Memoirs on Differential Equations and Mathematical Physics

Volume 94, 2025, 115–130

Arezou Soleimani Nia, Ghasem A. Afrouzi, Hadi Haghshenas

EXISTENCE AND MULTIPLICITY RESULTS FOR p-HAMILTONIAN SYSTEMS

Abstract. In this paper, we give some new criteria that guarantee the existence of at least one weak solution and two weak solutions for a p-Hamiltonian boundary value problem generated by impulsive effects. To ensure the existence of these solutions, we use variational methods and critical point theory as our main tools.

2020 Mathematics Subject Classification. 34B15, 34B37, 58E30.

Key words and phrases. Weak solution, p-Hamiltonian boundary value problem, impulsive effect, critical point theory, variational methods.

რეზიუმე. ნაშრომში მოცემულია ახალი კრიტერიუმები, რომლებიც უზრუნველყოფს მინიმუმ ერთი სუსტი ამონახსნის და მინიმუმ ორი სუსტი ამონახსნის არსებობას იმპულსური ეფექტებით წარმოქმნილი p-ჰამილტონური სასაზღვრო ამოცანისთვის. ამ ამონახსნების არსებობის უზრუნველსაყოფად მთავარ ინსტრუმენტად გამოყენებულია ვარიაციული მეთოდები და კრიტიკულ წერტილთა თეორია.

1 Introduction

In this research, we prove the existence of at least one weak solution and two weak solutions to the following second-order impulsive p-Hamiltonian system

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \mu\nabla G(t,u), \ \text{ a.e. } \ t \in J, \\ \triangle(|u_i'(t_j)|^{p-2}u_i'(t_j)) = I_{ij}(u_i(t_j)), \ \ i = 1,2,\dots,N, \ \ j = 1,2,\dots,m, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.1}$$

Here, we assume that

- $N \geq 1$, $m \geq 2$, $p > 1$, $T > 0$ and $\lambda > 0$;
- the function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in $[0,T]$ and is C^1 in $\mathbb{R}^N$;
- $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $G(\,\cdot\,,x)$ is continuous on $[0,T]$ for all $x \in \mathbb{R}^N$, and $G(t,\,\cdot\,)$ is C^1 on $\mathbb{R}^N$ for almost every $t \in [0,T]$;
- $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $J = [0,T] \setminus \{t_1,t_2,\dots,t_m\}$, $u(t) = (u_1(t),\dots,u_N(t))$ and $\triangle(u_i'(t_j)) = u_i'(t_j^+) - u_i'(t_j^-)$ such that $u_i'(t_j^{\pm}) = \lim\limits_{t\to t_j^{\pm}} u_i'(t)$;
- the functions $I_{ij} : \mathbb{R} \to \mathbb{R}$ ($i = 1,2,\dots,N$ and $j = 1,2,\dots,m$) satisfy $|I_{ij}(s)| \leq L_{ij}|s|^{p-1}$ for every $s \in \mathbb{R}$;
- $A(t) = (a_{ij}(t))_{N\times N}$ is an $N \times N$ continuous symmetric matrix and there is a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x,x) \geq \underline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0,T]$.

The study of the multiplicity of the solutions of Hamiltonian systems, as particular cases of dynamical systems, is mathematically important and interesting from a practical point of view. This is because these systems constitute a natural framework for the mathematical models of many natural phenomena in fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, etc. Inspired by the monographs [27] and [32], the existence and multiplicity of weak solutions for Hamiltonian systems have been investigated by many authors using variational methods (see, e.g., [13,14,16,18,20,28,30,39,43,46] and the references therein).

In recent years, critical points theorems were widely used to solve differential equations (see [3,7, 10–12,19,25] and references therein).

