Memoirs on Differential Equations and Mathematical Physics

Volume 96, 2025, 1–28

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BOSE–EINSTEIN CONDENSATION AND NON-EXISTENCE SOLUTION: DIFFERENTIAL EQUATION APPROACH

Abstract. In this work, we study the non-existence of a solution to an eigenvalue problem for the Gross–Pitaevskii equation for Bose–Einstein condensation with repulsive interaction and a trapping potential given as Borel regular measure which vanishes at infinity in some sense. By differential equation approach, we obtain various results on the nonexistence of the eigenvalue problem. Some celebrated Hardy type inequalities are obtained.

2020 Mathematics Subject Classification. 35A01, 35A23, 35D30.

Key words and phrases. Bose–Einstein condensate, Hardy inequality, killing potential Technique, Pokhozhaev's test function.

1 Introduction

One of the most promising discoveries of the 20^{th} century physics of matter is the Bose–Einstein condensation (BEC) phenomenon: at a very low temperature, close to the absolute zero, the de-Broglie wavelength of the bosons increases to the size comparable to the inter-particle spacing, so that the bosons behave coherently, occupy the same quantum state and form a condensate. The first mathematical model of BEC is of 1924, due to S. N. Bose [3] and A. Einstein [9]. However, the first experimental observation of the phenomenon appeared as late as 1995 by three separate experiments: due to M. H. Anderson et al. [1] on vapour of rubidium, K. B. Davis et al. [6] on vapour of sodium and C. C. Bradley et al. [4] on vapour of lithium (see [27] for a detailed history).

The GPE model, introduced by E. P. Gross [15] and L. P. Pitaevskii [26], describes an ultra cold dilute bosonic gases confined in an external trap, so that only binary interaction is important. The model was ultimately justified by E. H. Lieb [18,19] and H. T. Yau et al. [10]. In describing the model, we follow W. Bao [2].

We study this phenomenon using the GPE model. In Section 2, we have some results on Sobolev spaces, auxiliary for the further works. Some approximation results might be of interest to specialist. In Section 3, we consider Borel regular measures. In Section 4, we construct the energy space using the methods of Dirichlet forms (see, e.g., [11] and [12]). In Section 5, Hardy type inequalities are considered, where we obtain the celebrated Sobolevskii inequality (see Corollary 5.4) and Poincaré inequality (see Corollary 5.3) as corollaries. In Section 6, we show that the energy space D (see Definition 4.1) is isometrically isomorphic to a weighted Sobolev space. This isometrical isomorphism is a key tool in the killing potential technique (see Section 7).

Throughout this work, by solution we mean a weak solution unless otherwise stated, and dV refers to a measure ($dV = V\,dx$ if $V \in L^1_{loc}$). For brevity, we will omit the entire space $\mathbb{R}^N$ when writing, so instead of $W^{1,2}(\mathbb{R}^N)$ we will write $W^{1,2}$.

2 On Sobolev space

This section contains a toolbox related to Sobolev spaces and other auxiliary materials for the further works. In this section, $\Omega \subset \mathbb{R}^N$ is an open set unless otherwise stated.

2.1 A Sobolev space and its maps. Set capacity

Definition 2.1 (Cf. [28]) (Sobolev space)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set, $\rho \in L^1_{loc}(\Omega)$, $\rho > 0$ a.e. For $k = 1, 2, \dots,$ we define the Sobolev space $W_0^{k,2}(\Omega, \rho\,dx)$ as a completion of $C_c^\infty(\Omega)$ in the following norm:

$$\|f\|_{W^{k,2}(\rho\,dx)} = \Big(\sum_{|\alpha|\le k} \|D^\alpha f\|^2_{L^2(\rho\,dx)}\Big)^{\frac{1}{2}}.$$

Note that in general this completion is not a functional space, since it may contain the zero 'function' with non-zero derivative. This is known as the closability problem (see, e.g., [11, Section 1.3 and 3.1] for a discussion of the case $k = 1$). In this work, we mostly consider the case $\rho \equiv 1$ (no weight), in such a case we suppress the measure when referring to a space. In Section 7, we have to consider $\rho = w^2$ with $w \in W^{1,2}_{loc}(\Omega)$, $w > 0$ a.e. For this case, [11, Theorem 3.1.3] shows that $W_0^{1,2}(\Omega, \rho\,dx)$ is a well defined functional space. Later on, we consider only these two cases.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. There exists $K > 0$ such that for all $\phi \in \mathrm{Lip}(\Omega)$ and $w \in W_0^{1,2}(\Omega, \rho\,dx)$, one has*

$$\|\phi w\|_{W^{1,2}} \le K\|\phi\|_{\mathrm{Lip}}\|w\|_{W^{1,2}}.$$

Proof.

$$\|\phi w\|^2_{W^{1,2}} \le \|\phi\|^2_{L^\infty}\|w\|^2_{L^2} + \big(\|\phi\|^2_{L^\infty} + \|\nabla\phi\|^2_{L^\infty}\big)\|w\|^2_{W^{1,2}} \le 2\big(\|\phi\|^2_{L^\infty} + \|\nabla\phi\|^2_{L^\infty}\big)\|w\|^2_{W^{1,2}}. \qquad \square$$

Proposition 2.1. *Let $(X, (\rho_\alpha))$ be a gauge space (see [7, IX.10]), where (ρ_α) is a separating family of pseudometrics on X. Assume that a compact sequence x_n has a unique limit point x, that is, every subsequence x_k contains a sub-subsequence x_j such that $x_j \to x$ as $j \to \infty$. Then*

$$x_n \to x \ \text{ as } \ n \to \infty.$$

Proof. Assume to the contrary that $x_n \nrightarrow x$. That is, there exists a pseudometric ρ_α such that $\limsup_{n\to\infty} \rho_\alpha(x_n, x) > 0$. Then there exist $\sigma > 0$ and a subsequence x_k such that $\rho_\alpha(x_k, x) \geq \sigma$ for all x_k. However, x_k contains a sub-subsequence $x_j \to x$ as $j \to \infty$. Contradiction to assumption $\rho_\alpha(x_j, x) \geq \sigma > 0$. □

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $M > 0$,*

$$R_M(t) = \begin{cases} t, & |t| \leq M, \\ 0, & |t| > M. \end{cases}$$

Let $u_n \to u$ as $n \to \infty$ in $W_0^{1,2}(\Omega, \rho\, dx)$ with $w > 0$, $w \in W^{1,2}_{loc}(\Omega)$. Then

$$R_M(u_n) \to R_M(u) \ \text{ in } W_0^{1,2}(\rho\, dx) \text{ as } n \to \infty.$$

Proof. Recall that we are proving that R_M is a continuous mapping on the space $W_0^{1,2}(\rho\, dx)$ and that $u_n \to u$ in $W_0^{1,2}(\rho\, dx)$ as $n \to \infty$.

Since R_M is a contraction,

$$|R_M(u_n) - R_M(u)| \leq |u_n - u|.$$

So, $R_M(u_n) \to R_M(u)$ in $L^2(\rho\, dx)$. Next,

$$\begin{aligned} \nabla R_M(u_n) - \nabla R_M(u) &= \nabla u_n \, \mathbf{1}_{\{|u_n| \leq M\}} - \nabla u \, \mathbf{1}_{\{|u| \leq M\}} \\ &= (\nabla u_n - \nabla u) \, \mathbf{1}_{\{|u_n| \leq M\}} + \nabla u (\mathbf{1}_{\{|u_n| \leq M\}} - \mathbf{1}_{\{|u| \leq M\}}). \end{aligned}$$

Thus

$$\big|\nabla R_M(u_n) - \nabla R_M(u)\big| \leq |\nabla u_n - \nabla u| + |\nabla u| \, \mathbf{1}_{E_n},$$

where

$$E_n = \big\{|u| \leq M < |u_n|\big\} \cup \big\{|u_n| \leq M < |u|\big\}.$$

Hence

$$\limsup_{n\to\infty} \|R_M(u_n) - R_M(u)\|_{W^{1,2}} \leq \limsup_{n\to\infty} \|\nabla u \, \mathbf{1}_{E_n}\|_{L^2} \equiv \limsup_{n\to\infty} (\mu(E_n))^{\frac{1}{2}} \leq \Big(\mu\Big(\limsup_{n\to\infty} E_n\Big)\Big)^{\frac{1}{2}},$$

where $d\mu = |\nabla u|^2 \rho\, dx$. Now, observe that

$$\limsup_{n\to\infty} E_n \subseteq \Big\{|u| \leq M \leq \limsup_{n\to\infty} |u_n|\Big\} \cup \Big\{\liminf_{n\to\infty} |u_n| \leq M < |u|\Big\}.$$

Hence, if u_k is a subsequence such that $u_k \to u$ a.e., then

$$\limsup_{n\to\infty} \mu(E_k) \leq \mu\big(\{|u| = M\}\big) = 0.$$

since

$$\big\|\nabla u \, \mathbf{1}_{\{|u| = M\}}\big\|_{L^2} = 0.$$

Hence, for every subsequence u_n, we can subtract a sub-subsequence u_k such that

$$R_M(u_k) \to R_M(u) \ \text{ in } W_0^{1,2}(w^2\, dx) \text{ as } k \to \infty.$$

Hence the assertion follows by Proposition 2.1. □

Corollary 2.1. *For every $u \in W_0^{1,2}(\Omega, \rho\, dx)$, there exists a sequence $\theta_n \in \mathrm{Lip}_c(\Omega)$ such that $0 \le \theta_n \le 1$,*

$$\begin{gathered} \|\theta_n\|_{W^{1,2}} \to 0 \ \ as \ \ n \to \infty, \\ u(1-\theta_n) \in C_0(\Omega). \end{gathered} \tag{2.1}$$

Proof. Due to [11, Theorem 2.1.3] (the Lusin type theorem), for $n = 1, 2, 3, \dots$, there exists a growing sequence F_n of closed sets $F_1 \subset F_2 \subset \cdots \subset \Omega$ such that $\mathrm{cap}_1(\Omega \setminus F_n) \to 0$ as $n \to \infty$. Let $\theta_n \in \mathrm{Lip}_c(\Omega)$, $0 \le \theta_n \le 1$, $\theta_n = 1$ on $\Omega \setminus F_n$, be such that

$$\mathrm{cap}_1(\Omega \setminus F_n) + 2^{-n} > \|\theta_n\|^2_{W^{1,2}} \ge \mathrm{cap}_1(\Omega \setminus F_n).$$

Then (2.1) follows. Since $(1-\theta_n) = 0$ on $\Omega \setminus F_n$, one has $u(1-\theta_n) \in C_0(\Omega)$, by [11, Theorem 2.1.3] (the Lusin type theorem). For the definition of capacity, see, e.g., [23, Section 2.2.1]) and [20, Theorem 11.16]. □

2.2 Approximation in Sobolev spaces

The next proposition shows that bounded functions are dense in $W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx)$.

Proposition 2.2. *Let $q > 1$, $\beta > 0$ and let $u \in W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx)$. Then*

$$R_M(u) \to u \ \ as \ M \to \infty \ in \ W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx),$$

where $R_M(u)$ is as in Theorem 2.1.

Proof. For $M > 0$, $u_M = R_M(u)$. Then

$$|u - u_M| = (|u| - M)\, \mathbf{1}_{\{|u|>M\}} \le |u|\, \mathbf{1}_{\{|u|>M\}} \longrightarrow 0 \ \text{ as } \ M \to \infty$$

in $L^2(\rho\, dx) \cap L^q(\rho^\beta\, dx)$ by the Lebesgue dominated convergence theorem (LDCT).

Since $\nabla u_M = \nabla u\, \mathbf{1}_{\{|u| \le M\}}$, we have

$$|\nabla u - \nabla u_M| = |\nabla u|\, \mathbf{1}_{\{|u|>M\}} \longrightarrow 0 \ \text{ as } \ M \to \infty$$

in $L^2(\rho\, dx)$ by the LDCT. □

Proposition 2.3. *Bounded continuous functions are dense in $W_0^{1,2}(\Omega, \rho\, dx) \cap L^q(\Omega, \rho^\beta\, dx)$.*

Proof. By Proposition 2.2, it suffices to consider $u \in W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx) \cap L^\infty$. Let a sequence $\theta_n \in \mathrm{Lip}_c(\Omega)$ be as in Corollary 2.1. Then $u(1-\theta_n) \in C_0(\Omega)$. Now,

$$\begin{gathered} |u - u(1-\theta_n)| = |u|\theta_n, \\ \big|\nabla u - \nabla(u(1-\theta_n))\big| = |\nabla(u\theta_n)| \le \theta_n |\nabla u| + |u|\, |\nabla\theta_n| \le \theta_n |\nabla u| + \|u\|_{L^\infty} |\nabla \theta_n|. \end{gathered}$$

In its turn, $|u|\theta_n \to 0$ in $L^2(\rho\, dx) \cap L^q(\rho^\beta\, dx)$ and $|\nabla u|\theta_n \to 0$ in $L^2(\rho\, dx)$ as $n \to \infty$ by LDCT and (2.1).

Finally, $|\nabla \theta_n| \to 0$ in $L^2(\rho\, dx)$ as $n \to \infty$ by (2.1). □

Proposition 2.4. $\mathrm{Lip}_c(\Omega)$ *is dense in $W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx)$, that is, for every $u \in W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx)$, there exists a sequence $\widetilde{\phi}_j \in \mathrm{Lip}_c(\Omega)$ such that*

$$\widetilde{\phi}_j \to u \ \ in \ W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx) \ \ as \ j \to \infty.$$

Moreover,

$$\mathrm{supp}(\widetilde{\phi}_j) \subseteq \mathrm{supp}(u).$$

Proof. Let $u \in C_0 \cap W_0^{1,2}(\rho\, dx) \cap L^q(\rho^\beta\, dx)$.

