigg\{\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})\,d\xi\bigg\}\delta\tau_i+\int\limits_{t_{00}}^{t}Y(\xi;t)\Big[f_u[\xi]\delta u(\xi)+\sum_{i=1}^{\nu}f_{u_i}[\xi]\delta u(\xi-\theta_i)\Big]\,d\xi.\end{aligned} \tag{2.6}$$

Here,

$$\lim_{|\delta w|\to 0}\frac{o(t;\delta w)}{|\delta w|}=0 \ \ \textit{uniformly for}\ \ t\in[t_{00},t_1],$$

$$f_0^-=f\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}),u_0(t_{00}-),u_0(t_{00}-\theta_1-),\dots,u_0(t_{00}-\theta_\nu-)\big),$$

$$f_u[\xi]=\frac{\partial}{\partial u}\,f\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0}),u_0(\xi),u_0(\xi-\theta_1),\dots,u_0(\xi-\theta_\nu)\big);$$

$Y(\xi;t)$ *is the* $n\times n$ *matrix function satisfying the equation*

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-\sum_{i=1}^{s}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}],\ \ \xi\in[t_{00},t],\ \ t\in(t_{00},t_1], \tag{2.7}$$

and the conditions

$$Y(t;t)=E;\ Y(\xi;t)=\Theta,\ \xi>t; \tag{2.8}$$

E *is the identity matrix and* Θ *is the zero matrix.*

Theorem 2.1 corresponds to the case when the variation at the initial moment t_{00} occurs from the left.

Some comments

The function $\delta x(t;\delta w)$ is called the first variation of the solution $x_0(t)$ on the interval $[t_{00},t_1]$. Expression (2.5) is called the variation formula of the solution. The term "variation formula of the solution" was introduced by **Revaz Gamkrelidze**, who proved this for the ordinary differential equation in [5].

The expression

$$Y(t_{00};t)\big(\dot\varphi_0(t_{00})-f_0^-\big)\delta t_0$$

in formula (2.5) is the effect of perturbation of the moment t_{00} and the continuous initial condition.

The addend

$$Y(t_{00};t)\delta\varphi(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$

in formula (2.6) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$-\sum_{i=1}^{s}\bigg\{\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})\,d\xi\bigg\}\delta\tau_i$$

in formula (2.6) is the effect of perturbation of the delay parameters τ_{i0}, $i=1,\dots,s$.

The addend
$$\int_{t_0}^{t} Y(\xi;t)\Big[f_u[\xi]\delta u(\xi)+\sum_{i=1}^{\nu} f_{u_i}[\xi]\delta u(\xi-\theta_i)\Big]\,d\xi$$
in formula (2.6) is the effect of perturbation of the control function $u_0(t)$.

Based on the Cauchy formula [20, p. 31] we conclude that the function
$$\delta x(t)=\begin{cases}\delta\varphi(t), & t\in[\widehat{\tau},t_{00}),\\ (\dot\varphi_0(t_{00})-f_0^-)\delta t_0+\delta\varphi(t_{00}), & t=t_{00},\\ \delta x(t;\delta w), & t\in[t_{00},t_1],\end{cases}$$
satisfies the linear functional-differential equation
$$\frac{d}{dt}\,(\delta x(t))=f_x[t]\delta x(t)+\sum_{i=1}^{s} f_{x_i}[t]\delta x(t-\tau_{i0})$$
$$-\sum_{i=1}^{s} f_{x_i}[t]\dot x_0(t-\tau_{i0})\delta\tau_i+f_u[t]\delta u(t)+\sum_{i=1}^{\nu} f_{u_i}[t]\delta u(t-\theta_i),\ \ t\in(t_{00},t_1],\tag{2.9}$$
with the initial condition
$$\delta x(t)=\delta\varphi(t),\ \ t\in[\widehat{\tau},t_{00}),\quad \delta x(t_{00})=(\dot\varphi_0(t_{00})-f_0^-)\delta t_0+\delta\varphi(t_{00}).\tag{2.10}$$
Formula (2.4) allows us to construct on the interval $[t_{00},t_1]$ an approximate solution of the perturbed problem (2.2), (2.3)(see Proposition 2.1), where $\delta t_0<0$.