In contrast to Hamiltonian systems, for the general case $p > 1$, the study of the existence and multiplicity of periodic solutions is recent (see [21,40]). In [40], Xu and Tang proved the existence of periodic solutions for the problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \nabla F(t,u), \ \text{ a.e. } \ t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{1.2}$$

by minimax methods in the critical point theory. In [26], Ma and Zhang obtained some results on the existence and multiplicity of non-trivial periodic solutions for system (1.2). These results generalize the corresponding results in [34]. In [21], two existence results have been established by the least action principle and the Mountain-pass lemma for ordinary p-Laplacian systems with nonlinear boundary conditions.

In [25], based on two general three critical points theorems due respectively to Ricceri (see [33]) and Averna–Bonanno (see [4]), the authors proved the existence of three solutions for the p-Hamiltonian system

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \mu\nabla G(t,u), \ \text{ a.e. } \ t \in J, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

In this article, we use three theorems of Bonanno to prove the existence of one weak solution and two weak solutions for problem (1.1).

2 Preliminaries

For a given non-empty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions:

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(r_1,r_2)} \frac{\sup_{u \in \Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_2(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty,r_1)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, and

$$\rho(r) = \sup_{v \in \Phi^{-1}(r,\infty)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty,r)} \Psi(u)}{\Phi(v) - r}$$

for all $r \in \mathbb{R}$.

The following critical point theorems due to Bonanno will be used to prove our mail results.

Theorem 2.1 ([6, Theorem 5.1])**.** *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in (\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)})$, there is $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([6, Theorem 5.5])**.** *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there is $r \in \mathbb{R}$, with $\inf_X \Phi < r < \sup_X \Phi$, such that*

$$\rho(r) > 0,$$

and for each $\lambda > \frac{1}{\rho(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in (\frac{1}{\rho(r)}, +\infty)$, there is $u_{0,\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r, +\infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.3 ([5, Theorem 3.2])**.** *Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(r,+\infty)} \Psi(u) < +\infty$ and assume that for each $\lambda \in \big(0, \frac{r}{\sup_{u \in \Phi^{-1}(r,+\infty)} \Psi(u)}\big)$, the functional $J_\lambda = \Phi - \lambda\Phi$ satisfies the* (PS)*-condition and is unbounded from below. Then for each $\lambda \in \big(0, \frac{r}{\sup_{u \in \Phi^{-1}(r,+\infty)} \Psi(u)}\big)$, the functional J_λ admits two distinct critical points.*

Here, we recall some basic concepts that will be used in what follows. Let

$$W_T^{1,p} = \Big\{u : [0,T] \to \mathbb{R}^N : \ u \text{ is absolutely continuous}, \ u(0) = u(T), \ u' \in L^p([0,T], \mathbb{R}^N)\Big\},$$

be endowed with the norm

$$\|u\| = \Bigg(\int_0^T |u'(t)|^p + \big(A(t)|u(t)|^{p-2}u(t), u(t)\big)\, dt \Bigg)^{\frac{1}{p}}.$$

Observe that

$$\big(A(t)|x|^{p-2}x,x\big)=|x|^{p-2}\sum_{i,j=1}^{N}a_{ij}(t)x_ix_j\le|x|^{p-2}\sum_{i,j=1}^{N}|a_{ij}(t)|\,|x_i|\,|x_j|\le\Big(\sum_{i,j=1}^{N}\|a_{ij}(t)\|_\infty\Big)|x|^p.$$

Then there exists a constant $\overline{\lambda}\le\sum\limits_{i,j=1}^{N}\|a_{ij}(t)\|_\infty$ such that $(A(t)|x|^{p-2}x,x)\le\overline{\lambda}|x|^p$ for all $x\in\mathbb{R}^N$. So,

$$\min\{1,\underline{\lambda}\}|||u|||^p\le\|u\|^p\le\max\{1,\overline{\lambda}\}|||u|||^p, \tag{2.1}$$

where

$$|||u|||=\Big(\int_0^T|u(t)|^p\,dt+\int_0^T|u'(t)|^p\,dt\Big)^{\frac1p}$$

is the usual norm of $W_T^{1,p}$. Let

$$k_0=\sup_{u\in W_T^{1,p}\setminus\{0\}}\frac{\|u\|_\infty}{\|u\|},\quad \|u\|_\infty=\sup_{t\in[0,T]}|u(t)|, \tag{2.2}$$