1. For $n = 1, 2, \ldots$, denote $\xi_n(x)$ as

$$\xi_n(x) = \begin{cases} 1, & |u| \geq \dfrac{1}{n}, \\ \big(2n|u|(x) - 1\big)_+, & \dfrac{1}{2n} \leq |u| \leq \dfrac{1}{n}\,, \\ 0, & |u| \leq \dfrac{1}{2n}\,. \end{cases}$$

Then

$$\mathbf{1}_{\{|u|>\frac{1}{2n}\}} \geq \xi_n \geq \mathbf{1}_{\{|u|>\frac{1}{n}\}}, \quad \nabla\xi_n = \mathbf{1}_{\{\frac{1}{n}>|u|>\frac{1}{2n}\}} 2n\nabla|u|.$$

Hence

$$|\nabla\xi_n| = 2n\,\mathbf{1}_{\{\frac{1}{n}>|u|>\frac{1}{2n}\}}|\nabla u|.$$

Since $\xi_n \to \mathbf{1}_{\{|u|>0\}}$ pointwise as $n \to \infty$, it follows that $u\xi_n \to u$ in $L^2(\rho\,dx) \cap L^q(\rho^\beta\,dx)$ by the LDCT. In its turn,

$$\begin{aligned} |\nabla(u\xi_n) - \nabla u| &= \big|\xi_n \nabla u + u\nabla\xi_n - \nabla u\big| \leq |\xi_n - 1|\,|\nabla u| + |u|\,|\nabla\xi_n| \\ &= |\xi_n - 1|\,|\nabla u| + 2n\,\mathbf{1}_{\{\frac{1}{n}>|u|>\frac{1}{2n}\}}|u|\,|\nabla u| \longrightarrow 0 \ \text{ in } L^2(\rho\,dx) \text{ as } n \to \infty \end{aligned}$$

by the LDCT, since

$$\mathbf{1}_{\{\frac{1}{2n}<|u|<\frac{1}{n}\}} \leq 2n|u|\,\mathbf{1}_{\{\frac{1}{2n}<|u|<\frac{1}{n}\}} \leq 2\,\mathbf{1}_{\{\frac{1}{2n}<|u|<\frac{1}{n}\}},$$

So, $u\xi_n \to u$ in $W_0^{1,2}(\rho\,dx) \cap L^q(\rho^\beta\,dx)$ as $n \to \infty$.

2. Denote $F_n = \{|u| \geq \frac{1}{2n}\}$, $G_n = \{|u| > \frac{1}{3n}\}$. Observe that F_n is closed, G_n is open and that $\operatorname{supp}(u\xi_n) \subset F_n$. Let

$$\eta_n(x) = \frac{\operatorname{dist}(x, G_n^c)}{\operatorname{dist}(x, G_n^c) + \operatorname{dist}(x, F_n)}\,.$$

Then $\eta_n \in \operatorname{Lip}_c(\Omega)$, $\mathbf{1}_{G_n} \geq \eta_n \geq \mathbf{1}_{F_n}$. Denote $M = \|u\|_{L^\infty}$ and let $\phi_{nj} \in \operatorname{Lip}_c(\Omega)$ be such that $\phi_{nj} \to u\xi_n$ in $W_0^{1,2}(\rho\,dx)$ as $j \to \infty$ for all $n = 1, 2, \ldots$. Let $\widehat{\phi}_{nj} = R_M(\phi_{nj})$ and $\widetilde{\phi}_{nj} = \eta_n\widehat{\phi}_{nj}$. Observe that $\widetilde{\phi}_{nj} \to u\xi_n$ in $W_0^{1,2}(\rho\,dx)$ as $j \to \infty$ by Lemma 2.1 and Theorem 2.1. Now, we prove that $\widetilde{\phi}_{nj} \to u\xi_n$ in $L^q(\rho^\beta\,dx)$ as $j \to \infty$. Indeed, for every $n = 1, 2, \ldots$ and every subsequence of sequence $\widetilde{\phi}_{nj}$, we can find a sub-subsequence $\widetilde{\phi}_{nk}$ such that $\widetilde{\phi}_{nk} \to u\xi_n$ a.e. as $k \to \infty$. Note that

$$|\widetilde{\phi}_{nk}| \leq M\,\mathbf{1}_{G_n}, \ \ \forall\, n, k = 1, 2, 3, \ldots,$$

which implies that $\operatorname{supp}(\widetilde{\phi}_j) \subseteq \operatorname{supp}(u)$ and $\mathbf{1}_{G_n} \in L^q(\rho^\beta\,dx)$ by Markov–Chebyshev inequality. Hence

$$\widetilde{\phi}_{nk} \to u\xi_n \ \text{ in } L^q(\rho^\beta\,dx) \text{ as } k \to \infty, \ \ \forall\, n = 1, 2, \ldots,$$

by the LDCT. Then for every $n = 1, 2, \ldots$, $\widetilde{\phi}_{nj}$ is compact in $L^q(\rho^\beta\,dx)$ with the only limit point $u\xi_n$. Hence $\widetilde{\phi}_{nj} \to u\xi_n$ in $L^q(\rho^\beta\,dx)$ as $j \to \infty$ by Proposition 2.1. Summarizing

$$\begin{gathered} \widetilde{\phi}_{nj} \to u\xi_n \ \text{ in } W_0^{1,2}(\rho\,dx) \cap L^q(\rho^\beta\,dx) \text{ as } j \to \infty, \ \ \forall\, n = 1, 2, \ldots, \\ u\xi_n \to u \ \text{ in } W_0^{1,2}(\rho\,dx) \cap L^q(\rho^\beta\,dx) \text{ as } n \to \infty, \end{gathered}$$

so the assertion follows. □

Lemma 2.2 (Caccioppoli type identity)**.** *Let $u \in W^{1,1}_{loc}$, $\xi \in \operatorname{Lip}_{loc}$. Then*

$$\nabla u \cdot \nabla(u\xi^2) = |\nabla(u\xi)|^2 - |u|^2|\nabla\xi|^2.$$

Proof. Since $u, \xi, \xi^2 \in W^{1,1}_{loc}$, $u\xi, u\xi^2 \in L^1_{loc}$ and $\xi\nabla u + u\nabla\xi \in L^1_{loc}$, $\xi^2\nabla u + 2u\xi\nabla\xi \in L^1_{loc}$, it follows that $u\xi, u\xi^2 \in W^{1,1}_{loc}$ and the product rule applies:

$$\begin{aligned}\nabla u \cdot \nabla(u\xi^2) &= \xi\nabla u \cdot \nabla(u\xi) + u\xi\nabla u \cdot \nabla\xi \\ &= \nabla(u\xi)\cdot\nabla(u\xi) - u\nabla\xi\cdot\nabla(u\xi) + u\nabla(u\xi)\cdot\nabla\xi - u^2\nabla\xi\cdot\nabla\xi \\ &= |\nabla(u\xi)|^2 - |u|^2|\nabla\xi|^2.\end{aligned}$$

□

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in W^{1,2}_{loc}(\Omega)$. Assume that there exist a sequence $\xi_n \in \mathrm{Lip}_c(\Omega)$ and $M > 0$ such that $\xi_n \to 1$ a.e. and $\|u\xi_n\|_{L^2} \le M$, $\|\nabla(u\xi_n)\|_{L^2} \le M$. Then*

$$u \in W^{1,2}_0(\Omega) \ \text{ and } \ u\xi_n \xrightarrow{w} u \ \text{ in } W^{1,2}_0(\Omega) \text{ as } n \to \infty.$$

Proof. It follows from the assumption that the sequence $(u\xi_n)$ is weakly compact in $W^{1,2}_0(\Omega)$, by [14, Theorem 5.12]. Let w be a weak limit point of this sequence. Then there exists a sub-sequence ξ_k such that $u\xi_k \xrightarrow{w} w$ as $k \to \infty$ in $W^{1,2}_0(\Omega)$. Since $W^{1,2}_0(\Omega)$ is compactly embedded in $L^2_{loc}(\Omega)$ (see [14, Theorem 7.22] (Rellich–Kondrachov Theorem)), it follows that $u\xi_k \xrightarrow{s} w$ as $k \to \infty$ in $L^2_{loc}(\Omega)$. Hence there exists a sub-subsequence ξ_j such that

$$\lim_{j\to\infty} u\xi_j \xrightarrow{w} w \ \text{ in } W^{1,2}_0(\Omega), \quad \lim_{j\to\infty} u\xi_j \xrightarrow{s} w \ \text{ in } L^2_{loc}(\Omega).$$

By [8, III. 6, Corollary 3],

$$\lim_{j\to\infty} u\xi_j \xrightarrow{a.e.} w \ \text{ in } \Omega.$$

However, $\xi_j \xrightarrow{a.e.} 1$. Hence $u\xi_j \xrightarrow{a.e.} u$ as $j \to \infty$ in Ω. So, $u = w$, a.e. and hence $u \in W^{1,2}_0(\Omega)$ is the only weak limit point of the sequence. Then, by Proposition 2.1,

$$\lim_{n\to\infty} u\xi_n \xrightarrow{w} u \ \text{ in } W^{1,2}_0(\Omega).$$

□

3 Regularity of weak solution to elliptic PDE and regular, singular sets for a measure

At the beginning of this section we recall one important theorem (local elliptic regularity theorem), the rest will be devoted to the Borel regular measures. For the definition of weak solution, we follow [14, Section 8.3].

Theorem 3.1 (Cf. [14, Theorem 8.8], local elliptic regularity theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A : \Omega \to \mathbb{R}^{N\times N}$ and $B_1 : \Omega \to \mathbb{R}^N$ be locally Lipschitz continuous functions in Ω, $A > 0$. Let $B_2, C \in L^\infty_{loc}(\Omega)$, $f \in L^2_{loc}(\Omega)$ and let $u \in W^{1,2}_{loc}(\Omega)$ be a (local) weak solution to the equation*

$$-\operatorname{div}(A\nabla u + B_1 u) + B_2\nabla u + Cu = f.$$

Then $u \in W^{2,2}_{loc}(\Omega)$.

Lemma 3.1. *Given an open set $\Omega \subseteq \mathbb{R}^N$ for a Borel regular measure $V \ge 0$ on Ω, the regular set of points with respect to measure V $(\mathrm{Reg}(V))$ is open set and the singular set of points with respect to measure V $(\mathrm{Sing}(V))$ is closed.*

Proof. Let $x_0 \in \mathrm{Reg}(V)$, $r > 0$, be such that $V(B_r(x_0)) < \infty$. For every $x \in B_r(x_0)$, let $s < r - |x - x_0|$ (note that $r - |x - x_0| > 0$). Then

$$B_s(x) \subset B_r(x_0) \Longrightarrow V(B_s(x)) < V(B_r(x_0)) < \infty \ \ \forall\, x \in \mathrm{Reg}(V).$$

Now, note that $\mathrm{Sing}(V) = \Omega \setminus \mathrm{Reg}(V)$. So, $\mathrm{Sing}(V)$ is closed. □

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $V \geq 0$ be a Borel regular measure on Ω, and $u \in C(\Omega)$ be such that*
$$\int_\Omega |u|\,dV < \infty.$$
Then for all $\sigma > 0$,
$$\{|u| > \sigma\} \cap \operatorname{sing}(V) = \varnothing,$$
or, equivalently, $u = 0$ on $\operatorname{Sing}(V)$.

Proof. Assume that there exists $x \in \{|u| > \sigma\} \cap \operatorname{sing}(V)$. Since u is continuous, $\{|u| > \sigma\}$ is an open set, so there exists $r > 0$ such that $B_r(x) \subset \{|u| > \sigma\}$. Then
$$\int_\Omega |u|\,dV \geq \int_{B_r(x)} \sigma\,dV = \sigma V(B_r(x)) = \infty,$$
since $x \in \operatorname{sing}(V)$. Since $\{|u| > \sigma\} \cap \operatorname{Sing}(V) = \varnothing$ for all $\sigma > 0$, it follows that
$$|u|(x) \leq \sigma, \;\; \forall\, x \in \operatorname{Sing}(V), \;\; \forall\, \sigma > 0.$$
So,
$$0 \leq |u|(x) \leq \inf_{\sigma > 0} \sigma = 0.$$
□

Corollary 3.1. *Let $G \subseteq \Omega$ be an open set. Assume that $V\mathbf{1}_G \in \mathcal{D}'(G)$. Then $G \cap \operatorname{Sing}(V) = \varnothing$.*

Recall that $\mathcal{D}'(\Omega)$ is the dual space of $\mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ denotes $C_c^\infty(\Omega)$ with the topology of uniform convergence of all derivatives on compact subsets of Ω (i.e. the space of test functions).

Proof of Corollary 3.1. Consider $x_0 \in G \cap \operatorname{Sing}(V)$. Then there exists $\phi \in \mathcal{D}(G)$, $\phi \geq 0$, $\phi(x_0) = 1$. Since $V\,\mathbf{1}_G \in \mathcal{D}'(G)$,
$$(V\,\mathbf{1}_G)(\phi) = \int_G \phi\,dV < \infty.$$
Hence, by Lemma 3.2, $\phi = 0$ on $\operatorname{Sing}(V)$. This contradicts the fact that $\phi(x_0) = 1$. □

Theorem 3.2. *Let $G \subseteq \Omega \subseteq \mathbb{R}^N$ be an open set such that $V\,\mathbf{1}_G$ is a Radon measure on G. Then $G \subset$ Reg (V). Moreover, $V\,\mathbf{1}_{\operatorname{Reg}(V)}$ is a Radon measure on Reg (V).*

Proof. Recall that $\operatorname{Reg}(V)$ is open by Lemma 3.1. If $V\,\mathbf{1}_G$ is a Radon measure on G, then $V\,\mathbf{1}_G \in \mathcal{D}'(G)$.

By Corollary 3.1, it follows that $G \cap \operatorname{Sing}(V) = \varnothing$. Hence $G \subseteq \operatorname{Reg}(V)$ as follows:
$$G \cap \operatorname{Reg}(V) = G \cap (\Omega \setminus \operatorname{Sing}(V)) = (G \cap \Omega) \setminus (G \cap \operatorname{Sing}(V)) = G \setminus \varnothing = G,$$
that is, $G \subseteq \operatorname{Reg}(V)$. Now, we prove the second assertion. Let $K \subset \operatorname{Reg}(V)$ be a compact. For every $x \in K$, let $r_x > 0$ be such that $V(B_{r_x}(x)) < \infty$. Then the collection $\{B_{r_x}(x)\}_{x\in K}$ is an open covering of K. Hence there are finitely many $x_1, x_2, \ldots, x_M$ such that
$$K \subseteq \bigcup_{m=1}^{M} B_{r_{x_m}}(x_m).$$
So,
$$V(K) \leq \sum_{m=1}^{M} V(B_{r_{x_m}}(x_m)) < \infty.$$
□

4 Energy space

One of our fundamental tools in this work is the construction of an energy space which is carried our in this section.