In fact, for a small $|\delta w|$ from (2.4), for the solution $x(t;w_0+\delta w)$ of the perturbed problem (2.2), (2.3), we have
$$x(t;w_0+\delta w)\approx x_0(t)+\delta x(t;\delta w),\ \ t\in[t_{00},t_1].\tag{2.11}$$
Thus, $x_0(t)+\delta x(t;\delta w)$ can be considered as an approximate solution on the interval $[t_{00},t_1]$. It is clear that the first variation $\delta x(t;\delta w)$ can be calculated in two ways: first, by finding the solution $Y(\xi;t)$ of problem (2.7), (2.8); second, by finding the solution of problem (2.9), (2.10).

Theorem 2.2. *Let $x_0(t)=x(t;w_0)$ be the solution corresponding to the element $w_0\in W$ and defined on the interval I_1. Then for each fixed $\widehat{t}_0\in(t_{00},t_{00}+\delta)$, where $\delta>0$ and $t_{00}+\delta<t_{02}$, there exists a number $\varepsilon_2\in(0,\varepsilon_1)$ such that for arbitrary*
$$\delta w\in\delta W_{\varepsilon_2}^+=\big\{\delta w\in W-w_0:\ |\delta w|\le\varepsilon_2,\ \delta t_0>0\big\},$$
on the interval $[\widehat{t}_0,t_1]$, representation (2.4) holds, where
$$\delta x(t;\delta w)=Y(t_{00};t)\big(\dot\varphi_0(t_{00})-f_0^+\big)\delta t_0+\beta(t;\delta w).\tag{2.12}$$

Theorem 2.2 corresponds to the case when the variation at the point t_{00} occurs from the right.

Theorem 2.3. *Let $x_0(t)=x(t;w_0)$ be the solution corresponding to the element $w_0\in W$ and defined on the interval I_1. Moreover, let*
$$f_0^+=f_0^-:=f_0.$$
Then for each fixed $\widehat{t}_0\in(t_{00},t_{00}+\delta)$, where $\delta>0$ and $t_{00}+\delta<t_{02}$, there exists a number $\varepsilon_2\in(0,\varepsilon_1)$ such that for arbitrary
$$\delta w\in\delta W_{\varepsilon_2}=\big\{\delta w\in W-w_0:\ |\delta w|\le\varepsilon_2\big\},$$
on the interval $[\widehat{t}_0,t_1]$, representation (2.4) holds, where
$$\delta x(t;\delta w)=Y(t_{00};t)\big(\dot\varphi_0(t_{00})-f_0\big)\delta t_0+\beta(t;\delta w).$$

Theorem 2.3 corresponds to the case when the variation at the point t_{00} occurs from both sides and is a corollary to Theorems 2.1 and 2.2.

3 Auxiliary assertions

Theorem 3.1 ([20, p. 18]). *Let $x_0(t) = x(t; w_0)$ be the solution corresponding to the element $w_0 \in W$ and defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $w = w_0 + \delta w \in W_{\varepsilon_1}(w_0)$, there corresponds the solution $x(t) := x(t; w) = x(t; w_0 + \delta w)$ defined on the interval I_1 with $x(t) \in K_0$ and $u_0(t) + \delta u(t) \in U_0$, where $K_0 \subset O$ is a compact set containing a neighborhood of the set $x_0(I_1)$ and $U_0 \subset U$ is a compact set containing a neighborhood of the set $\operatorname{cl} u_0(I_1)$.*

Theorem 3.1 allows us to introduce the increment of the solution $x_0(t)$ on the interval I_1:

$$\Delta x(t) := \Delta x(t; \delta w) = x(t; w_0 + \delta w) - x_0(t), \ \ t \in I_1, \quad \delta w = w - w_0 \in \delta W^-_{\varepsilon_1}.$$

Theorem 3.2. *There exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that*

$$\max_{t\in I_1} |\Delta x(t)| = \max_{t\in I_1} |\Delta x(t; \delta w)| \le O(\delta w) \tag{3.1}$$

for an arbitrary $\delta w \in \delta W^-_{\varepsilon_2}$. Moreover,

$$\Delta x(t_{00}) = \delta\varphi(t_{00}) + (\dot\varphi_0(t_{00}) - f_0^-)\delta t_0 + o(\delta w). \tag{3.2}$$

Here,

$$\lim_{|\delta w|\to 0} \frac{O(\delta w)}{|\delta w|} < \infty.$$

Theorem 3.3. *There exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that*

$$\max_{t\in I_1} |\Delta x(t)| \le O(\delta w)$$

for an arbitrary $\delta w \in \delta W^+_{\varepsilon_2}$. Moreover,

$$\Delta x(t_0) = \delta\varphi(t_{00}) + \big(\dot\varphi_0(t_{00}) - f_0^+\big)\delta t_0 + o(\delta w).$$

Remark. Theorems 3.2 and 3.3 can be proved without principle changes by the analogous scheme given in [2] for the functional-differential equation, where $s = 1$ and $\nu = 1$.