where $|\cdot|$ is the usual norm of $\mathbb{R}^N$. Since $W_T^{1,p}\hookrightarrow C^0$ is compact, one has $k_0<+\infty$ and for each $u\in W_T^{1,p}$, there exists $\xi\in[0,T]$ such that $|u(\xi)|=\min_{t\in[0,T]}|u(t)|$. Hence, by Hölder's inequality, one has

$$\begin{aligned}|u(t)|&=\Big|\int_\xi^t u'(s)\,ds+u(\xi)\Big|\le\int_0^T|u'(s)|\,ds+\frac1T\int_0^T|u(\xi)|\,ds\\&\le\int_0^T|u'(s)|\,ds+\frac1T\int_0^T|u(s)|\,ds\le T^{\frac1q}\Big(\int_0^T|u'(s)|^p\,ds\Big)^{\frac1p}+T^{-\frac1p}\Big(\int_0^T|u(s)|^p\,ds\Big)^{\frac1p}\\&\le\max\{T^{\frac1q},T^{-\frac1p}\}\Big(\Big(\int_0^T|u'(s)|^p\,ds\Big)^{\frac1p}+\Big(\int_0^T|u(s)|^p\,ds\Big)^{\frac1p}\Big)\\&\le\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}\Big(\int_0^T|u'(s)|^p\,ds+\int_0^T|u(s)|^p\,ds\Big)^{\frac1p}=\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}|||u|||\end{aligned}$$

for each $t\in[0,T]$ and $q=\frac{p}{p-1}$. So, by (2.1) and the above expression, we obtain

$$\|u\|_\infty\le\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}|||u|||\le\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}\big(\min\{1,\underline{\lambda}\}\big)^{-\frac1p}\|u\|.$$

From this and (2.2) it follows that

$$k_0\le k=\sqrt[q]{2}\,\max\{T^{\frac1q},T^{-\frac1p}\}\big(\min\{1,\underline{\lambda}\}\big)^{-\frac1p}.$$

For all $v\in W_T^{1,p}$ we have

$$\begin{aligned}&-\int_0^T\big(|u'(t)|^{p-2}u'(t)\big)'v(t)\,dt+\int_0^T\big(A(t)|u(t)|^{p-2}u(t),v(t)\big)\,dt\\&\qquad-\lambda\int_0^T\big(\nabla F(t,u(t)),v(t)\big)\,dt-\mu\int_0^T\big(\nabla G(t,u(t)),v(t)\big)\,dt=0,\end{aligned}$$

according to the condition of problem (1.1),

$$\int_0^T \Big[\big(|u'(t)|^{p-2}u'(t), v'(t)\big) + \big(A(t)|u(t)|^{p-2}u(t), v(t)\big)\Big]\,dt$$
$$+\sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) - \lambda\int_0^T \big(\nabla F(t,u(t)), v(t)\big)\,dt - \mu\int_0^T \big(\nabla G(t,u(t)), v(t)\big)\,dt = 0 \quad (2.3)$$

for all $v \in W_T^{1,p}$. As usual, a weak solution to problem (1.1) is any $u \in W_T^{1,p}$ that satisfies in (2.3).