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $V \geq 0$ be a Borel regular measure on Ω.

(1) We define a Hilbert space $\widehat{D}(\Omega) = \widehat{D}(\Omega, V)$ as a completion of the set

$$\widetilde{D}(\Omega) = \left\{ \phi \in \mathrm{Lip}_c(\Omega) \mid \int\limits_{\Omega} \phi^2 \, dV < \infty \right\} = \mathrm{Lip}_c(\Omega) \cap L^2(\Omega, dV)$$

in the norm $\|\cdot\|_D$,

$$\|\phi\|_D^2 = \|\phi\|_{W^{1,2}}^2 + \int \phi^2 \, dV.$$

(2) Let J denote an extension to $\widehat{D}(\Omega)$ of the identity operator

$$id : \widetilde{D}(\Omega) \subset \widehat{D}(\Omega) \to W_0^{1,2}(\Omega).$$

So, $J : \widehat{D}(\Omega) \to W_0^{1,2}(\Omega)$.

Note that J is a contraction, since $\|\phi\|_{W^{1,2}} \leq \|\phi\|_D$ for all $\phi \in \widetilde{D}$.

(3) In general, $\mathrm{Ker}(J) \neq \{0\}$ is a closed subspace in the Hilbert space $\widehat{D}(\Omega)$. Then

$$D(\Omega) := \widehat{D}(\Omega) \ominus \mathrm{Ker}(J) \equiv (\mathrm{Ker}(J))^{\perp}.$$

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, V be a Borel regular measure on Ω. Then:*

(1) $\widetilde{D}(\Omega) = \widetilde{D}(\mathrm{Reg}(V)) = \mathrm{Lip}_c(\mathrm{Reg}(V))$.

(2) $\widehat{D}(\Omega) = \widehat{D}(\mathrm{Reg}(V))$.

(3) $D(\Omega) = D(\mathrm{Reg}(V))$.

Proof. It follows from Lemma 3.2 that every $\phi \in \widetilde{D}(\Omega)$ satisfies $\phi \equiv 0$ on $\mathrm{Sing}(V)$. Hence

$$\widetilde{D}(\Omega) = \widetilde{D}(\mathrm{Reg}(V)).$$

Now observe that, by Theorem 3.2, $V\,\mathbf{1}_{\mathrm{Reg}(V)}$ is a Radon measure on $\mathrm{Reg}(V)$. Hence

$$\mathrm{Lip}_c(\mathrm{Reg}(V)) \subset L^2(\mathrm{Reg}(V), dV).$$

So,

$$\widetilde{D}(\mathrm{Reg}(V)) = \mathrm{Lip}_c(\mathrm{Reg}(V)).$$

Assertions (2) and (3) follow from the first assertion. $\square$

Theorem 4.1 (See [11, Theorem 2.2.4] and [12, Lemma 2.1])**.** *Let $G \subset \mathbb{R}^N$ be an open set, V be a Radon measure on G. Then there exist a Borel set $\mathcal{R} \subset G$, $\mathrm{cap}_1(\mathcal{R}, G) = 0$, and an increasing sequence K_n of compacts $K_1 \subset K_2 \subset \cdots \subset G$ such that*

(1) $V\Big(G \setminus \Big(\mathcal{R} \cup \bigcup\limits_{n=1}^{\infty} K_n\Big)\Big) = 0.$

(2) $\mathrm{cap}_1(K \setminus K_n) \to 0$ *as $n \to \infty$ for all compact $K \subset G$.*

(3) *For every $n = 1, 2, 3, \dots$, there exists $C_n > 0$ such that*

$$\int\limits_{K_n \setminus \mathcal{R}} |\phi| \, dV \le C_n \|\phi\|_{W^{1,2}(G)}, \quad \forall \, \phi \in \mathrm{Lip}_c(G).$$

Definition 4.2.

(1) Let $\Omega \subset \mathbb{R}^N$ be an open set. A Radon measure μ on Ω is called smooth if it charges no set of zero capacity, that is, for every Borel set $E \subset \Omega$,

$$\mu(E) = 0 \;\text{ whenever } \mathrm{cap}_1(E, \Omega) = 0.$$

(2) For a Borel regular measure V on Ω, define V_s as $V_s = V \, \mathbf{1}_{\mathrm{Reg}(V) \setminus \mathcal{R}}$, where $\mathcal{R}$ is as in Theorem 4.1. It follows from Theorem 4.1 (3) that V_s is a smooth measure. V_s is called the smooth part of measure V.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set, V be a Borel regular measure on $\mathbb{R}^N$. Let J be as in Definition 4.1, and let $\mathcal{R}$ be as in Theorem 4.1 with $\Omega = \mathrm{Reg}(V)$. Then*

$$\mathrm{Ker}(J) = L^2(\mathcal{R}, dV) \equiv \Big\{ u \in L^2(\mathrm{Reg}(V), dV) \mid \quad u = 0 \textit{ on } \mathrm{Reg}(V) \setminus \mathcal{R}, \; V\textit{-a.e.} \Big\}.$$

Proof. Since $\mathrm{cap}_1(\mathcal{R}, \Omega) = 0$, it follows that $L^2(\mathcal{R}, dV) \subseteq \mathrm{Ker}(J)$. Hence we are left to prove that

$$\mathrm{Ker}(J) \subseteq L^2(\mathcal{R}, dV).$$

Let $u \in \mathrm{Ker}(J)$, $u \neq 0$. Note that $\widetilde{D}(\Omega, V) = \mathrm{Lip}_c(\Omega)$, since V is Radon measure. Then there exists $\phi_j \in \mathrm{Lip}_c(\mathrm{Reg}(V))$ such that

$$\phi_j \xrightarrow{s} 0 \;\text{ in } W_0^{1,2}(\mathrm{Reg}(V)) \text{ as } j \to \infty$$

and

$$\phi_j \xrightarrow{s} u \;\text{ in } L^2(\mathrm{Reg}(V), dV) \text{ as } j \to \infty.$$

For $n = 1, 2, 3, \dots$,

$$\phi_j \xrightarrow{s} u \;\text{ in } L^2(K_n \setminus \mathcal{R}, dV) \text{ as } j \to \infty,$$

since $K_n \setminus \mathcal{R} \subset \mathrm{Reg}(V)$. Then, by Theorem 4.1 (3),

$$\int\limits_{K_n \setminus \mathcal{R}} \phi_j \, dV \le C_n \|\phi_j\|_{W^{1,2}} \to 0 \;\text{ as }\; j \to \infty.$$

So, $u = 0$ on $K_n \setminus \mathcal{R}$, V-a.e. for every $n = 1, 2, 3, \dots$. Hence $u = 0$ on $\bigcup\limits_{n=1}^{\infty} K_n \setminus \mathcal{R}$, V-a.e. By Theorem 4.1 (1),

$$V\Big(\mathrm{Reg}(V) \setminus \Big(\mathcal{R} \cup \bigcup_{n=1}^{\infty} K_n \Big) \Big) = 0.$$

We conclude that $u = 0$ on $\mathrm{Reg}(V) \setminus \mathcal{R}$, V-a.e. □

Corollary 4.1.

$$D(\Omega) = W_0^{1,2}(\mathrm{Reg}(V)) \cap L^2(\mathrm{Reg}(V), dV_s).$$

Moreover, if $V(\mathcal{R}) = 0$, then $\widehat{D} = D$.

5 Hardy type inequality

This section demonstrates some recently developed Hardy type inequalities, as they will become an important tool in the following sections.

Definition 5.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, V be a (signed) Borel regular measure on Ω. By a positive local weak solution w to the equation

$$-\Delta w + Vw = 0 \tag{5.1}$$

we consider a function $w \in D_{loc}(\Omega, |V|)$ such that $w_- = 0$ in $W^{1,2}_{loc}(\mathrm{Reg}(|V|))$ and $L^2_{loc}(\mathrm{Reg}(V), d|V|_s)$, and

$$\int \nabla\xi \nabla w \, dx + \int \xi w \, dV = 0, \quad \forall\, \xi \in \mathrm{Lip}_c(\mathrm{Reg}(|V|)).$$

Proposition 5.1. *Let $w \in D_{loc}(\Omega)$ be a positive solution of equation (5.1). Then the following inequality for self-adjoint operators holds:*

$$\frac{1}{4}\frac{|\nabla w|^2}{w^2} \leq -\Delta + \frac{1}{2}V.$$

That is, one has

$$\frac{1}{4}\int \phi^2 \frac{|\nabla w|^2}{w^2}\, dx \leq \int |\nabla\phi|^2 \, dx + \frac{1}{2}\int \phi^2 \, dV, \ \ \forall\, \phi \in W^{1,2}_0(\mathrm{Reg}(|V|)) \cap L^2(\mathrm{Reg}(|V|), d|V|_s). \tag{5.2}$$

Proof. Without loss of generality, we assume that $\Omega = \mathrm{Reg}(|V|)$ and $|V|\,\mathbf{1}_{\{\mathrm{Reg}(V)\}}$ is a smooth measure on Ω. It suffices to prove the assertion for $\phi \in \mathrm{Lip}_c(\Omega) \cap L^2(\Omega, d|V|)$. For $\epsilon > 0$, let $\theta = \theta_\epsilon = \frac{\phi}{\sqrt{w+\epsilon}}$ be weak differentiable, so $\phi = \theta\sqrt{w+\epsilon}$ and

$$|\nabla\phi|^2 = (w+\epsilon)|\nabla\theta|^2 + \frac{1}{2}\nabla\theta^2 \cdot \nabla w + \frac{1}{4}\frac{\theta^2}{w+\epsilon}|\nabla w|^2. \tag{5.3}$$

Now, we show that $\theta^2 \in D_c(\Omega)$. Hence it is a test function for (5.1),

$$\theta^2 = \frac{\phi^2}{w+\epsilon} \leq \frac{1}{\epsilon}\phi^2 \in L^2(\Omega, d|V|) \cap L^\infty_c(\Omega),$$
$$|\nabla\theta^2| = \Big|\frac{2\phi\nabla\phi}{w+\epsilon} - \phi^2\frac{\nabla w}{w+\epsilon}\Big| \leq \frac{1}{\epsilon}\big(2|\phi\nabla\phi| + \phi^2|\nabla w|\big) \in L^2_c(\Omega).$$

Thus

$$\int \nabla\theta^2 \nabla w \, dx + \int \theta^2 w \, dV = 0,$$

so

$$\int \frac{1}{2}\nabla\theta^2 \cdot \nabla w \, dx = -\frac{1}{2}\int \theta^2 w \, dV.$$

It follows from integrating (5.3) that

$$\int |\nabla\phi|^2 \, dx = \int (w+\epsilon)|\nabla\theta|^2 \, dx - \frac{1}{2}\int w\theta^2 \, dV + \frac{1}{4}\int \frac{\theta^2}{w+\epsilon}|\nabla w|^2 \, dx. \tag{5.4}$$

Note that

$$-\frac{1}{2}\int w\theta^2 \, dV = \frac{1}{2}\int w\theta^2 \, dV_- - \frac{1}{2}\int w\theta^2 \, dV_+ \,.$$

So, from (5.4) we have

$$\int |\nabla\phi|^2 \, dx + \frac{1}{2}\int w\theta^2 \, dV_+ = \int (w+\epsilon)|\nabla\theta|^2 \, dx + \frac{1}{2}\int w\theta^2 \, dV_- + \frac{1}{4}\int \frac{\theta^2}{w+\epsilon}|\nabla w|^2 \, dx.$$

Therefore,

$$\int |\nabla\phi|^2\,dx + \frac{1}{2}\int w\theta^2\,dV_+ \geq \frac{1}{2}\int w\theta^2\,dV_- + \frac{1}{4}\int \frac{\theta^2}{w+\epsilon}\,|\nabla w|^2\,dx.$$

Now, observe that

$$\theta^2 w = \phi^2\,\frac{w}{w+\epsilon} \leq \phi^2.$$

So,

$$\frac{1}{2}\int w\theta^2\,dV_+ \leq \frac{1}{2}\int \phi^2\,dV_+\,.$$

On the other hand, $\theta^2 w = \phi^2 \frac{w}{w+\epsilon} \uparrow \phi^2$ and

$$\frac{|\nabla w|^2}{w+\epsilon}\,\theta^2 = \phi^2\,\frac{|\nabla w|^2}{(w+\epsilon)^2} \uparrow \phi^2\Big|\frac{\nabla w}{w}\Big|^2 \;\text{ as }\; \epsilon \downarrow 0.$$

Hence, by the Beppo Levi Lemma, we have

$$\frac{1}{2}\int \theta^2 w\,dV_- + \frac{1}{4}\int \frac{\theta^2}{w+\epsilon}\,|\nabla w|^2\,dx \uparrow \frac{1}{2}\int \phi^2\,dV_- + \frac{1}{4}\int \phi^2\,\Big|\frac{\nabla w}{w}\Big|^2\,dx \;\text{ as }\; \epsilon \downarrow 0.$$

Finally,

$$\int |\nabla\phi|^2\,dx + \frac{1}{2}\int \phi^2\,dV_+ \geq \frac{1}{2}\int \phi^2\,dV_- + \frac{1}{4}\int \phi^2\,\Big|\frac{\nabla w}{w}\Big|^2\,dx$$

which is equivalent to (5.2). □

Corollary 5.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set.*

(1) *Let $w \in W^{1,2}_{loc}(\Omega)$ be such that $w > 0$ a.e. on Ω and $\Delta w \in L^1_{loc}(\Omega)$. Then*

$$\int\limits_\Omega |\nabla\phi|^2\,dx + \frac{1}{2}\int\limits_\Omega \phi^2\,\frac{(\Delta w)_+}{w}\,dx \geq \frac{1}{4}\int\limits_\Omega \phi^2\,\frac{|\nabla w|^2}{w^2}\,dx + \frac{1}{2}\int\limits_\Omega \phi^2\,\frac{(\Delta w)_-}{w}\,dx, \quad \forall\,\phi \in \mathrm{Lip}_c(\Omega). \tag{5.5}$$

(2) *Let $u \in W^{1,2}_{loc}(\Omega)$ be such that $u > 0$ a.e. on Ω and there exists $\lambda > 0$ such that $-\Delta u = \lambda u$. Then*

$$\int\limits_\Omega |\nabla\phi|^2\,dx \geq \lambda \int\limits_\Omega \phi^2\,dx, \quad \forall\,\phi \in \mathrm{Lip}_c(\Omega).$$

Proof.
(1) For all $\epsilon > 0$, one has $\frac{1}{w+\epsilon} \in L^\infty(\Omega)$, since $w > 0$ a.e. Hence $V_\epsilon = \frac{\Delta w}{w+\epsilon} \in L^1_{loc}(\Omega)$ and

$$-\Delta(w+\epsilon) + V_\epsilon(w+\epsilon) = 0.$$

So, it follows from Proposition 5.1 that

$$\int\limits_\Omega |\nabla\phi|^2\,dx + \frac{1}{2}\int\limits_\Omega \phi^2\,\frac{\Delta w}{w+\epsilon}\,dx \geq \frac{1}{4}\int\limits_\Omega \phi^2\,\frac{|\nabla w|^2}{(w+\epsilon)^2}\,dx, \quad \forall\,\phi \in \mathrm{Lip}_c(\Omega).$$

Hence

$$\int\limits_\Omega |\nabla\phi|^2\,dx + \frac{1}{2}\int\limits_\Omega \phi^2\,\frac{(\Delta w)_+}{w+\epsilon}\,dx \geq \frac{1}{4}\int\limits_\Omega \phi^2\,\frac{|\nabla w|^2}{(w+\epsilon)^2}\,dx + \frac{1}{2}\int\limits_\Omega \phi^2\,\frac{(\Delta w)_-}{w+\epsilon}\,dx, \quad \forall\,\phi \in \mathrm{Lip}_c(\Omega).$$

Then (5.5) follows by the Beppo Levi monotone convergence theorem.