4 Proof of Theorem 2.1

Let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that for an arbitrary $\delta w \in \delta W^-_{\varepsilon_2}$ the inequality

$$t_{00} - \tau_i < t_0, \ \ i = 1, \dots, s, \tag{4.1}$$

holds, where $\tau_i = \tau_{i0} + \delta\tau_i$ and $t_0 = t_{00} + \delta t_0$. On the interval $[t_{00}, t_1]$, the function $\Delta x(t)$ satisfies the equation

$$\begin{aligned} \dot{\Delta} x(t) &= a(t; \delta w) = f_x[t]\Delta x(t) \\ &+ \sum_{i=1}^{s} f_{x_i}[t]\Delta x(t - \tau_{i0}) + f_u[t]\delta u(t) + \sum_{i=1}^{\nu} f_{u_i}[t]\delta u(t - \theta_i) + b(t; \delta w), \ \ t \in [t_{00}, t_1], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} a(t; \delta w) &= f\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_s), u(t), u(t - \theta_1), \dots, u(t - \theta_\nu)\big) - f[t], \\ x(t) &= x(t; w_0 + \delta w) = x_0(t) + \Delta x(t), \quad u(t) = u_0(t) + \delta u(t), \\ f[t] &= f\big(t, x_0(t), x_0(t - \tau_{10}), \dots, x_0(t - \tau_{s0}), u_0(t), u_0(t - \theta_1), \dots, u_0(t - \theta_\nu)\big), \\ b(t; \delta w) &= a(t; \delta w) - f_x[t]\Delta x(t) - \sum_{i=1}^{s} f_{x_i}[t]\Delta x(t - \tau_{i0}) - f_u[t]\delta u(t) - \sum_{i=1}^{\nu} f_{u_i}[t]\delta u(t - \theta_i). \end{aligned} \tag{4.3}$$

Using the Cauchy formula [20, p. 31], one can represent the solution of equation (4.2) in the form

$$\Delta x(t)=Y(t_{00};t)\Delta x(t_{00})+\int\limits_{t_{00}}^{t}Y(\xi;t)\Big(f_u[\xi]\delta u(\xi)+\sum_{i=1}^{\nu}f_{u_i}[\xi]\delta u(\xi-\theta_i)\Big)\,d\xi \\ +b_1(t;t_{00},\delta w)+b_2(t;t_{00},\delta w), \tag{4.4}$$

where

$$b_1(t;t_{00},\delta w)=\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi,$$

$$b_2(t;t_{00},\delta w)=\int\limits_{t_{00}}^{t}Y(\xi;t)b(\xi;\delta w)\,d\xi$$

and $Y(\xi;t)$ is the matrix function satisfying equation (2.7) and condition (2.8). The function $Y(\xi;t)$ is continuous on the set

$$\Pi=\big\{(\xi,t):\ \xi\in[t_{00},t],\ t\in[t_{00},t_1]\big\}$$

(see [20, Lemma 2.6]). Therefore,

$$Y(t_{00};t)\Delta x(t_{00})=Y(t_{00};t)\Big[\delta\varphi(t_{00})+\Big(\dot\varphi_0(t_{00})-f_0^-\Big)\delta t_0\Big]+o(t;\delta w) \tag{4.5}$$

(see (3.2)). It can be readily seen that

$$\begin{aligned} b_1(t;t_{00},\delta w)&=\sum_{i=1}^{s}\Bigg(\int\limits_{t_{00}-\tau_{i0}}^{t_0}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\Bigg)\\ &=\sum_{i=1}^{s}\Bigg(\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi\\ &\qquad-\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\Bigg)\\ &=\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0}];t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+o(t;\delta w) \end{aligned} \tag{4.6}$$

(see (3.1)). We introduce the notation:

$$\begin{gathered} f[t;\xi,\delta w]=f\Big(t,x_0(t)+\xi\Delta x(t),x_0(t-\tau_{10})+\xi(x_0(t-\tau_1)-x_0(t-\tau_{10})+\Delta x(t-\tau_1)),\dots,\\ x_0(t-\tau_{s0})+\xi\big(x_0(t-\tau_s)-x_0(t-\tau_{s0})+\Delta x(t-\tau_s)\big),\\ u_0(t)+\xi\delta u(t),u_0(t-\theta_1)+\xi\delta u(t-\theta_1),\dots,u_0(t-\theta_\nu)+\xi\delta u(t-\theta_\nu)\Big),\\ \sigma_x(t;\xi,\delta w)=f_x[t;\xi,\delta w]-f_x[t]. \end{gathered}$$