3 Main results

For two given non-negative constants θ_i for $i = 1, 2$ and a given positive constant d with $\theta_i^p \neq (\frac{1-s}{1+s})\overline{\lambda}Tk^pd^p$, put

$$a_d(\theta_i) := \frac{\int_0^T \max_{|u|<\theta_i}\big[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\big]\,dt - \int_0^T F(t,d)\,dt}{\theta_i^p - (\frac{1-s}{1+s})\overline{\lambda}Tk^pd^p},$$

$$\mu_1 := \frac{(1-s)\theta_1^p - (1+s)\overline{\lambda}Tk^pd^p - \lambda pk^p\int_0^T \max_{|u|<\theta_1} F(t,u)\,dt + \lambda pk^p\int_0^T F(t,d)\,dt}{pk^p\int_0^T \max_{|u|<\theta_1} G(t,u)\,dt},$$

$$\mu_2 := \frac{(1-s)\theta_2^p - (1+s)\overline{\lambda}Tk^pd^p - \lambda pk^p\int_0^T \max_{|u|<\theta_2} F(t,u)\,dt + \lambda pk^p\int_0^T F(t,d)\,dt}{pk^p\int_0^T \max_{|u|<\theta_2} G(t,u)\,dt}$$

and

$$s := k^p\sum_{j=1}^{m}\sum_{i=1}^{N} L_{ij} < 1.$$

Now, we present an application of Theorem 2.1 that we will used to obtain one nontrivial weak solution.

Theorem 3.1. *Assume that there exist three nonnegative constants θ_1, θ_2, and d with*

$$\theta_1^p < \Big(\frac{1+s}{1-s}\Big)\overline{\lambda}Tk^pd^p < \theta_2^p \quad (3.1)$$

such that

(A$_1$) $\int_0^T F(t,d)\,dt \geq 0$ *for every* $t \in [0,T]$;

(A$_2$) $a_d(\theta_2) < a_d(\theta_1)$.

Moreover, $\lambda \in \frac{(1-s)}{pk^p}\left(\frac{1}{a_d(\theta_1)}, \frac{1}{a_d(\theta_2)}\right)$ *and potential* $G(t,x)$ *for all* $(t,x) \in [0,T]\times(0,+\infty)$*, is nonnegative. Then for every* $\mu \in (\mu_1, \mu_2)$*, problem (1.1) admits at least one nontrivial weak solution* $u_1 \in W_T^{1,p}$.

Proof. Let $X = W_T^{1,p}$ be endowed with $\|\cdot\|$. We introduce the functionals ϕ, $\psi : X \to \mathbb{R}$ for each u in X as follows:

$$\phi(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(t)\,dt$$

and
$$\psi(u) = \int_0^T F(t,u(t))\,dt + \frac{\mu}{\lambda}\int_0^T G(t,u(t))\,dt,$$
and put $J_\lambda(u) := \phi(u) - \lambda\psi(u)$. Let us prove that the functionals ϕ and ψ satisfy the conditions. It is well known that ψ is a differentiable functional whose differential at the point $u \in X$ is
$$\psi'(u)(v) = \int_0^T \big(\nabla F(t,u(t)), v(t)\big)\,dt + \frac{\mu}{\lambda}\int_0^T \big(\nabla G(t,u(t)), v(t)\big)\,dt$$
for every $v \in X$ as well as being sequentially weakly upper semicontinuous. Furthermore, $\psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that ψ' is strongly continuous on X. To this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$; then $\{u_n\}$ converges uniformly to u on T as $n \to \infty$ (see [44]). Since ∇F, ∇G are continuous functions in $\mathbb{R}$ for every $t \in T$,
$$\nabla F(t,u_n) + \frac{\mu}{\lambda}\nabla G(t,u_n) \to \nabla F(t,u) + \frac{\mu}{\lambda}\nabla G(t,u)$$
as $n \to \infty$. Hence $\psi'(u_n) \to \psi'(u)$ as $n \to \infty$. Thus we have proved that ψ' is strongly continuous on X, which implies that ψ' is a compact operator by Proposition 26.2 of [44]. Furthermore, $\phi' : X \to X^*$ admits a continuous inverse, where
$$\phi'(u)(v) = \int_0^T \Big[\, |u'(t)|^{p-2}u'(t)v'(t) + A(t)|u(t)|^{p-2}u(t)v(t)\Big]\,dt$$
for every $v \in X$. Clearly, the weak solutions of problem (1.1) are exactly the solutions of the equation $J'_\lambda(u) = 0$. Now, put
$$r_1 := \frac{(1-s)}{p}\left(\frac{\theta_1}{k}\right)^p, \quad r_2 := \frac{(1-s)}{p}\left(\frac{\theta_2}{k}\right)^p \text{ and } w(t) := d.$$
It is easy to verify that $w \in X$ and
$$\frac{(1-s)\underline{\lambda}T}{p}\,d^p \le \phi(w) \le \frac{(1-s)\overline{\lambda}T}{p}\,d^p.$$
In particular, from (3.1) we conclude that
$$r_1 < \phi(w) < r_2.$$
On the other hand, for all $u \in X$, we have
$$\phi^{-1}(-\infty, r_2) = \big\{u \in X:\ \phi(u) < r_2\big\} = \big\{u \in X:\ |u| < c_2\big\},$$
from which it follows that
$$\sup_{u\in\phi^{-1}(-\infty,r_2)} \psi(u) = \sup_{u\in\phi^{-1}(-\infty,r_2)} \left[\int_0^T \Big(F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big)\,dt\right]$$
$$\le \int_0^T \max_{|u(t)|<\theta_2}\Big[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big]\,dt.$$
Arguing as before, we obtain
$$\sup_{u\in\phi^{-1}(-\infty,r_1)} \psi(u) = \sup_{u\in\phi^{-1}(-\infty,r_1)} \left[\int_0^T \Big(F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big)\,dt\right]$$
$$\le \int_0^T \max_{|u(t)|<\theta_1}\Big[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big]\,dt.$$