(2) Set first $w = u$. Then $\Delta w = -\lambda u \in L^1_{loc}(\Omega)$ and $(\Delta w)_+ = 0$, $\frac{(\Delta w)_-}{w} = \lambda$. So, by (5.5),

$$\int\limits_\Omega |\nabla\phi|^2\,dx \geq \frac{1}{4}\int\limits_\Omega \phi^2\,\frac{|\nabla u|^2}{u^2}\,dx + \frac{\lambda}{2}\int\limits_\Omega \phi^2\,dx, \quad \forall\,\phi \in \mathrm{Lip}_c(\Omega).$$

In particular,

$$\int_\Omega \phi^2 \, \frac{|\nabla u|^2}{u^2} \, dx < \infty, \ \ \forall \, \phi \in \mathrm{Lip}_c(\Omega). \tag{5.6}$$

Now, set $w = u^2$. Then

$$\Delta w = 2u\Delta u + 2|\nabla u|^2 = -2\lambda u^2 + 2|\nabla u|^2.$$

So $\Delta w \in L^1_{loc}(\Omega)$, since $u \in W^{1,2}_{loc}(\Omega)$. Hence (5.5) holds. Note that $\frac{(\Delta w)_+}{w} \le 2 \, \frac{|\nabla u|^2}{u^2}$. Hence, by (5.6),

$$\int_\Omega \phi^2 \, \frac{(\Delta w)_+}{w} \, dx < \infty, \ \ \forall \, \phi \in \mathrm{Lip}_c(\Omega).$$

It follows from (5.5) that $\phi^2 \, \frac{\Delta w}{w} \in L^1(\Omega)$ for all $\phi \in \mathrm{Lip}_c(\Omega)$ and

$$\int_\Omega |\nabla \phi|^2 \, dx \ge \int_\Omega \phi^2 \Big(\frac{1}{2} \, \frac{(-\Delta w)}{w} + \frac{1}{4} \, \frac{|\nabla w|^2}{w^2} \Big) \, dx, \ \ \forall \, \phi \in \mathrm{Lip}_c(\Omega).$$

Since

$$-\frac{1}{2} \, \frac{\Delta w}{w} + \frac{1}{4} \, \frac{|\nabla w|^2}{w^2} = \lambda - \frac{|\nabla u|^2}{u^2} + \frac{|\nabla u|^2}{u^2} = \lambda,$$

the assertion is follows. □

Remark 5.1. Inequality (5.5) holds true if $-\Delta w \ge 0$ is a positive Radon measure. The only change in the proof is a demonstration that $-\frac{\Delta w}{w+\epsilon}$ is a positive Radon measure for all $\epsilon > 0$. This follows from the following proposition and corollary with Ω and ρ as in Definition 2.1, $W_0^{1,2} = W_0^{1,2}(\Omega, \rho \, dx)$, $E \subset \Omega$ is a Borel set, defining $\mathrm{cap}_0(E, \Omega)$ in the same way as $\mathrm{cap}_1(E, \Omega)$ is defined in, e.g., [23, Section 2.2.1]) and, e.g., [20, Theorem 11.16]) with the only difference being $\|\nabla \theta\|^2_{L^2}$ instead of $\|\theta\|^2_{W^{1,2}}$.

Proposition 5.2 (Cf. [11, Lemma 2.2.3])**.** *Let $w \in W^{1,2}(\Omega)$, $-\Delta w = \mu \ge 0$. Then, for every compact $K \subset \Omega$*

$$\|\phi\|_{L^1(d\mu)} \le \|\nabla \phi\|_{L^2} \|\nabla w\|_{L^2}, \ \ \forall \, \phi \in \mathrm{Lip}_c(\Omega), \tag{5.7}$$

and

$$\mu(K) \le \sqrt{\mathrm{cap}_0(K, \Omega)} \, \|\nabla w\|_{L^2(\Omega)}.$$

Proof. Let $\phi \in \mathrm{Lip}_c(\Omega)$. Then

$$\|\phi\|_{L^1(d\mu)} = \int |\phi| \, d\mu = \int \nabla |\phi| \cdot \nabla w \, dx \le \|\nabla |\phi|\|_{L^2} \|\nabla w\|_{L^2} = \|\nabla \phi\|_{L^2} \|\nabla w\|_{L^2}.$$

If in addition $0 \le \phi \le 1$ and $\phi = 1$ on K, then

$$\mu(K) \le \|\phi\|_{L^1(d\mu)} \le \|\nabla \phi\|_{L^2} \|\nabla w\|_{L^2(\Omega)}.$$

The result follows by applying infimum over ϕ. □

Corollary 5.2. *Let $w \in W^{1,2}(\Omega)$, $-\Delta w = \mu \ge 0$. Then $W^{1,2}_0(\Omega)$ continuously embedded in $L^1(\Omega, d\mu)$.*

Proof. In addition to (5.7), it suffices to observe that $\mathrm{Lip}_c(\Omega)$ is dense in $W^{1,2}_0(\Omega)$. □

To obtain Poincaré inequality, we need the following proposition.

Proposition 5.3. *Let $N_1, N_2, \ldots, N_d \in \mathbb{N}$ and $\Omega_k \subset \mathbb{R}^{N_k}$ be a bounded domain. Let $\lambda_k \ge 0$, $\phi_k \in W^{1,2}_0(\Omega_k)$ be the first eigenvalue and eigenfunction for the Dirichlet Laplacian on Ω_k. Let $N = \sum\limits_{k=1}^d N_k$, $\Omega = \prod\limits_{k=1}^d \Omega_k \subset \mathbb{R}^N$. Then $\lambda = \sum\limits_{k=1}^d \lambda_k$ and $\phi(x) = \prod\limits_{k=1}^d \phi_k(x_k)$ are the first eigenvalue and the eigenfunction for the Dirichlet Laplacian on Ω.*

Proof. Let $u \in W_0^{1,2}(\Omega)$. Then

$$\|\nabla u\|_{L^2}^2 = \sum_{k=1}^{d} \|\nabla_k u\|_{L^2}^2. \tag{5.8}$$

For $k = 1, 2, \ldots, d$, let $S_k = \prod_{m \neq k} \Omega_m$. For $x \in \Omega$, we write $x = (x_k, x')$ with $x' \in S_k$, $x_k \in \Omega_k$,

$$\|\nabla_k u\|_{L^2}^2 = \int_{S_k} \int_{\Omega_k} |\nabla_k u|^2(x_k, x')\, dx_k\, dx' \geq \int_{S_k} \lambda_k \int_{\Omega_k} |u|^2(x_k, x')\, dx_k\, dx'.$$

So,

$$\lambda_k \|u\|_{L^2}^2 \leq \|\nabla_k u\|_{L^2}^2, \quad k = 1, 2, \ldots, d.$$

Summing up, by (5.8), we get

$$\Big(\sum_{k=1}^{d} \lambda_k\Big) \|u\|_{L^2}^2 \leq \sum_{k=1}^{d} \|\nabla_k u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2.$$

Let $\phi = \prod_{k=1}^{d} \phi_k(x_k)$. Then $\phi \in W_0^{1,2}(\prod_{k=1}^{d} \Omega_k) \equiv W_0^{1,2}(\Omega)$,

$$\begin{aligned}\|\nabla_k \phi\|_{L^2}^2 &= \int_{S_k} \Big| \prod_{m \neq k} \phi_m(x_m) \Big|^2 dx' \int_{\Omega_k} \big|\nabla_k \phi_k(x_k)\big|^2 dx_k \\ &= \int_{S_k} \Big| \prod_{m \neq k} \phi_m(x_m) \Big|^2 dx' \int_{\Omega_k} \lambda_k \phi_k^2(x_k)\, dx_k = \lambda_k \|\phi\|_{L^2}^2.\end{aligned}$$

Thus

$$\|\nabla \phi\|_{L^2}^2 = \sum_{k=1}^{d} \|\nabla_k \phi\|_{L^2}^2 = \Big(\sum_{k=1}^{d} \lambda_k\Big) \|\phi\|_{L^2}^2 \equiv \lambda \|\phi\|_{L^2}^2.$$

Finally,

$$\min \left\{ \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} \;\Big|\; u \in W_0^{1,2}(\Omega),\ u \neq 0 \right\} = \lambda$$

and $u = \phi$ is a minimiser. Hence the assertion follows. □

Corollary 5.3.

(1) *Let $I \subset \mathbb{R}$ be an interval. Then the Poincaré inequality*

$$\int_I |\phi'|^2\, ds \geq \frac{\pi^2}{|I|^2} \int_I \phi^2\, ds, \quad \forall\, \phi \in W_0^{1,2}(I)$$

holds. Moreover, $\lambda_I = \frac{\pi^2}{|I|^2}$ is the first eigenvalue for the Dirichlet Laplacian on I with the eigenfunction $u(s) = \sin(\pi\, \frac{s - \inf I}{|I|})$ (Cf. [5, Section 8.1]).

(2) *Let $Q = I_1 \times I_2 \times \cdots \times I_N$. Then*

$$\int_Q |\nabla \phi|^2\, dx \geq \pi^2 \Big(\sum_{k=1}^{N} \frac{1}{|I_k|^2}\Big) \int_Q \phi^2\, dx, \quad \forall\, \phi \in W_0^{1,2}(Q).$$

Moreover, $\lambda_Q = \pi^2 \big(\sum_{k=1}^{N} \frac{1}{|I_k|^2}\big)$ is the first eigenvalue for the Dirichlet Laplacian on Q with the eigenfunction $u(s) = \prod_{k=1}^{N} \sin(\pi\, \frac{x_k - a_k}{|I_k|})$ where $a_k = \inf I_k$.

Proof. The first assertion follows from Corollary 5.1 (2). Proposition 5.3 implies the second assertion. □

Corollary 5.4. *For $I \subset \mathbb{R}$, the Sobolevskii inequality [22]*

$$\int_I |\phi'|^2(x)\,dx \geq \frac{1}{4}\,|I|^2 \int_I \frac{\phi^2\,dx}{(x-\inf I)^2(\sup I - x)^2}\,, \quad \forall\,\phi \in W_0^{1,2}(I) \tag{5.9}$$

holds. For $I_k \subset \mathbb{R}$, $k = 1, 2, \ldots, N$, $Q = \prod_{k=1}^{N} I_k$,

$$\int_Q |\nabla\phi|^2(x)\,dx \geq \frac{1}{4}\int_Q \sum_{k=1}^{N} \frac{|I_k|^2\phi^2}{(x_k - \inf I_k)^2(\sup I_k - x_k)^2}\,dx, \quad \forall\,\phi \in W_0^{1,2}(Q).$$

Proof. First, let $I = (0,1)$. Let $w(x) = x(1-x)$. Then

$$\frac{-w''}{2w} + \frac{1}{4}\Big(\frac{w'}{w}\Big)^2 = \frac{1}{x(1-x)} + \frac{(1-2x)^2}{4x^2(1-x)^2} = \frac{1}{4x^2(1-x)^2}\,.$$

So, by Corollary 5.1,

$$\int_0^1 |\phi'|^2\,ds \geq \frac{1}{4}\int_0^1 \frac{\phi^2}{s^2(1-s)^2}\,ds.$$

Now, let $I = (a,b)$, $a < b$,

$$\begin{aligned}\int_a^b |\phi'|^2\,dx &= \frac{1}{b-a}\int_0^1 |\phi'|^2\,ds \geq \frac{1}{4}\,\frac{1}{b-a}\int_0^1 \frac{\phi^2}{s^2(1-s)^2}\,ds\\ &= \frac{1}{4}\,\frac{1}{(b-a)^2}\int_a^b \frac{(b-a)^4\phi^2}{(x-a)^2(b-x)^2}\,dx = \frac{1}{4}\,(b-a)^2\int_a^b \frac{\phi^2}{(x-a)^2(b-x)^2}\,dx.\end{aligned}$$

Next, for the cuboid Q, by (5.9), we have

$$\begin{aligned}\int_Q |\nabla\phi|^2\,dx &= \sum_{k=1}^{N} \int\cdots\int_{\prod_{m\neq k} I_m} \int_{I_k} \Big|\frac{\partial\phi}{\partial x_k}\Big|^2\,dx_k\,dx^k\\ &\geq \sum_{k=1}^{N} \int\cdots\int_{\prod_{m\neq k} I_m} \frac{1}{4}\,|I_k|^2 \int_{I_k} \frac{\phi^2}{(x_k - \inf I_k)^2 - (\sup I_k - x_k)^2}\,dx_k\,dx^k\\ &= \frac{1}{4}\int_Q \Big(\sum_{k=1}^{N} \frac{|I_k|^2\phi^2}{(x_k - \inf I_k)^2 - (\sup I_k - x_k)^2}\Big)\,dx.\end{aligned}$$

□

The following lemma is a particular case of the inequality proved in [21, Lemma A.1].