It is not difficult to see that

$$
\begin{aligned}
a(t;\delta w) &= \int_0^1 \frac{d}{d\xi}\, f[t;\xi,\delta w]\, d\xi \\
&= \int_0^1 \Big\{ f_x[t;\xi,\delta w]\Delta x(t) + \sum_{i=1}^{s} f_{x_i}[t;\xi,\delta w]\big(x_0(t-\tau_i) - x_0(t-\tau_{i0}) + \Delta x(t-\tau_i)\big) \\
&\qquad + f_u[t;s,\delta w]\delta u(t) + \sum_{i=1}^{\nu} f_{u_i}[t;\xi,\delta w]\delta u(t-\theta_i)\Big\}\, d\xi \\
&= \Big[\int_0^1 \sigma_x(t;\xi,\delta w)\, d\xi\Big]\Delta x(t) + \sum_{i=1}^{s}\Big[\int_0^1 \sigma_{x_i}(t;\xi,\delta w)\, d\xi\Big]\big(x_0(t-\tau_i) - x_0(t-\tau_{i0}) + \Delta x(t-\tau_i)\big) \\
&\qquad + \sum_{i=1}^{\nu}\Big[\int_0^1 \sigma_u(t;\xi,\delta w)\, d\xi\Big]\delta u(t) + \sum_{i=1}^{\nu}\Big[\int_0^1 \sigma_{u_i}(t;\xi,\delta w)\, d\xi\Big]\delta u(t-\theta_i) + f_x[t]\Delta x(t) \\
&\qquad + \sum_{i=1}^{s} f_{x_i}[t]\big(x_0(t-\tau_i) - x_0(t-\tau_{i0}) + \Delta x(t-\tau_i)\big) + f_u[t]\delta u(t) + \sum_{i=1}^{\nu} f_{u_i}[t]\delta u(t-\theta_i).
\end{aligned}
$$

Taking into account the last relation for $t \in [t_{00}, t_1]$, we have

$$
\begin{aligned}
b_2(t;t_{00},\delta w) &= b_{21}(t;\delta w) \\
&\qquad + \sum_{i=1}^{s} b_{22}^{(i)}(t;\delta w) + b_{23}(t;\delta w) + \sum_{i=1}^{\nu} b_{24}^{(i)}(t;\delta w) + \sum_{i=1}^{s} b_{25}^{(i)}(t;\delta w) + \sum_{i=1}^{s} b_{26}^{(i)}(t;\delta w),
\end{aligned}
$$

where

$$
\begin{aligned}
b_{21}(t;\delta w) &= \int_{t_{00}}^{t} Y(\xi;t)\sigma_x(\xi;\delta w)\Delta x(\xi)\, d\xi, \sigma_x(\xi;\delta w) = \int_0^1 \sigma_x(\xi;\varsigma,\delta w)\, d\varsigma, \\
b_{22}^{(i)}(t;\delta w) &= \int_{t_{00}}^{t} Y(\xi;t)\sigma_{x_i}(\xi;\delta w)(x_0(\xi-\tau_i) - x_0(\xi-\tau_{i0}) + \Delta x(\xi-\tau_i))\, d\xi, \\
\sigma_{x_i}(\xi;\delta w) &= \int_0^1 \sigma_{x_i}(\xi;\varsigma,\delta w)\, d\varsigma, \quad i = 1,\dots,s, \\
b_{23}(t;\delta w) &= \int_{t_{00}}^{t} Y(\xi;t)\sigma_u(\xi;\delta w)\delta u(\xi)\, d\xi, \\
\sigma_u(\xi;\delta w) &= \int_0^1 \sigma_u(\xi;\varsigma,\delta w)\, d\varsigma; b_{24}^{(i)}(t;\delta w) = \int_{t_{00}}^{t} Y(\xi;t)\sigma_{u_i}(\xi;\delta w)\delta u(\xi-\theta_i)\, d\xi, \\
\sigma_{u_i}(\xi;\delta w) &= \int_0^1 \sigma_{u_i}(\xi;\varsigma,\delta w)\, d\varsigma, \quad i = 1,\dots,\nu, \\
b_{25}^{(i)}(t;\delta w) &= \int_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi][x_0(\xi-\tau_i) - x_0(\xi-\tau_{i0})]\, d\xi, \quad i = 1,\dots,s, \\
b_{26}^{(i)}(t;\delta w) &= \int_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi][\Delta x(\xi-\tau_i) - \Delta x(\xi-\tau_{i0})]\, d\xi, \quad i = 1,\dots,s
\end{aligned}
$$