Since $w(t) > 0$ for each $t \in T$, assumption (A_1) ensures that

$$\psi(w) \geq \int_0^T F(t,d)\,dt.$$

Then, due to the fact that $G \geq 0$, we get

$$\int_0^T \max_{|u|<\theta_2} \Big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\Big]\,dt \geq \int_0^T F(t,d)\,dt,$$

and thus $a_d(\theta_2) \geq 0$. At this point, we have

$$\begin{aligned}
\beta(r_1,r_2) &\leq \frac{\sup\limits_{u\in\phi^{-1}(-\infty,r_2)} \psi(u) - \psi(w)}{r_2 - \phi(w)} \\
&\leq \frac{\int_0^T \max\limits_{|u|<\theta_2} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt - \int_0^T F(t,d)\,dt}{\frac{(1-s)}{p}\,(\frac{\theta_2}{k})^p - \frac{(1+s)\overline{\lambda}T}{p}\,d^p} \\
&= \frac{pk^p}{(1-s)}\,\frac{\int_0^T \max\limits_{|u|<\theta_2} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt - \int_0^T F(t,d)\,dt}{\theta_2^p - (\frac{1+s}{1-s})\overline{\lambda}Tk^p d^p} \\
&= \frac{pk^p}{(1-s)}\,a_d(\theta_2).
\end{aligned}$$

Since $a_d(\theta_2) \geq 0$, hypothesis (A_2) implies that

$$\int_0^T \max_{|u|<\theta_1} \Big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\Big]\,dt < \int_0^T F(t,d)\,dt.$$

So,

$$\begin{aligned}
\rho_2(r_1,r_2) &\geq \frac{\psi(w) - \sup\limits_{u\in\phi^{-1}(-\infty,r_1)} \psi(u)}{\phi(w) - r_1} \\
&\geq \frac{\int_0^T F(t,d)\,dt - \int_0^T \max\limits_{|u|<\theta_1} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt}{\frac{(1+s)\overline{\lambda}T}{p}\,d^p - \frac{(1-s)}{p}\,(\frac{\theta_1}{k})^p} \\
&= \frac{pk^p}{(1-s)}\,\frac{\int_0^T F(t,d)\,dt - \int_0^T \max\limits_{|u|<\theta_1} \big[F(t,u(t)) + \frac{\mu}{\lambda}\,G(t,u(t))\big]\,dt}{(\frac{1+s}{1-s})\overline{\lambda}Tk^p d^p - \theta_1^p} \\
&= \frac{pk^p}{(1-s)}\,a_d(\theta_1).
\end{aligned}$$