Lemma 5.1.

$$\int |\nabla\theta|^2 r^{2\alpha}\,dx \geq \frac{(N+2\alpha-2)^2}{4}\int \frac{\theta^2}{r^2}\,r^{2\alpha}\,dx, \quad \forall\,\theta \in W^{1,2}(r^{2\alpha}\,dx), \quad \alpha > -\frac{N-2}{2}\,.$$

Proof. First, let $\theta \in \mathrm{Lip}_c$, $\phi = \theta r^\alpha$, $\alpha > -\frac{N-2}{2}$, $r = |x|$. Then

$$|\nabla\phi|^2 = r^{2\alpha}|\nabla\theta|^2 + 2\alpha r^{2\alpha-2}\theta\nabla\theta\cdot x + \alpha^2\theta^2 r^{2\alpha-2},$$

$$\int r^{2\alpha}|\nabla\theta|^2\,dx = \int|\nabla\phi|^2\,dx - 2\alpha\int\theta\nabla\theta\cdot(xr^{2\alpha-2})\,dx - \alpha^2\int\frac{\theta^2}{r^2}r^{2\alpha}\,dx.$$

Note that, by the Hardy inequality,

$$\int|\nabla\phi|^2\,dx \ge \frac{(N-2)^2}{4}\int\frac{\phi^2}{r^2}\,dx = \frac{(N-2)^2}{4}\int\frac{\theta^2}{r^2}r^{2\alpha}\,dx.$$

Now, since

$$\begin{aligned}-2\alpha\int\theta\nabla\theta\cdot(xr^{2\alpha-2})\,dx &= -\alpha\int\nabla\theta^2\cdot(xr^{2\alpha-2})\,dx\\ &= \alpha\int\theta^2\operatorname{div}(xr^{2\alpha-2})\,dx = \alpha(N+2\alpha-2)\int\theta^2r^{2\alpha-2}\,dx = \alpha(N+2\alpha-2)\int\frac{\theta^2}{r^2}r^{2\alpha}\,dx\end{aligned}$$

we have

$$\int r^{2\alpha}|\nabla\theta|^2\,dx \ge \Big(\frac{(N-2)^2}{4} - \alpha^2 + \alpha(N+2\alpha-2)\Big)\int\frac{\theta^2}{r^2}r^{2\alpha}\,dx.$$

So, we have demonstrated the assertion for $\theta\in\mathrm{Lip}_c$. The general case follows by approximation. $\square$

6 Energy space isometry

Given an open set $\Omega\subset\mathbb{R}^N$ and a Borel regular measure V on Ω, in this section we construct an isometrical isomorphism of the energy space $D(\Omega)$ (constructed in Section 4) with a weighted Sobolev space on $\mathrm{Reg}(V)$. So, without loss of generality, we assume that V is a smooth Radon measure on Ω, so that $D(\Omega) = W_0^{1,2}(\Omega)\cap L^2(\Omega,dV)$ and $\widetilde{D}(\Omega) = \mathrm{Lip}_c(\Omega)$ (see Corollary 4.1 and Proposition 4.1).

Theorem 6.1. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set, V be a smooth Radon measure on Ω and let w be a positive local solution to equation $-\Delta w + Vw = 0$ in sense of Definition 5.1. Consider a multiplication operator $\mathcal{S}\theta = w\theta$. Then*

$$\mathcal{S}: W_0^{1,2}(\Omega, w^2\,dx)\to W_0^{1,2}(\Omega)\cap L^2(\Omega,dV) = D(\Omega)$$

is an isometric bijection. In particular,

$$\|w\theta\|_D^2 \equiv \int|\nabla(\theta w)|^2\,dx + \int\theta^2w^2\,dV = \int|\nabla\theta|^2w^2\,dx \equiv \|\nabla\theta\|_{L^2(w^2\,dx)} \tag{6.1}$$

for all $\theta\in W_0^{1,2}(\Omega,w^2\,dx)$. We delegate the proof to several propositions.

Lemma 6.1. *With $w\in D_{loc}(\Omega)$, equation (6.1) holds for $\theta\in\mathrm{Lip}_c(\Omega)$.*

Proof.

$$\begin{aligned}\int|\nabla(\theta w)|^2\,dx + \int\theta^2w^2\,dV &= \int\big|\theta\nabla w + w\nabla\theta\big|^2\,dx + \int(\theta^2w)w\,dV\\ &= \int\big(\theta^2|\nabla w|^2 + 2\theta w\nabla\theta\nabla w + w^2|\nabla\theta|^2\big)\,dx + \int(\theta^2w)w\,dV.\end{aligned}$$

Note that

$$\theta^2|\nabla w|^2 + 2\theta w\nabla\theta\nabla w = (\theta^2\nabla w + 2\theta w\nabla\theta)\cdot\nabla w = (\theta^2\nabla w + w\nabla\theta^2)\cdot\nabla w = \nabla(\theta^2w)\cdot\nabla w.$$

So,

$$\begin{aligned}&\int|\nabla(\theta w)|^2\,dx + \int\theta^2w^2\,dV\\ &\qquad= \int w^2|\nabla\theta|^2\,dx + \int\nabla(\theta^2w)\cdot\nabla w\,dx + \int(\theta^2w)w\,dV = \int|\nabla\theta|^2w^2\,dx,\end{aligned}$$

since $\theta^2w\in D_c(\Omega)$ is a test function for (5.1). $\square$

Corollary 6.1. *The map $\mathcal{S}$ as in Theorem 6.1 is an isometry.*

Proof. By Lemma 6.1, we have

$$\|S\theta\|_D^2 = \|\theta\|^2_{W^{1,2}(w^2\,dx)}, \quad \forall\, \theta \in \mathrm{Lip}_c(\Omega).$$

Since $\mathrm{Lip}_c(\Omega)$ is dense in $W_0^{1,2}(\Omega, w^2\,dx)$, it follow that the same identity holds for all $\theta \in W_0^{1,2}(\Omega, w^2\,dx)$. □

Proposition 6.1. *The range of the map $\mathcal{S}$ as in Theorem 6.1 is dense in $D(\Omega)$.*

Proof. It suffices to show that $\mathrm{Lip}_c(\Omega) \subset \overline{\mathrm{Range}\, S}$, since $\mathrm{Lip}_c(\Omega)$ is dense in $D(\Omega)$. That is, a function $\phi \in \mathrm{Lip}_c(\Omega)$ can be approximated by $\phi_\epsilon = \theta_\epsilon w$ for some $\theta_\epsilon \in W_0^{1,2}(\Omega, w^2\,dx)$. Let $\phi \in \mathrm{Lip}_c(\Omega)$. For $\epsilon > 0$, let

$$\phi_1 = \begin{cases} w, & w \geq \epsilon, \\ \epsilon, & w \leq \epsilon. \end{cases}$$

Let $\theta_\epsilon = \frac{\phi}{\phi_1}$, $\theta_\epsilon \in L_c^\infty \subset L_c^2(\Omega, w^2\,dx)$. Then

$$\nabla\theta_\epsilon = \frac{\nabla\phi}{\phi_1} - \phi\,\frac{\nabla w}{w^2}\,\mathbf{1}_{\{w>\epsilon\}} \in L_c^2(\Omega, w^2\,dx).$$

Indeed,

$$\Big\|\frac{\nabla\phi}{\phi_1}\Big\|^2_{L^2(w^2\,dx)} = \int |\nabla\phi|^2\,\frac{w^2}{(\phi_1)^2}\,dx \leq \int |\nabla\phi|^2\,dx < \infty,$$

since $\phi \in \mathrm{Lip}_c(\Omega)$, and

$$\Big\|\phi\,\frac{\nabla w}{w^2}\,\mathbf{1}_{\{w>\epsilon\}}\Big\|^2_{L^2(w^2\,dx)} = \int \phi^2\,\frac{|\nabla w|^2}{w^2}\,\mathbf{1}_{\{w>\epsilon\}}\,dx < \infty,$$

since $w \in W^{1,2}_{loc}(\Omega)$, $\phi \in \mathrm{Lip}_c(\Omega)$ and $\mathbf{1}_{\{w>\epsilon\}}\,\frac{1}{w^2} < \frac{1}{\epsilon^2}$. So, $\theta_\epsilon \in W_0^{1,2}(\Omega, w^2\,dx)$. Hence $\phi_\epsilon = \theta_\epsilon w \in \mathrm{Range}\, S$. We show that $\phi_\epsilon = \phi\,\frac{w}{\phi_1} \to \phi$ as $\epsilon \to 0$ in $D(\Omega)$.

Note that

$$\phi(x) - \phi(x)\,\frac{w}{\phi_1}(x) = \phi\,\frac{\phi_1 - w}{\phi_1}(x) \leq \phi\,\mathbf{1}_{w<\epsilon} \longrightarrow 0 \ \text{ a } \quad \epsilon \to 0$$

and

$$0 \leq \frac{\phi_1 - w}{\phi_1} = \begin{cases} 0, & w \geq \epsilon, \\ \dfrac{\epsilon - w}{\epsilon}, & w < \epsilon \end{cases} = \frac{\epsilon - w}{\epsilon}\,\mathbf{1}_{\{w<\epsilon\}} \leq \mathbf{1}_{\{w<\epsilon\}}.$$

for a.a. $x \in \Omega$. So, $\phi_\epsilon \to \phi$ a.e., since $w(x) > 0$ a.e. Hence $\phi_\epsilon \to \phi$ in $L^2(\Omega) \cap L^2(\Omega, dV)$ by the LDCT. Consider now the gradient

$$\begin{aligned}
\nabla\Big(\phi - \phi\,\frac{w}{\phi_1}\Big) &= \nabla\phi - \frac{w}{\phi_1}\,\nabla\phi - \phi\,\frac{\phi_1\nabla w - w\,\mathbf{1}_{\{w>\epsilon\}}\nabla w}{(\phi_1)^2} \\
&= \nabla\phi\Big(1 - \frac{w}{\phi_1}\Big) - \phi\,\frac{w\,\mathbf{1}_{\{w>\epsilon\}}\nabla w + \epsilon\,\mathbf{1}_{\{w<\epsilon\}}\nabla w - w\,\mathbf{1}_{\{w>\epsilon\}}\nabla w}{(\phi_1)^2} \\
&= \nabla\phi\Big(1 - \frac{w}{\phi_1}\Big) - \phi\,\frac{1}{\epsilon}\,\mathbf{1}_{\{w<\epsilon\}}\nabla w \in L^2(\Omega).
\end{aligned}$$

Indeed, $(1 - \frac{w}{\phi_1}) \to 0$ as $\epsilon \to 0$ a.e. So, $\nabla\phi(1 - \frac{w}{\phi_1}) \to 0$ in $L^2(\Omega, dx)$ as $\epsilon \to 0$ by the LDCT. We have

$$\Big\|\phi\,\frac{1}{\epsilon}\,\mathbf{1}_{\{w<\epsilon\}}\nabla w\Big\|^2_{L^2(\Omega)} = \int \phi^2\,\frac{|\nabla w|^2}{\epsilon^2}\,\mathbf{1}_{\{w<\epsilon\}}\,dx \leq \int \phi^2\,\frac{|\nabla w|^2}{w^2}\,\mathbf{1}_{\{w<\epsilon\}}\,dx \longrightarrow 0 \ \text{ as } \ \epsilon \to 0.$$

Here, we use the fact that $\phi^2\,\frac{|\nabla w|^2}{w^2} \in L^1(\Omega)$ by Proposition 5.1. Since $\mathbf{1}_{\{w<\epsilon\}} \to 0$ a.e. as $\epsilon \to 0$ (since $w > 0$), we conclude that the proposition is valid by the LDCT. □

The following folklore result completes preparations for the proof of Theorem 6.1.

Theorem 6.2. *Let X, Y be Banach spaces and $\mathcal{S} : X \to Y$ be linear isometric. Then Range $\mathcal{S}$ is closed in Y.*

Proof. Let $u_n \in \mathrm{Range}(\mathcal{S})$, $u_n \to u$. Then there exists a sequence $v_n \in X$ such that $u_n = \mathcal{S}v_n$, $n = 1, 2, 3, \dots$. For $m, n = 1, 2, 3, \dots$,

$$\|v_n - v_m\|_X = \|\mathcal{S}(v_n - v_m)\|_Y = \|\mathcal{S}v_n - \mathcal{S}v_m\|_Y = \|u_n - u_m\|_Y,$$

since $\mathcal{S}$ is an isometry. Hence $\{v_n\}$ is a Cauchy sequence, since $\{u_n\}$ is a Cauchy sequence. So, $v_n \to v \in X$. Since $\mathcal{S}$ is continuous, $\mathcal{S}v_n \to \mathcal{S}v \in Y$. However, $\mathcal{S}v_n = u_n \to u \in Y$. So, $\mathcal{S}v = u$ and $u \in \mathrm{Range}(\mathcal{S})$. □

Proof Theorem 6.1. By Corollary 6.1, $\mathcal{S}$ is an isometry. Hence $\mathcal{S}$ is an injection and Range $\mathcal{S}$ is closed in $D(\Omega)$ by Theorem 6.2. By Proposition 6.1, Range $\mathcal{S}$ is also dense in $D(\Omega)$, so $\mathrm{Range}(\mathcal{S}) = D(\Omega)$. Thus, $\mathcal{S}$ is a bijection. □

7 NLS equation eigenvalue problem. PDE approach

In this section, we consider non-existence of a global weak solution for the eigenvalue problem to the nonlinear Schrödinger (NLS) equation with singular potential V of the form

$$-\Delta u + Vu + u|u|^{p-1} = \lambda u, \quad u \in \mathbb{D}(\mathbb{R}^N), \tag{7.1}$$

where u is the solution of the eigenvalue problem. If $u > 0$, then u presents the amplitude of the wave and λ presents the frequency of the wave. This λ (the eigenvalue of problem (7.1)) retains the name of chemical potential for the BEC (see, e.g., [25, Section 6.1]). $\mathbb{D}(\mathbb{R}^N) = D(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ with $p > 1$ and some $\lambda \in \mathbb{R}$,

$$D(\mathbb{R}^N) = W_0^{1,2}(\mathrm{Reg}(V)) \cap L^2(\mathrm{Rev}(V), dV_s),$$

where $\mathrm{Reg}(V)$ is the maximal open set, where V is a Radon measure (see Theorem 3.2), V_s is the smooth component of $V\,\mathbf{1}_{\mathrm{Reg}(V)}$ (see Corollary 4.1). We prove that u is a global weak solution (see Definition 7.1) of (7.1) if it is a local weak solution and belongs to $L^2(\mathbb{R}^N)$. The a priori boundedness of global weak solution u is also shown in Section 7.1. The problem with equation (7.1) is that V is a non-smooth function and may not be a function (measure). In this section, we demonstrate the killing potential technique to transfer equation (7.1) into an equivalent equation without potential V:

$$-\operatorname{div}(w^2 \nabla v) + w^{p+1} v|v|^{p-1} = \lambda w^2 v, \tag{7.2}$$

where $v = \frac{u}{w}$ is a global (local) weak solution of (7.2) if and only if u is a global (local) weak solution of (7.1) and w is as in Definition 5.1. We have to mention that the killing potential technique changes the nature of the unknown function (solution) from a standard Sobolev space to a weighted Sobolev space, all this is discussed in Section 7.2. In Section 7.3, we frequently use the Pokhozhaev type test function technique. For detailed information on this technique, see, e.g., [24] and [13].