(see (4.3)). The function $x_0(t)$, $t\in I_1$, is absolutely continuous. For each Lebesgue point $\xi\in(t_{00},t_1]$ of the functions $\dot x_0(\xi-\tau_{i0})$, $i=1,\dots,s$, we get

$$x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})=\int\limits_{\xi}^{\xi-\delta\tau_i}\dot x_0(\varsigma-\tau_{i0})\,d\varsigma=-\dot x_0(\xi-\tau_{i0})\delta\tau_i+\gamma(\xi;\delta\tau_i),\ \ i=1,\dots,s, \tag{4.7}$$

with

$$\lim_{|\delta\tau_i|\to 0}\Big|\frac{\gamma(\xi;\delta\tau_i)}{\delta\tau_i}\Big|=0.$$

We now denote $\gamma(\xi;\delta\tau_i)$ by $\gamma_i(\xi;\delta w)$. It is clear that

$$\lim_{|\delta w|\to 0}\frac{|\gamma_i(\xi;\delta w|}{|\delta w|}\le\lim_{|\delta\tau_i|\to 0}\Big|\frac{\gamma(\xi;\delta\tau_i)}{\delta\tau_i}\Big|=0. \tag{4.8}$$

Thus, (4.8) is valid at almost all points on the interval (t_{00},t_1). It is to see that

$$|\dot x_0(t)|\le M_0\ \text{ a.e. on } I_1$$

(see (2.1), Definition 2.1, Theorem 3.1) and there exists a number $L>0$ such that

$$\begin{gathered}\big|f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_\nu)-f(t,y,y_1,\dots,y_s,v,v_1,\dots,v_\nu)\big|\\ \le L\Big(|x-y|+\sum_{i=1}^{s}|x_i-y_i|+|u-v|+\sum_{i=1}^{\nu}|u_i-v_i|\Big)\\ \forall\, t\in I,\ (x,y,x_1,\dots,x_s,y_1,\dots,y_s)\in K_0^{2+2s},\ (u,v,u_1,\dots,u_\nu,v_1,\dots,v_\nu)\in U_0^{2+2\nu}\end{gathered}$$

(see [20, p. 29]). From (4.7), taking into account the boundedness of the function $\dot x_0(t)$, $t\in I_1$, it follows that

$$|x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})|\le M_0|\delta\tau_i|\le M_0|\delta w|=O(\delta w),\ \ i=1,\dots,s, \tag{4.9}$$

and

$$\frac{|\gamma_i(\xi;\delta w|}{|\delta w|}\le\Big|\frac{\gamma(\xi;\delta\tau_i)}{\delta\tau_i}\Big|=\bigg|\dot x_0(\xi-\tau_{i0})+\frac{1}{\delta\tau_i}\int\limits_{\xi}^{\xi-\delta\tau_i}\dot x_0(\varsigma-\tau_{i0})\,d\varsigma\bigg|\le const. \tag{4.10}$$

We introduce the notation

$$\varrho_{i1}=\min\big\{t_0+\tau_i,t_{00}+\tau_{i0}\big\},\quad \varrho_{i2}=\max\big\{t_0+\tau_i,t_{00}+\tau_{i0}\big\},\ \ i=1,\dots,s.$$

It is not difficult to see that

$$\varrho_{i1}>t_{00},\ \ i=1,\dots,s$$

(see (4.1)) and

$$|\varrho_{i2}-\varrho_{i1}|=O(\delta w),\ \ i=1,\dots,s.$$

It is clear that for $\xi\in[t_{00},\varrho_{i1}]$,

$$\big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big|=\big|\delta\varphi(\xi-\tau_i)-\delta\varphi(\xi-\tau_{i0})\big|\le\bigg|\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}|\dot{\delta\varphi}(\varsigma)|\,d\varsigma\bigg|=o(\delta w) \tag{4.11}$$

and for $\xi\in[\varrho_{i1},\varrho_{i2}]$, we have

$$|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})|\le O(\delta w) \tag{4.12}$$