Hence, from assumption (A_2), $\beta(r_1,r_2) < \rho_2(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in \frac{(1-s)}{pk^p}\,(\frac{1}{a_d(\theta_1)}, \frac{1}{a_d(\theta_2)})$, the functional J_λ admits at least one critical point u_1 such that

$$r_1 < \phi(u_1) < r_2. \qquad \square$$

Theorem 3.2. *Assume that there exist two constants θ and $\overline{d}$ with*

$$\Big(\frac{1+s}{1-s}\Big)\overline{\lambda}Tk^p\overline{d}^{\,p} < \theta^p$$

such that

(A$_3$) $\int_0^T F(t,\overline{d})\,dt \geq 0$ *for every* $t \in [0,T]$;

(A$_4$) $\lim_{|x|\to 0} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = \lim_{|x|\to+\infty} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly, for almost every* $t \in [0,T]$.

(A$_5$) *There exist the constants* $c > 0$ *and* $1 \leq q < p$ *such that*

$$|\nabla F(t,x)| \leq c(1+|x|^{q-1})$$

for all $x \in \mathbb{R}^N$ *and almost every* $t \in [0,T]$.

(A$_6$) *For any* $i \in \{1,2,\dots,N\}$ *and* $j \in \{1,2,\dots,m\}$, *there exist the constants* $a_{ij} > 0$, $b_{ij} > 0$ *and* $\gamma_{ij} \in [0,1]$ *such that*

$$I_{ij}(y) \geq -a_{ij} - b_{ij}y^{\gamma_{ij}} \quad (y \geq 0) \quad \text{and} \quad I_{ij}(y) \leq a_{ij} + b_{ij}(-y)^{\gamma_{ij}} \quad (y \leq 0).$$

Let $\lambda > \lambda_3$, *where*

$$\lambda_3 := \frac{(1-s)}{pk^p} \frac{(\frac{1+s}{1-s})\overline{\lambda}Tk^p\overline{d}^p - \theta^p}{\int_0^T F(t,\overline{d})\,dt - \int_0^T \max_{|u|<\theta}\left(F(t,u) + \frac{\mu}{\lambda}G(t,u)\right)dt},$$

whose potential $G(t,x)$ *for all* $(t,x) \in [0,T]\times(0,+\infty)$ *is nonnegative. Then for every* $\mu \in (0,\mu_3)$, *where*

$$\mu_3 := \frac{(1-s)\theta^p - (1+s)\overline{\lambda}Tk^p d^p - \lambda pk^p \int_0^T \max_{|u|<\theta} F(t,u)\,dt + \lambda pk^p \int_0^T F(t,d)\,dt}{pk^p \int_0^T \max_{|u|<\theta} G(t,u)\,dt},$$

problem (1.1) *admits at least one nontrivial weak solution* $u_3 \in W_T^{1,p}$.

Proof. Since the critical points of the functional $J := \phi - \lambda\psi$ on X are exactly the weak solutions of problem (1.1), our aim is to apply Theorem 2.1 to ϕ and ψ. It is well-known that ϕ is a continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, ψ is continuously Gateaux differentiable and sequentially weakly continuous. Owning to the assumption (A$_6$), we have

$$\int_0^z I_{ij}(t)\,dt \geq -a_{ij}z - \frac{b_{ij}}{\gamma_{ij}+1}\,z^{\gamma_{ij}+1} = -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}\,|z|^{\gamma_{ij}+1} \quad (z \geq 0)$$

and

$$\int_z^0 I_{ij}(t)\,dt \leq -a_{ij}z - \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij}+1}\,z^{\gamma_{ij}+1} = a_{ij}|z| + \frac{b_{ij}}{\gamma_{ij}+1}\,|z|^{\gamma_{ij}+1} \quad (z < 0).$$