7.1 Global, local solution and a priori boundedness

Definition 7.1. A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is called a global (local) solution of equation (7.1) in $\mathbb{R}^N$ if $u \in \mathbb{D} = D \cap L^{p+1}$ ($u \in \mathbb{D}_{loc} = D_{loc} \cap L^{p+1}_{loc}$), where D is as in Definition 4.1 and

$$\int \nabla u \cdot \nabla \eta \, dx + \int u\eta \, dV + \int u\,|u|^{p-1}\,\eta \, dx = \int \lambda u\eta \, dx, \quad \forall\, \eta \in \mathbb{D} \ \ (\eta \in \mathbb{D}_c). \tag{7.3}$$

Remark 7.1. Following the arguments of Propositions 2.2–2.4, one can show that Lip_c is dense in $\mathbb{D}$.

Theorem 7.1. *Let u be a local solution of equation (7.1) in $\mathbb{R}^N$. Assume in addition that $u \in L^2$. Then $u \in \mathbb{D}$ and u is a global solution of equation (7.1).*

Proof. For $\xi \in C_c^1$, one has $u\xi^2 \in \mathbb{D}_c$ and by Definition 7.1,

$$\int \nabla u \nabla(u\xi^2)\,dx + \int u^2\xi^2\,dV + \int \xi^2|u|^{p+1}\,dx = \lambda \int \xi^2 u^2\,dx.$$

So, by Lemma 2.2,

$$\int |\nabla(u\xi)|^2\,dx + \int u^2\xi^2\,dV + \int \xi^2|u|^{p+1}\,dx = \int u^2|\nabla\xi|^2\,dx + \lambda \int \xi^2 u^2\,dx. \tag{7.4}$$

Let $\xi_1 \in C_c^1$, $\mathbf{1}_{B_1} \geq \xi_1 \geq \mathbf{1}_{B_{\frac{1}{2}}}$ and we construct a sequence $\xi_n(x) \in C_c^1$ such that

$$\xi_n(x) = \xi_1\Big(\frac{x}{n}\Big).$$

Thus $\mathbf{1}_{B_n} \geq \xi_n \geq \mathbf{1}_{B_{\frac{n}{2}}}$, also $\xi_n(x) \uparrow 1$ as $n \uparrow \infty$, $\|\nabla\xi_n\|_\infty \downarrow 0$ as $n \uparrow \infty$. That is,

$$|\nabla\xi_n(x)| = \frac{1}{n}\Big|\nabla\xi_1\Big(\frac{x}{n}\Big)\Big|.$$

By the Beppo Levi Lemma and the LDCT, passing to the limit in (7.4), we get

$$\begin{aligned}\lim_{n\to\infty}\Big[\int |\nabla(u\xi_n)|^2\,dx + \int u^2\xi_n^2\,dV + \int \xi_n^2|u|^{p+1}\,dx\Big] &= \lim_{n\to\infty}\Big[\int u^2|\nabla\xi_n|^2\,dx + \lambda\int \xi_n^2u^2\,dx\Big]\\ \Longrightarrow \lim_{n\to\infty}\int |\nabla(u\xi_n)|^2\,dx + \int u^2\,dV &+ \int |u|^{p+1}\,dx = \lambda\int u^2\,dx.\end{aligned}$$

So, $u \in L^2(dV)$, $u \in L^{p+1}$. By Proposition 2.5, we conclude that $u\xi_n \xrightarrow{w} u$ in $W^{1,2}$. Hence $u \in \mathbb{D}$.

The second assertion follows trivially as $-\Delta u + u\,dV + u|u|^{p-1} + \lambda u$ is a bounded linear functional on $\mathbb{D}$. Indeed, let

$$l(\omega) = \int \nabla u \cdot \nabla\omega\,dx + \int u\omega\,dV + \int \omega u|u|^{p-1}\,dx - \lambda\int \omega u\,dx \tag{7.5}$$

be a bounded linear functional on $\mathbb{D}$, we have $l(\omega) = 0$ for all $\omega \in \mathbb{D}_c$ (local solution). Since $\mathbb{D}_c$ is a dense set in $\mathbb{D}$, it follows that $l(\omega) = 0$ for all $\omega \in \mathbb{D}$ (u is a global solution). ☐

In the rest of this section, we show the a priori boundedness of a global solution u for (7.1). First, we need the following lemma.

Lemma 7.1. *Let u be a global solution of equation (7.1). For $s > 0$, let $u_s = |u| \wedge s$. Then, for $q > 2$,*

$$\int |u|^{p+1}u_s^{q-2}\,dx \leq \lambda\int u^2u_s^{q-2}\,dx.$$

Proof. Let $q > 2$, $\eta = u_s^{q-2}u$. We are going to use η as a test function in (7.3). Hence, we show that $\eta \in \mathbb{D}$. Since $u \in \mathbb{D}$ and $u_s \in L^\infty$, it suffices to show that $\nabla\eta \in L^2$. However,

$$\begin{aligned}\nabla\eta = \nabla(u_s^{q-2}u) &= u(q-2)u_s^{q-3}\,\mathbf{1}_{\{|u|<s\}}\,\frac{u}{|u|}\,\nabla u + u_s^{q-2}\nabla u\\ &= (q-2)u_s^{q-2}\,\mathbf{1}_{\{|u|<s\}}\nabla u + u_s^{q-2}\nabla u = \big((q-2)\,\mathbf{1}_{\{|u|<s\}} + 1\big)u_s^{q-2}\nabla u \in L^2\end{aligned}$$

noting that $u_s = |u|\,\mathbf{1}_{\{|u|<s\}} + s\,\mathbf{1}_{\{u\geq s\}}$. Now, we will apply η to (7.3) as follows:

$$\int \nabla u \cdot \nabla(u_s^{q-2}u)\,dx + \int uu_s^{q-2}u\,dV + \int |u|^{p-1}uu_s^{q-2}u\,dx = \lambda\int uu_s^{q-2}u\,dx. \tag{7.6}$$

Note that

$$\int \nabla u \cdot \nabla(u_s^{q-2}u)\,dx = \int \big((q-2)\,\mathbf{1}_{\{|u|<s\}} + 1\big)u_s^{q-2}|\nabla u|^2 \geq 0, \quad q \geq 2.$$

Then (7.6) becomes

$$\int \big((q-2)\,\mathbf{1}_{\{|u|<s\}}+1\big)u_s^{q-2}|\nabla u|^2+\int u^2u_s^{q-2}\,dV+\int |u|^{p+1}u_s^{q-2}\,dx=\lambda\int u^2u_s^{q-2}\,dx.$$

Note that the first and the second parts are positive, so we can drop them and get the required result. □

Theorem 7.2. *Let u be a global solution of equation (7.1). Then $u\in L^\infty$ and $\|u\|_{L^\infty}\le\lambda^{\frac{1}{p-1}}$.*

Proof. Let $q_0=2,\quad q_n=q_{n-1}+(p-1)$, so that $q_n=2+n(p-1)$. By Lemma 7.1, we have (since $|u|\ge u_s$)

$$\int u_s^{q_{n+1}}\,dx\le\lambda\int u^2u_s^{q_n-2}\,dx. \tag{7.7}$$

By induction, we prove that $u\in L^{q_n}$ for $n=0,1,2,3,\dots$ and

$$\int |u|^{q_{n+1}}\,dx\le\lambda\int |u|^{q_n}\,dx,\quad n=0,1,2,3,\dots.$$

Indeed, for $n=0$, we have $q_n=2$, so $u\in L^{q_0}$, since u is a global solution to (7.1). Now, given $n=0,1,2,3,\dots$, assume that $u\in L^{q_n}$. Then it follows from (7.7) that

$$\lim_{s\to\infty}\int u_s^{q_{n+1}}\,dx\le\lambda\lim_{s\to\infty}\int u^2u_s^{q_n-2}\,dx.$$

Thus, by the Beppo Levi Lemma,

$$\int |u|^{q_{n+1}}\,dx\le\lambda\int |u|^{q_n}\,dx<\infty.$$

So, $u\in L^{q_{n+1}}$, and therefore $u\in L^\infty$, since $q_n=2+n(p-1)\to\infty$ as $n\to\infty$. By induction, it also follows that

$$\int |u|^{q_n}\,dx\le\lambda^n\int u^2\,dx,$$

$$\|u\|_{L^\infty}=\lim_{n\to\infty}\|u\|_{L^{q_n}}\le\lim_{n\to\infty}\lambda^{\frac{n}{q_n}}\|u\|_{L^2}^{\frac{2}{q_n}}=\lim_{n\to\infty}\lambda^{\frac{n}{2+n(p-1)}}\|u\|_{L^2}^{\frac{2}{2+n(p-1)}}=\lambda^{\frac{1}{p-1}}.$$ □

Corollary 7.1. *Let u be a global solution to equation (7.1). Then equation (7.3) holds for any $\eta\in D$.*

Proof. Since u is a global solution, we have $u\in L^2$, $|u|^{p-1}\in L^\infty$ by Theorem 7.2, and $|u|^{p-1}u\in L^2$. Let l be as in (7.5),

$$l_u=-\Delta u+uV+|u|^{p-1}u-\lambda u\in W^{-1,2}+L^2(dV)+L^2.$$

Hence l_u is a bounded linear functional on D, using the same arguments as in Theorem 7.1. □

7.2 Killing Potential technique

In this section, we transfer (7.1) to (7.2) by the following theorem.

Theorem 7.3. *Let V be a Borel regular measure and let w be a positive solution to $-\Delta w+Vw=0$ in the sense of Definition 5.1.*

(1) *A function $u\in\mathbb{D}(\mathrm{Reg}(V))$ is a global solution to equation (7.1) if and only if the function*

$$v=\frac{u}{w}\in W_0^{1,2}(\mathrm{Reg}(V),w^2\,dx)\cap L^{p+1}(\mathrm{Reg}(V),w^{p+1}\,dx)$$

is a global solution to equation (7.2).

(2) *A function $u \in \mathbb{D}_{loc}(\mathrm{Reg}(V))$ is a local solution to equation (7.1) if and only if the function*

$$v = \frac{u}{w} \in W^{1,2}_{loc}(\mathrm{Reg}(V), w^2\,dx) \cap L^{p+1}_{loc}(\mathrm{Reg}(V), w^{p+1}\,dx)$$

is a local solution to equation (7.2).

The following lemma is more general than Lemma 3.4 in [17] because we remove all the restrictions on the solution w besides the positivity and consider more general potential as a measure.

Lemma 7.2 (See [17, Lemma 3.4.])**.** *Suppose that there exists a solution $w \in D_{loc}$ to the equation $-\Delta w + Vw = 0$ in the sense of Definition 5.1. Then $u \in \mathbb{D}_{loc}(\mathrm{Reg}(V))$ satisfies*

$$-\Delta u + Vu + |u|^{p-1}u - \lambda u = 0 \ \ \textit{in } \mathbb{D}'(\mathrm{Reg}(V))$$

if and only if $v = \frac{u}{w} \in W^{1,2}_{loc}(\mathrm{Reg}(V), w^2\,dx) \cap L^{p+1}_{loc}(\mathrm{Reg}(V), w^{p+1}\,dx)$ satisfies

$$-\operatorname{div}(w^2\nabla v) + w^{p+1}|v|^{p-1}v - \lambda w^2 v = 0 \ \ \textit{in } \mathbb{D}'(\mathrm{Reg}(V)).$$

Proof. It follows from Theorem 6.1 that $u \in \mathbb{D}_{loc}(\mathrm{Reg}(V))$ if and only if $v = \frac{u}{w} \in W^{1,2}_{loc}(\mathrm{Reg}(V), w^2\,dx) \cap L^{p+1}_{loc}(\mathrm{Reg}(V), w^{p+1}\,dx)$. Let $\eta \in \mathrm{Lip}_c(\mathrm{Reg}(V))$. If we multiply (7.1) by ηw and (5.1) by ηu, then integrating and subtracting, them we obtain

$$\int \big(\nabla u \cdot \nabla(\eta w) - \nabla w \cdot \nabla(u\eta)\big)\,dx + \int (Vu\eta w - Vwu\eta)\,dx + \int \eta w u |u|^{p-1}\,dx = \lambda \int u\eta w\,dx. \tag{7.8}$$

Now, for the first part in the left-hand side of (7.8), we have

$$\begin{aligned}
&\int \big(\nabla u \cdot \nabla(\eta w) - \nabla w \cdot \nabla(u\eta)\big)\,dx \\
&\qquad = \int \big(\eta(\nabla u \cdot \nabla w) + w\nabla u \cdot \nabla\eta - \eta(\nabla w \cdot \nabla u) - u\nabla w \cdot \nabla \eta\big)\,dx \\
&\qquad\quad = \int \nabla\eta \cdot (w\nabla u - u\nabla w)\,dx = \int \nabla\eta \cdot \Big(w^2 \nabla \frac{u}{w}\Big)\,dx = \int \nabla\eta \cdot \nabla v\, w^2\,dx.
\end{aligned}$$

Thus (7.8) becomes

$$\int \nabla v \nabla \eta w^2\,dx + \int \eta v|v|^{p-1}w^{p+1}\,dx = \lambda \int v\eta w^2\,dx, \ \ \forall\, \eta \in \mathrm{Lip}_c(\mathrm{Reg}(V)). \tag{7.9}$$

Hence, (7.9) implies that v is a solution of (7.2) (in the distribution sense). □

Proof of Theorem 7.3. Assume that u is a global solution to (7.1). Then

$$m(\eta) = \int \nabla\eta\nabla u\,dx + \int \eta u\,dV + \int \eta|u|^{p-1}u\,dx - \lambda \int \eta u\,dx = 0, \ \ \forall\, \eta \in \mathbb{D}(\mathrm{Reg}(V)).$$

In particular, $m(\eta) = 0$ for all $\eta \in \mathrm{Lip}_c(\mathrm{Reg}(V)) \subset \mathbb{D}(\mathrm{Reg}(V))$.