(see (3.1)). Let $\xi\in[\varrho_{i2},t_1]$, then $\xi-\tau_i\ge t_{00}$, $\xi-\tau_{i0}\ge t_{00}$. Therefore,

$$
\big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big| = \Big|\int\limits_{\xi-\tau_i}^{\xi-\tau_{i0}} |\dot{\Delta}x(\varsigma)|\,d\varsigma\Big|
$$

$$
\leq \Big|\int\limits_{\xi-\tau_i}^{\xi-\tau_{i0}} L\Big[\,|\Delta x(\varsigma)| + \sum_{i=1}^{s}\big|x_0(\varsigma-\tau_i)-x_0(\varsigma-\tau_{i0})\big| + |\delta u(\varsigma)| + \sum_{i=1}^{\nu}|\delta u(\varsigma-\theta_i)|\,\Big]\,d\varsigma\Big| \leq o(\delta w) \quad (4.13)
$$

(see (3.1), (4.9)). According to (3.1), (4.7) for the expressions $b_{21}(t;\delta w)$, $b_{22}^{(i)}(t;\delta w)$, $i=1,\dots,s$; $b_{23}(t;\delta w)$, $b_{24}^{(i)}(t;\delta w)$, $i=1,\dots,\nu$, we obtain

$$
|b_{21}(t;\delta w)| \leq \| \, Y \, \| \, \sigma_x(\delta w), \quad \sigma_x(\delta w)=\int\limits_{t_{00}}^{t_1}|\sigma_x(\xi;\delta w)|\,d\xi,
$$

$$
|b_{22}^{(i)}(t;\delta w)| \leq \|Y\|\sigma_{x_i}(\delta w), \quad \sigma_{x_i}(\delta w)=\int\limits_{t_{00}}^{t_1}|\sigma_{x_i}(\xi;\delta w)|\,d\xi, \;\; i=1,\dots,s,
$$

$$
|b_{23}(t;\delta w)| \leq \|Y\|\sigma_u(\delta w), \quad \sigma_u(\delta w)=\int\limits_{t_{00}}^{t_1}|\sigma_u(\xi;\delta w)|\,d\xi,
$$

$$
|b_{24}^{(i)}(t;\delta w)| \leq \|Y\|\sigma_{u_i}(\delta w), \quad \sigma_{u_i}(\delta w)=\int\limits_{t_{00}}^{t_1}|\sigma_{u_i}(\xi;\delta w)|\,d\xi, \;\; i=1,\dots,\nu,
$$

where

$$
\|Y\| = \sup\big\{|Y(\xi;t)| : \; (\xi,t)\in\Pi\big\}.
$$

Next, for $b_{25}^{(i)}(t;\delta w)$, $i=1,\dots,s$, we have

$$
b_{25}^{(i)}(t;\delta w) = -\Big[\int\limits_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\dot{x}_0[\xi]\,d\xi\Big]\delta\tau_i + \widehat{\gamma}_i(t;\delta w),
$$

where

$$
\widehat{\gamma}_i(t;\delta w) = \int\limits_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\gamma_i(\xi;\delta w)\,d\xi
$$

(see (4.7)). By the Lebesgue theorem, on the passage to the limit under the integral sign, we have

$$
\lim_{\delta w\to 0}\sigma_x(\delta w)=0, \quad \lim_{\delta w\to 0}\sigma_{x_i}(\delta w)=0, \;\; i=1,\dots,s,
$$
$$
\lim_{\delta w\to 0}\sigma_u(\delta w)=0, \quad \lim_{\delta w\to 0}\sigma_{u_i}(\delta w)=0, \;\; i=1,\dots,\nu,
$$

and

$$
\lim_{\delta w\to 0}\frac{|\widehat{\gamma}_i(t;\delta w|)}{|\delta w|}=0
$$

uniformly for $t\in[t_{00},t_1]$ (see (4.8), (4.10)). Thus

$$
b_{21}(t;\delta w)=o(\delta w), \quad b_{22}^{(i)}(t;\delta w)=o(\delta w), \;\; i=1,\dots,s,
$$
$$
b_{23}(t;\delta w)=o(\delta w), \quad b_{24}^{(i)}(t;\delta w)=o(\delta w), \;\; i=1,\dots,\nu,
$$
$$
b_{25}^{(i)}(t;\delta w) = -\Big[\int\limits_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\dot{x}_0[\xi]\,d\xi\Big]\delta\tau_i + o(\delta w).
$$