Therefore, for every $i \in \{1,2,\dots,N\}$, $j \in \{1,2,\dots,m\}$ and $z \in \mathbb{R}$,

$$\int_0^z I_{ij}(t)\,dt \geq -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}\,|z|^{\gamma_{ij}+1}. \tag{3.2}$$

Thanks to (A$_4$), fixing $0 < \varepsilon < \frac{\min\{1,\lambda\}}{\mu}$ small enough, we can find a constant $C_\varepsilon > 0$ such that

$$|G(t,x)| \leq C_\varepsilon + \frac{\varepsilon}{p}\,|x|^p \tag{3.3}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. Also, taking (A_5) into account, we get

$$|F(t,x)| \le c|x| + \frac{c}{q}|x|^q \tag{3.4}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. Now, by (3.2), (3.3) and (3.4), for all $u \in X$ and $\lambda \in \mathbb{R}^+$, we obtain

$$\begin{aligned}
\phi(u) - \lambda\psi(u) &= \frac{1}{p}\|u\|^p - \lambda\int_0^T F(t,u(t))\,dt - \mu\int_0^T G(t,u(t))\,dt + \sum_{j=1}^m\sum_{i=1}^N \int_0^{u_i(t_j)} I_{ij}(t)\,dt \\
&\ge \frac{1}{p}\|u\|^p - \lambda\int_0^T \Big(c|u(t)| + \frac{c}{q}|u(t)|^q\Big)\,dt - \mu\int_0^T \Big(C_\varepsilon + \frac{\varepsilon}{p}|u(t)|^p\Big)\,dt \\
&\quad - \sum_{j=1}^m\sum_{i=1}^N a_{ij}|u(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1} \\
&\ge \frac{1}{p}\Big(1 - \frac{\mu\varepsilon}{\min\{1,\underline{\lambda}\}}\Big)\|u\|^p - \frac{1}{q}\big(\min\{1,\underline{\lambda}\}\big)^{-\frac{q}{p}}\lambda c\|u\|^q - \big(\min\{1,\underline{\lambda}\}\big)^{-\frac{1}{p}} T^{\frac{1}{q}}\lambda c\|u\| \\
&\quad - \mu C_\varepsilon T - \sum_{j=1}^m\sum_{i=1}^N a_{ij}|u(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1}.
\end{aligned}$$

Since $p > q$ and ε is small enough,

$$\lim_{\|u\|\to+\infty} \big[\phi(u) - \lambda\psi(u)\big] = +\infty, \tag{3.5}$$

which means that the functional J_λ is coercive. Let $r := \frac{(1-s)}{p}\,(\frac{\theta}{k})^p$ and $\overline{w}(x) = \overline{d}$. We obtain

$$\rho(r) \ge \frac{pk^p}{(1-s)} \frac{\int_0^T F(t,\overline{d})\,dt - \int_0^T \max_{|u|<\theta}\big(F(t,u) + \frac{\mu}{\lambda}G(t,u)\big)\,dt}{(\frac{1+s}{1-s})\overline{\lambda}Tk^p\overline{d}^{\,p} - \theta^p}.$$

So, from our assumption it follows that $\rho(r) > 0$. Hence, from Theorem 2.2 for each $\lambda > \lambda_3$, the functional J_λ admits at least one local minimum u_3 such that

$$\phi(u_3) > r,$$

and the conclusion is achieved. □

Now, we present an application of Theorem 2.2 which will be used to obtain two nontrivial weak solutions.