Hence

$$l(\eta) = \int \nabla\eta\nabla v w^2\,dx + \int \eta|v|^{p-1}v w^{p+1}\,dx - \lambda \int \eta v w^2\,dx = 0, \ \ \eta \in \mathrm{Lip}_c(\mathrm{Reg}(V))$$

by Lemma 7.2. Note that $v \in L^{p+1}(\mathrm{Reg}(V), w^{p+1}\,dx)$ by the definition and $v \in W^{1,2}_0(\mathrm{Reg}(V), w^2\,dx)$ by Theorem 6.1. Hence l is bounded linear functional on $W^{1,2}_0(\mathrm{Reg}(V), w^2\,dx) \cap L^{p+1}(\mathrm{Reg}(V), w^{p+1}\,dx)$. Since $\mathrm{Lip}_c(\mathrm{Reg}(V))$ is dense in $W^{1,2}_0(\mathrm{Reg}(V), w^2\,dx) \cap L^{p+1}(\mathrm{Reg}(V), w^{p+1}\,dx)$, by Proposition 2.4, we conclude that $l(\eta) = 0$ for all $\eta \in W^{1,2}_0(\mathrm{Reg}(V), w^2\,dx) \cap L^{p+1}(\mathrm{Reg}(V), w^{p+1}\,dx)$. That is, v is a global solution for (7.2). The reverse and the 'local' statements follow by similar arguments. □

7.3 Pokhozhaev's type test function

As an application, in this section, we obtain some result on the non-existence of a global weak solution to equation (7.1) (see Theorem 7.4) by using a Pokhozhaev type test function.

Theorem 7.4. *Let $V = \frac{(N-2)^2}{4} \frac{c}{r^2}$ with $N > 2$, $c > -1$. Then there is no non-trivial global solution to equation (7.1) for any $\lambda \in \mathbb{R}$.*

In proving this result, we demonstrate the killing potential method. The next lemma shows that in this case a local solution w to (5.1) satisfies $w(x) = |x|^\alpha$ with $\alpha \in \mathbb{R}$. Before we begin, we need the following lemmas.

Lemma 7.3. *$f(r) = r^\mu \in L^p_{loc}$ if and only if $\mu p > -N$.*

Proof.

$$\int_{B_1} |f|^p \, dx = |S_{N-1}| \int_0^1 r^{\mu p} r^{N-1} \, dr < \infty \Longleftrightarrow \mu p + N > 0,$$

i.e., $\mu p > -N$. □

Corollary 7.2. *$f(r) = r^\mu \in W^{1,2}_{loc}$ if and only if $2(\mu - 1) > -N$.*

Proof. By Lemma 7.3,

$$|\nabla r^\mu| = |\mu| r^{\mu-1} \in L^2_{loc} \Longleftrightarrow 2(\mu - 1) > -N.$$

Or $\mu > -\frac{N-2}{2}$. □

Lemma 7.4. *Let*

$$V = \frac{(N-2)^2}{4} \frac{c}{r^2}$$

with $N > 2$, $c > -1$. Then $w = r^\alpha$ is a local solution to equation (5.1) with

$$\alpha = \frac{(N-2)}{2} \left(\sqrt{1+c} - 1 \right).$$

Proof. Note that in this case, $dV = \frac{(N-2)^2}{4} \frac{c}{r^2} \, dx$. So, $L^2(dV) = L^2(r^{-2} \, dx)$, $r^\alpha \in L^2_{loc}(r^{-2} \, dx)$ for $2\alpha - 2 + N > 0 \Longleftrightarrow \alpha > -\frac{N-2}{2}$. Thus, by Corollary 7.2, $r^\alpha \in W^{1,2}_{loc} \cap L^2_{loc}(dV)$ if and only if $\alpha > -\frac{N-2}{2}$.

Now, consider (5.1) for a radial w and $V = \frac{(N-2)^2}{4} \frac{c}{r^2}$. Then

$$-w'' - \frac{N-1}{r} w' + \frac{(N-2)^2}{4} \frac{c}{r^2} w = 0. \tag{7.10}$$

This second order ordinary differential equation has two linearly independent solutions (see, e.g., [16, Chapter 4 Section 1, Theorem 1.1 and Section 8.]). We will look for them in the form $w(r) = r^\alpha$, then (7.10) implies

$$\begin{aligned} -\alpha(\alpha-1) r^{\alpha-2} - (N-1)\alpha r^{\alpha-2} + \frac{(N-2)^2}{4} c r^{\alpha-2} &= 0, \\ \alpha(\alpha-1) + (N-1)\alpha - \frac{(N-2)^2}{4} c &= 0, \\ \alpha^2 + (N-2)\alpha - \frac{(N-2)^2}{4} c &= 0, \\ \left(\alpha + \frac{(N-2)}{2} \right)^2 = \frac{(N-2)^2}{4} (1+c). \end{aligned}$$

So,

$$\alpha_1 = \frac{(N-2)}{2} \left(\sqrt{1+c} - 1 \right) \text{ and } \alpha_2 = \frac{(N-2)}{2} \left(-\sqrt{1+c} - 1 \right).$$

By Corollary 7.2, $w = r^{\alpha_2} \notin W^{1,2}_{loc}$, and on the contrary $w = r^{\alpha_1} \in W^{1,2}_{loc}$, since

$$\alpha_1 = -\frac{N-2}{2} + \frac{N-2}{2}\sqrt{1+c} > -\frac{N-2}{2}\,, \ \ \forall\, c > -1.$$

For simplicity, from now we will denote α_1 by α. □

Corollary 7.3. *Let $u \in \mathbb{D}_{loc}$ be a local solution to equation (7.1) with $V = \frac{(N-2)^2}{4}\frac{c}{r^2}$, $c > -1$, $N > 2$. Then $v = \frac{u}{r^\alpha}$ is a local solution to equation (7.2) with $w = r^\alpha$ and α, as in Lemma 7.4.*

Proof. The proof is straightforward from Lemma 7.2 and Lemma 7.4. □

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $\rho > 0$, $\rho^{\pm 1} \in L^\infty_{loc}(\Omega)$. Then*

$$W^{m,p}_{loc}(\Omega) = W^{m,p}_{loc}(\Omega, \rho\, dx), \ \ \forall\, m \geq 0, \ \ p \geq 1.$$

Proof. To prove the statement it suffices to show that

$$W^{m,p}_0(G) = W^{m,p}_0(G, \rho\, dx)$$

for all open G such that $G \subset \bar{G} \subset \Omega$. To this end, one observes that $C^\infty_c(G)$ is dense in both spaces. Sos it suffices to prove that the norms are equivalent, that is, for every G, there are the constants $c_a > c_b > 0$ such that

$$c_a\|\phi\|_{W^{m,p}_0(G)} \geq \|\phi\|_{W^{m,p}_0(G,\rho\, dx)} \geq c_b\|\phi\|_{W^{m,p}_0(G)}, \ \ \forall\, \phi \in C^\infty_c(G).$$

Now, let $c_a = \|\rho\|^{\frac{1}{p}}_{L^\infty(G)}$, $c_b = \|\rho^{-1}\|^{\frac{-1}{p}}_{L^\infty(G)}$. Then

$$\begin{aligned}\|\phi\|^p_{W^{m,p}_0(G,\rho\, dx)} &= \sum_{k=0}^m \|D^k\phi\|^p_{L^p(G,\rho\, dx)} \\ &= \sum_{k=0}^m \int_G |D^k\phi|^p \rho\, dx \leq c_a^p \sum_{k=0}^m \int_G |D^k\phi|^p\, dx = c_a^p\|\phi\|^p_{W^{m,p}_0(G)}.\end{aligned}$$

Similarly we get the estimate from below:

$$\begin{aligned}\|\phi\|^p_{W^{m,p}_0(G,\rho\, dx)} &= \sum_{k=0}^m \|D^k\phi\|^p_{L^p(G,\rho\, dx)} \\ &= \sum_{k=0}^m \int_G |D^k\phi|^p \rho\, dx \geq c_b^p \sum_{k=0}^m \int_G |D^k\phi|^p\, dx = c_b^p\|\phi\|^p_{W^{m,p}_0(G)}.\end{aligned}$$ □

Lemma 7.5. *Let $w = r^\alpha$, $\alpha \geq \frac{N-2}{2}$. Let $v = \frac{u}{w}$ be a global solution of (7.2). Then*

$$v \in W^{2,2}_{loc}(\mathbb{R}^N \setminus (0), w^2\, dx).$$

Proof. We apply Theorem 3.1 with $A = w^2 Id$, where Id is the identity matrix, $B_1 = B_2 = 0$, $C = w^{p+1}|v|^{p-1} - \lambda w^2$, $f = 0$ on $\Omega = \mathbb{R}^N \setminus \{0\}$. Note that $w^2 = r^{2\alpha} \in C^\infty(\Omega)$, so A is locally Lipschitz on Ω. Note that $C = w^2(|u|^{p-1} - \lambda) \in L^\infty_{loc}(\Omega)$, since $|u|^{p-1} \leq \lambda$, by Theorem 7.2, and $w^2 \in C^\infty(\Omega)$. Finally, by Proposition 7.1, $W^{1,2}_{loc}(\Omega) = W^{1,2}_{loc}(\Omega, w^2\, dx)$, since $w^2 \in C^\infty(\Omega)$ and $w^2 > 0$. So, $v \in W^{1,2}_{loc}(\Omega)$ is a local weak solution to

$$-\operatorname{div}(w^2\nabla v) + Cv = 0.$$

Hence $v \in W^{2,2}_{loc}(\Omega) = W^{2,2}_{loc}(\Omega, w^2\, dx)$ by Theorem 3.1 and Proposition 7.1. □

Now in (7.2) we will use a Pokhozhaev type test function η described below. It will eliminate the right-hand side of (7.2) leaving a non-negative term in the left-hand side.

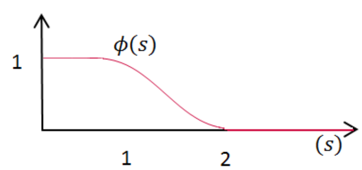
Definition 7.2. Define a Pokhozhaev type test function η by

$$\eta = \xi^2\big(2\nabla v \cdot x + (N+2\alpha)v\big),$$

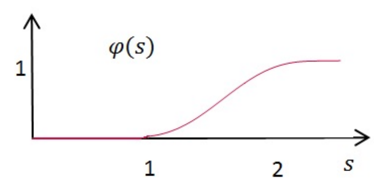
where v is a global solution to (7.2), the cut-off function $\xi \in \mathcal{D}$ is constructed as follows:

$$\xi_{\rho,P}(x) = \xi_\rho(x)\xi_P(x), \;\; \xi_P(x) = \phi\Big(\frac{|x|}{P}\Big), \;\; \xi_\rho(x) = \varphi\Big(\frac{|x|}{\rho}\Big),$$

where $\phi, \varphi \in C^\infty([0,\infty))$, such that $\mathbf{1}_{[0,1]} \le \phi \le \mathbf{1}_{[0,2)}$, $\mathbf{1}_{[2,\infty)} \le \varphi \le \mathbf{1}_{(1,\infty)}$ (see Figure 1).



(a) Function $\phi(s)$



(b) Function $\varphi(s)$

Figure 1: Functions $\phi(s)$ and $\varphi(s)$

Lemma 7.6. *Let ξ be as in Definition 7.2, $\xi \to 1$ as $P \to \infty$, $\rho \to 0$. Then*

$$|x|\,|\nabla\xi| \le 2\max\big\{\|\phi'\|_{L^\infty}, \|\varphi'\|_{L^\infty}\big\}\big(\mathbf{1}_{\{P<|x|<2P\}} + \mathbf{1}_{\{\rho<|x|<2\rho\}}\big).$$

In particular,

$$\big\|\,|x|\,|\nabla\xi|\,\big\|_{L^\infty} \le 2\max\big\{\|\phi'\|_{L^\infty}, \|\varphi'\|_{L^\infty}\big\} \tag{7.11}$$

and

$$|x|\,|\nabla\xi|(x) \to 0, \;\; \forall\, x \in \mathbb{R}^N \;\; \text{as} \;\; P \to \infty, \;\; \rho \to 0, \;\; \text{pointwise.} \tag{7.12}$$

Proof.

$$|\nabla\xi_P(x)| = \begin{cases} |\phi'|\Big(\dfrac{|x|}{P}\Big)\,\dfrac{1}{P}, & P \le |x| \le 2P, \\ 0, & \text{othewise,} \end{cases}, \quad |\nabla\xi_\rho(x)| = \begin{cases} |\varphi'|\Big(\dfrac{|x|}{\rho}\Big)\,\dfrac{1}{\rho}, & \rho \le |x| \le 2\rho, \\ 0, & \text{othewise.} \end{cases}$$

Now,

$$|x|\,|\nabla\xi_\rho| \le \|\varphi'\|_{L^\infty}\,\frac{|x|}{\rho}\,\mathbf{1}_{\{1<\frac{|x|}{\rho}<2\}}, \quad |x|\,|\nabla\xi_P| \le \|\phi'\|_{L^\infty}\,\frac{|x|}{P}\,\mathbf{1}_{\{1<\frac{|x|}{P}<2\}},$$
$$0 \le \xi_\rho(x) \le \mathbf{1}_{\{|x|>\rho\}}, \;\; 0 \le \xi_P(x) \le \mathbf{1}_{\{|x|<2P\}},$$
$$\begin{aligned} |x|\,|\nabla\xi| &\le |x|\,|\xi_\rho\nabla\xi_P| + |x|\,|\xi_P\nabla\xi_\rho| \\ &\le \|\phi'\|_{L^\infty}\,\frac{|x|}{P}\,\mathbf{1}_{\{P<|x|<2P\}}\,\mathbf{1}_{\{|x|>\rho\}} + \|\varphi'\|_{L^\infty}\,\frac{|x|}{\rho}\,\mathbf{1}_{\{|x|<2P\}}\,\mathbf{1}_{\{\rho<|x|<2\rho\}}. \end{aligned}$$