Further,

$$|b_{26}^{(i)}(t;\delta w)| \le \|Y\| \int\limits_{t_{00}}^{t_1} |f_{x_i}[\xi]||\Delta x(\xi-\tau_i) - \Delta x(\xi-\tau_{i0})|\,d\xi = o(\delta w)$$

(see (4.11)–(4.13)). Consequently,

$$b_2(t;t_{00},\delta w) = -\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t} Y(\xi;t) f_{x_i}[\xi]\dot{x}_0[\xi]\,d\xi\bigg]\delta\tau_i + o(\delta w). \tag{4.14}$$

From (4.4), by virtue of (4.5), (4.6) and (4.14), we obtain (2.4), where $\delta x(t;\delta w)$ has the form (2.5).

5 Proof of Theorem 2.2

Let $\delta>0$ be insomuch small number that $t_{00}+\delta<t_{02}$ and let $\widehat{t}_0 \in (t_{00}, t_{00}+\delta)$ be a fixed point. Moreover, let $\varepsilon_2 \in (0,\varepsilon_1)$ be insomuch small that $t_0 = t_{00}+\delta t_0 < \widehat{t}_0$ for an arbitrary

$$\delta w \in \delta W_{\varepsilon_2}^{+} = \big\{\delta w \in W - w_0 :\ |\delta w| \le \varepsilon_2,\ \delta t_0 > 0\big\}.$$

On the interval $[t_0,t_1]$, the function $\Delta x(t)$ satisfies equation (4.2). Therefore, using the Cauchy formula, we can represent it in the form

$$\begin{aligned}\Delta x(t) &= Y(t_0;t)\Delta x(t_0)\\ &\quad + \int\limits_{t_0}^{t} Y(\xi;t)\Big(f_u[\xi]\delta u(\xi) + \sum_{i=1}^{\nu} f_{u_\nu}[\xi]\delta u(\xi-\theta_i)\Big)\,d\xi + b_1(t;t_0,\delta w) + b_2(t;t_0,\delta w),\end{aligned} \tag{5.1}$$

where $Y(\xi;t)$ is the matrix function satisfying equation (2.7) and condition (2.8). The function $Y(s;t)$ is continuous on the set $[t_{00},\widehat{t}_0)\times[\widehat{t}_0,t_1]$, therefore,

$$Y(t_0;t)\Delta x(t_0) = Y(t_{00};t)[\delta\varphi(t_{00}) + (\dot{\varphi}_0(t_{00}) - f_0^{+})\delta t_0] + o(t;\delta w) \tag{5.2}$$

(see Theorem 3.3). It is not difficult to see that

$$\begin{aligned} b_1(t;t_0,\delta w) &= \sum_{i=1}^{s}\bigg(\int\limits_{t_0-\tau_{i0}}^{t_{00}} Y(\xi;t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + \int\limits_{t_{00}}^{t_0} Y(\xi;t) f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\bigg)\\ &= \sum_{i=1}^{s}\bigg(\int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi;t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi - \int\limits_{t_{00}-\tau_{i0}}^{t_0-\tau_{i0}} Y(\xi;t) f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\bigg) + o(t;\delta w)\\ &= \sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi;t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi. \end{aligned} \tag{5.3}$$

In a similar way, with nonessential changes, for $t\in[\widehat{t}_0,t_1]$ one can prove

$$b_2(t;t_0,\delta w) = -\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t} Y(\xi;t) f_{x_i}[\xi]\dot{x}_0[\xi]\,d\xi\bigg]\delta\tau_i + o(t;\delta w). \tag{5.4}$$

Taking into account (5.2)–(5.4), from (5.1) we obtain formula (2.4) on the interval $[\widehat{t}_0,t_1]$, where $\delta x(t;\delta w)$ is of the form (2.12).

References

[1] R. Gabasov and F. Kirillova, *The Qualitative Theory of Optimal Processes.* (Russian) Monographs in Theoretical Foundations of Technical Cybernetics. Izdat. "Nauka", Moscow, 1971.

[2] K. Berdawood, G. Berikelashvili, Ph. Dvalishvili and T. Tadumadze, Representation formulas of solution for a perturbed controlled functional-differential equation considering variation of the initial moment and continuity of the initial condition. *Semin. I. Vekua Inst. Appl. Math. Rep.* **50** (2024), 18–34.