Since $\rho \ll P$,

$$\mathbf{1}_{\{P<|x|<2P\}}\,\mathbf{1}_{\{|x|>\rho\}} = \mathbf{1}_{\{P<|x|<2P\}}, \quad \mathbf{1}_{\{|x|<2P\}}\,\mathbf{1}_{\{\rho<|x|<2\rho\}} = \mathbf{1}_{\{\rho<|x|<2\rho\}},$$
$$\begin{aligned} |x|\,|\nabla\xi| &\le \|\phi'\|_{L^\infty}\,\frac{|x|}{P}\,\mathbf{1}_{\{P<|x|<2P\}} + \|\varphi'\|_{L^\infty}\,\frac{|x|}{\rho}\,\mathbf{1}_{\{\rho<|x|<2\rho\}} \\ &\le 2\max\big\{\|\phi'\|_{L^\infty}, \|\varphi'\|_{L^\infty}\big\}\big(\mathbf{1}_{\{P<|x|<2P\}} + \mathbf{1}_{\{\rho<|x|<2\rho\}}\big) \le 2\max\big\{\|\phi'\|_{L^\infty}, \|\varphi'\|_{L^\infty}\big\}. \end{aligned}$$

Note that (7.11) follows by the maximum norm, and (7.12) also follows pointwise as $P \to \infty$ and $\rho \to 0$. ∎

Lemma 7.7. *Let v be a global solution to equation (7.2) and let η be as in Definition 7.2. Then*

$$\lambda \int r^{2\alpha} v\eta \, dx \to 0 \quad \text{as} \quad P \to \infty \quad \text{and} \quad \rho \to 0.$$

Proof.

$$\begin{aligned}
&\lambda \int r^{2\alpha} v\xi^2 \left(2\nabla v \cdot x + (N+2\alpha)v\right) dx \\
&\qquad = \lambda \int \Big(2r^{2\alpha} v\xi^2 \nabla v \cdot x + r^{2\alpha} v^2 \xi^2 (N+2\alpha)\Big) \, dx \\
&\qquad = \lambda \int \Big(r^{2\alpha} \nabla |v|^2 \xi^2 \cdot x + r^{2\alpha} v^2 \xi^2 (N+2\alpha)\Big) \, dx \\
&\qquad = \lambda \int \Big(- |v|^2 \operatorname{div}(r^{2\alpha}\xi^2 \cdot x) + r^{2\alpha} v^2 \xi^2 (N+2\alpha)\Big) \, dx \\
&\qquad = \lambda \int \Big(- |v|^2 \Big(2\xi(\nabla\xi \cdot x) r^{2\alpha} + (N+2\alpha)\xi^2 r^{2\alpha}\Big) + r^{2\alpha} v^2 \xi^2 (N+2\alpha)\Big) \, dx \\
&\qquad = -2\lambda \int |v|^2 \xi (\nabla \xi \cdot x) r^{2\alpha} \, dx \longrightarrow 0
\end{aligned}$$

by Lemma 7.6 and the LDCT. □

Lemma 7.8. *Let v be a global solution to equation (7.2) and let η be as in Definition 7.2. Then*

$$\int r^{(p+1)\alpha} |v|^{p-1} v\eta \, dx \longrightarrow \frac{N(p-1)}{p+1} \int |v|^{p+1} \xi^2 r^{(p+1)\alpha} \, dx > 0, \quad \forall\, p > 1$$

as $P \to \infty$ and $\rho \to 0$.

Proof.

$$\begin{aligned}
&\int r^{(p+1)\alpha} |v|^{p-1} v\xi^2 \left(2\nabla v \cdot x + (N+2\alpha)v\right) dx \\
&\quad = \int 2r^{(p+1)\alpha} |v|^{p-1} v\xi^2 \nabla v \cdot x \, dx + \int (N+2\alpha)|v|^{p+1}\xi^2 r^{(p+1)\alpha} \, dx \\
&\quad = \int \frac{2}{p+1} \nabla |v|^{p+1} \xi^2 \cdot x r^{(p+1)\alpha} \, dx + \int (N+2\alpha)|v|^{p+1}\xi^2 r^{(p+1)\alpha} \, dx \\
&\quad = -\int \frac{2}{p+1} |v|^{p+1} \operatorname{div}(\xi^2 \cdot x r^{(p+1)\alpha}) \, dx + \int (N+2\alpha)|v|^{p+1}\xi^2 r^{(p+1)\alpha} \, dx \\
&\quad = \int \frac{-2(N+(p+1)\alpha)}{p+1} |v|^{p+1}\xi^2 r^{(p+1)\alpha} \, dx - \int \frac{2}{p+1} |v|^{p+1} r^{(p+1)\alpha} (2\xi\nabla\xi \cdot x) \, dx \\
&\qquad\qquad + \int (N+2\alpha)|v|^{p+1}\xi^2 r^{(p+1)\alpha} \, dx \\
&\quad = \int \Big(\frac{-2(N+(p+1)\alpha)}{p+1} + N + 2\alpha\Big) |v|^{p+1}\xi^2 r^{(p+1)\alpha} \, dx \\
&\qquad\qquad - \int \frac{4}{p+1} |v|^{p+1} r^{(p+1)\alpha} (\xi\nabla\xi \cdot x) \, dx \\
&\quad \longrightarrow \int \frac{N(p-1)}{p+1} |v|^{p+1} r^{(p+1)\alpha} \, dx + 0
\end{aligned}$$

by Lemma 7.6 and the LDCT. □

Lemma 7.9. *Let v be a global solution to equation (7.2) and let η be as in Definition 7.2. Then*

$$\int r^{2\alpha} \nabla v \nabla \eta \, dx \longrightarrow 2 \int |\nabla v|^2 r^{2\alpha} \quad \text{as} \quad P \to \infty \quad \text{and} \quad \rho \to 0.$$

Proof. Note that $v \in W^{2,2}_{loc}(\mathbb{R}^N \setminus \{0\})$ by Lemma 7.5. Then

$$\begin{aligned}
\int r^{2\alpha}\nabla v\nabla\big(\xi^2\big(2\nabla v\cdot x+(N+2\alpha)v\big)\big)\,dx \\
&= \int r^{2\alpha}\xi^2\Big(2\nabla v\cdot D^2v\cdot x+2\nabla v\cdot I\cdot\nabla v+(N+2\alpha)\nabla v\cdot\nabla v\Big)\,dx \\
&\qquad + \int r^{2\alpha}2(\nabla v\cdot\nabla\xi)\xi\big(2\nabla v\cdot x+(N+2\alpha)v\big)\,dx \\
&= \int r^{2\alpha}\xi^2\Big(\nabla|\nabla v|^2\cdot x+2|\nabla v|^2+(N+2\alpha)|\nabla v|^2\Big)\,dx \\
&\qquad + \int r^{2\alpha}2(\nabla v\cdot\nabla\xi)\xi\big(2\nabla v\cdot x+(N+2\alpha)v\big)\,dx \\
&= I_1+I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int r^{2\alpha}\xi^2\Big(\nabla|\nabla v|^2\cdot x+(2+N+2\alpha)|\nabla v|^2\Big)\,dx \\
&= -\int|\nabla v|^2\operatorname{div}(r^{2\alpha}\xi^2\cdot x)\,dx+(2+N+2\alpha)\int r^{2\alpha}\xi^2|\nabla v|^2\,dx \\
&= -\int|\nabla v|^2\Big(2\xi(\nabla\xi\cdot x)r^{2\alpha}+(N+2\alpha)\xi^2r^{2\alpha}\Big)\,dx+(2+N+2\alpha)\int r^{2\alpha}\xi^2|\nabla v|^2\,dx \\
&= -2\int|\nabla v|^2\xi(\nabla\xi\cdot x)r^{2\alpha}\,dx+2\int r^{2\alpha}\xi^2|\nabla v|^2\,dx \longrightarrow -0+2\int|\nabla v|^2r^{2\alpha}\,dx
\end{aligned}$$

by Lemma 7.6 and the LDCT, and

$$\begin{aligned}
|I_2| = \left|\int r^{2\alpha}2(\nabla v\cdot\nabla\xi)\xi\big(2\nabla v\cdot x+(N+2\alpha)v\big)\,dx\right| &\le 4\int r^{2\alpha}|\nabla v|^2\big(\xi|x|\,|\nabla\xi|\big)\,dx \\
&+2\int(N+2\alpha)r^{2\alpha}|\nabla v|\,\frac{|v|}{r}\,\big(\xi|x|\,|\nabla\xi|\big)\,dx\longrightarrow 0\ \text{ as }\ P\to\infty,\ \ \rho\to 0
\end{aligned}$$

by Lemma 7.6 and the LDCT. Observe that $|\nabla v|\frac{|v|}{r}\in L^1$ due to the Hardy inequality (see Lemma 5.1). □

Proof of Theorem 7.4. It follows from (7.2) and Lemmas 7.7–7.9 that for a global solution v of (7.2),

$$2\int|\nabla v|^2r^{2\alpha}\,dx+\frac{N(p-1)}{p+1}\int|v|^{p+1}r^{(p+1)\alpha}\,dx=0.$$

So, $v=0$. □

Acknowledgments

The referee's thorough reading and insightful comments on the note are especially appreciated by the authors.

References

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Observation of Bose–Einstein condensation in a dilute atomic vapor. *Science* **269** (1995), 198–201.

[2] W. Bao and Y. Cai, Mathematical theory and numerical methods for Bose–Einstein condensation. *Kinet. Relat. Models* **6** (2013), no. 1, 1–135.

[3] S. N. Bose, Plancks Gesetz und Lichtquantenhypothese. *Z. Physik* **26** (1924), 178–181.

[4] C. C. Bradley, C. A. Sackett, J. J. Tollett and R. G. Hulet, Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions. *Phys. Rev. Lett.* **75**, 1687–1691; *Phys. Rev. Lett.* **79** (1997), 1170.

[5] E. A. Coddington and R. Carlson, *Linear Ordinary Differential Equations.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[6] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn and W. Ketterle, Bose–Einstein condensation in a gas of sodium atoms. *Phys. Rev. Lett.* **75** (1995), 3969–3974.

[7] J. Dugundji, *Topology.* Allyn and Bacon, Inc., Boston, MA, 1966.

[8] N. Dunford, J. T. Schwartz, W. G. Bade and R. G. Bartle, *Linear Operators*, Part I. *General Theory*John Wiley & Sons, Inc., Hoboken, New Jersey, 1957.

[9] A. Einstein, Quantentheorie des einatomigen idealen gases. Zweite Abhandlung. *Akademie der Wissenshaften, in Kommission bei W. de Gruyter*, 1924, 261–267.

[10] L. Erdős, B. Schlein and H.-T. Yau, Derivation of the Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensate. *Ann. of Math. (2)* **172** (2010), no. 1, 291–370.

[11] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes. 2nd. ed., Studies in Mathematics, 19. Walter de Gruyter, Göttingen, 2010.

[12] M. Fukushima, K.-iti Sato and S. Taniguchi, On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures. *Osaka J. Math.* **28** (1991), no. 3, 517–535.

[13] V. A. Galaktionov, E. L. Mitidieri and S. I. Pohozaev, *Blow-Up for Higher-Order Parabolic, Hyperbolic, Dispersion and Schrödinger Equations.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.

[14] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer Berlin, Heidelberg, 2015.

[15] E. P. Gross, Structure of a quantized vortex in boson systems. *Nuovo Cimento (10)* **20** (1961), 454–477.

[16] Ph. Hartman, *Ordinary Differential Equations.* John Wiley & Sons, Inc., New York–London–Sydney, 1964.

[17] V. Kondratiev, V. Liskevich and Z. Sobol, Second-order semilinear elliptic inequalities in exterior domains. *J. Differential Equations* **187** (2003), no. 2, 429–455.

[18] E. H. Lieb, R. Seiringer, J. Ph. Solovej and J. Yngvason, *The Mathematics of the Bose Gas and its Condensation.* Oberwolfach Seminars, 34. Birkhäuser Verlag, Basel, 2005.

[19] E. H. Lieb, R. Seiringer and J. Yngvason, Bosons in a trap: A rigorous derivation of the Gross–Pitaevskii energy functional. In: *Thirring, W. (eds) The Stability of Matter: From Atoms to Stars*, 33–72. *Springer, Berlin, Heidelberg*, 2001.

[20] E. H. Lieb and M. Loss, *Analysis.* Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[21] V. Liskevich, A. Shishkov and Z. Sobol, Singular solutions to the heat equations with nonlinear absorption and Hardy potentials. *Commun. Contemp. Math.* **14** (2012), no. 2, 1250013, 28 pp.

[22] T. Matskewich and P. E. Sobolevskii, The best possible constant in generalized Hardy's inequality for convex domain in $\mathbf{R}^n$. *Nonlinear Anal.* **28** (1997), no. 9, 1601–1610.

[23] V. G. Maz'ya, *Sobolev Spaces. With Applications to Elliptic Partial Differential Equations.* Transl. from the Russian by T. O. Shaposhnikova. 2nd revised and augmented ed. Grundlehren der Mathematischen Wissenschaften 342. Springer, Berlin, 2011.

[24] È. Mitidieri and S. I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. (Russian) *Tr. Mat. Inst. Steklova* **234** (2001), 1–384; translation in *Proc. Steklov Inst. Math.* **2001**, no. 3(234), 1–362.

[25] C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases.* 2nd. ed. Cambridge University Press, UK, 2008.

[26] L. P. Pitaevskii, Vortex lines in an imperfect bose gas. *Soviet Phys. JETP* **13** (1961), no. 2, 451–454.

[27] L. Pitaevskii and S. Stringari, *Bose–Einstein Condensation.* International Series of Monographs on Physics, 116. The Clarendon Press, Oxford University Press, Oxford, 2003.

[28] W. P. Ziemer, *Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation.* Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

(Received 07.09.2024; revised 14.02.2025; accepted 20.02.2025)

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