[3] R. V. Gamkrelidze, On some extremal problems in the theory of differential equations with applications to the theory of optimal control. *J. SIAM Control Ser. A* **3** (1965), 106–128.

[4] R. V. Gamkrelidze, Necessary first order conditions, and the axiomatics of extremal problems. (Russian) *Trudy Mat. Inst. Steklov.* **112** (1971), 152–180, 387.

[5] R. V. Gamkrelidze, *Principles of Optimal Control Theory.* Revised edition. Mathematical Concepts and Methods in Science and Engineering, Vol. 7. Plenum Press, New York-London, 1978.

[6] R. V. Gamkrelidze and G. L. Kharatishvili, Extremal problems in linear topological spaces. I. *Math. Systems Theory* **1** (1967), 229–256.

[7] R. V. Gamkrelidze and G. L. Haratishvili, Extremal problems in linear topological spaces. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **33** (1969), 781–839.

[8] G. L. Kharatishvili, *Optimal Processes with Delays.* (Russian) Metsniereba, Tbilisi, 1966.

[9] G. L. Kharatishvili, Z. A. Machaidze, N. I. Markozashvili, and T. A. Tadumadze, *Abstract Variational Theory and its Applications to Optimal Problems with Delays.* Metsniereba, Tbilisi, 1973.

[10] G. L. Kharatishvili and T. A. Tadumadze, Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments. (Russian) *Sovrem. Mat. Prilozh.* no. 25, Optimal. Upr. (2005), 3–166; translation in *J. Math. Sci. (N.Y.)* **140** (2007), no. 1, 1–175.

[11] K. B. Mansimov, *Singular Controls in Systems with Delay.* (Russian) *Elm, Baku*, 1999.

[12] M. D. Mardanov, *Certain Problems of the Mathematical Theory of Optimal Processes with Delay.* (Russian) *Azerb. State Univ., Baku* 1987.

[13] T. K. Melikov, *Singular Controls in Hereditary Systems.* (Russian) *ELM, Baku*, 2002.

[14] A. Nachaoui, T. Shavadze and T. Tadumadze, The local representation formula of solution for the perturbed controlled differential equation with delay and discontinuous initial condition. *Mathematics* **8** (2020), no. 10, 12 pp.

[15] L. W. Neustadt, *Optimization. A Theory of Necessary Conditions.* With a chapter by H. T. Banks. Princeton University Press, Princeton, NJ, 1976.

[16] N. M. Ogustoreli, *Time-Delay Control Systems.* Academic Press, New York–London, 1966.

[17] L. S. Pontrjagin, V. G. Boltjanskiĭ, R. V. Gamkrelidze and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes.* (Russian) Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961.

[18] T. Shavadze, Local variation formulas of solutions for nonlinear controlled functional differential equations with constant delays and the discontinuous initial condition. *Georgian Math. J.* **27** (2020), no. 4, 617–628.

[19] T. Tadumadze, Formulas of variation for solutions for some classes of functional differential equations and their applications. *Nonlinear Anal.* **71** (2009), no. 12, e706–e710.

[20] T. Tadumadze, Variation formulas of solutions for functional differential equations with several constant delays and their applications in optimal control problems. *Mem. Differ. Equ. Math. Phys.* **70** (2017), 7–97.

[21] T. Tadumadze and L. Alkhazishvili, Formulas of variation of solution for non-linear controlled delay differential equations with continuous initial condition. *Mem. Differential Equations Math. Phys.* **31** (2004), 83–97.

[22] T. Tadumadze, Ph. Dvalishvili and T. Shavadze, On the representation of solution of the perturbed controlled differential equation with delay and continuous initial condition. *Appl. Comput. Math.* **18** (2019), no. 3, 305–315.

[23] T. Tadumadze and N. Gorgodze, Variation formulas of a solution and initial data optimization problems for quasi-linear neutral functional differential equations with discontinuous initial condition. *Mem. Differ. Equ. Math. Phys.* **63** (2014), 1–77.

[24] T. Tadumadze and N. Gorgodze, Variation formulas of solution for a functional differential equation with delay function perturbation. (Russian) *Izv. Nats. Akad. Nauk Armenii Mat.* **49** (2014), no. 2, 65–80; translation in *J. Contemp. Math. Anal.* **49** (2014), no. 2, 98–108.

[25] T. Tadumadze and A. Nachaoui, Variation formulas of solution for a controlled functional-differential equation considering delay perturbation. *TWMS J. Appl. Eng. Math.* **1** (2011), no. 1, 58–68.

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