Theorem 3.3. *Suppose F and G satisfy the assumptions (A_i) for $i = 4,5,6$ and there are $M > 0$ and $\sigma > p$ such that*

(A_7) $0 < \sigma F(t,x) \le \langle \nabla F(t,x), x\rangle$ *for all $x \in \mathbb{R}^N$ with $|x| \ge M$ and a.e. $t \in [0,T]$.*

Let $\lambda \in (0,\lambda_4)$, where

$$\lambda_4 := \frac{(1-s)}{pk^p} \frac{\theta^p}{\int_0^T \max_{|u|<\theta}\big(F(t,u) + \frac{\mu}{\lambda}G(t,u)\big)\,dt},$$

whose potential $G(t,x)$ for all $(t,x) \in [0,T]\times(0,+\infty)$ is non-negative. Then for every $\mu \in (0,\mu_4)$, where

$$\mu_4 := \frac{(1-s)\theta^p - \lambda pk^p \int_0^T \max_{|u|<\theta} F(t,u)\,dt}{pk^p\int_0^T \max_{|u|<\theta} G(t,u)\,dt},$$

problem (1.1) admits two distinct critical points.

Proof. We prove this theorem by using the same reasoning as in the proof of Theorem 2.3. First, we show that J_λ satisfies the (PS)-condition. Suppose that $\{u_n\}_{n=1}^\infty$ is a (PS)-sequence of J_λ, that is, there exists $C>0$ such that

$$J_\lambda(u_n)\to C,\quad J'_\lambda(u_n)\to 0 \ \text{ as } \ n\to\infty.$$

Assume that $\|u_n\|\to+\infty$. Then (3.5) contradicts $J_\lambda(u_n)\to C$; hence $\{u_n\}_{n=1}^\infty$ is bounded in $W_T^{1,p}$. We may assume that there exists $u_0\in W_T^{1,p}$ satisfying $u_n\to u_0$ weakly in $W_T^{1,p}$, $u_n\to u_0$ in $L^p[0,T]$, $u_n(t)\to u_0(t)$ for almost every $t\in[0,T]$. Observe that

$$\begin{aligned}J'_\lambda(u_n)(u_n-u_0)=\int\limits_0^T\Big[\big(|u'_n(t)|^{p-2}u'_n(t),u'_n(t)-u'_0(t)\big)+\big(A(t)|u_n(t)|^{p-2}u_n(t),u_n(t)-u_0(t)\big)\Big]\,dt\\ -\lambda\int\limits_0^T\big(\nabla F(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt-\mu\int\limits_0^T\big(\nabla G(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt\\ +\sum_{j=1}^m\sum_{i=1}^N I_{ij}\big((u_n)_i(t_j)\big)\big((u_n)_i(t_j)-(u_0)_i(t_j)\big).\end{aligned}$$

We already know that

$$J'_\lambda(u_n)(u_n-u_0)\to 0 \ \text{ as } \ n\to\infty.$$

By (A_4). given $\varepsilon>0$, we can find a constant $C_\varepsilon>0$ such that

$$|\nabla G(t,x)|\le C_\varepsilon+\varepsilon|x|^{p-1}$$

for every $x\in\mathbb{R}^N$ and almost every $t\in[0,T]$. So,

$$\int\limits_0^T\big(\nabla G(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt\to 0 \ \text{ as } \ n\to\infty.$$

Moreover, by (A_5),

$$\int\limits_0^T\big(\nabla F(t,u_n(t)),u_n(t)-u_0(t)\big)\,dt\to 0 \ \text{ as } \ n\to\infty.$$

Also,

$$\sum_{j=1}^m\sum_{i=1}^N I_{ij}\big((u_n)_i(t_j)\big)\big((u_n)_i(t_j)-(u_0)_i(t_j)\big)\to 0 \ \text{ as } \ n\to\infty.$$

Therefore,

$$\int\limits_0^T\Big[\big(|u'_n(t)|^{p-2}u'_n(t),u'_n(t)-u'_0(t)\big)+\big(A(t)|u_n(t)|^{p-2}u_n(t),u_n(t)-u_0(t)\big)\Big]\,dt\to 0 \ \text{ as } \ n\to\infty.$$

This, together with the weak convergence of $u_n\to u_0$ in $W_T^{1,p}$, implies that

$$u_n\to u_0 \ \text{ in } \ W_T^{1,p} \ \text{ as } \ n\to\infty.$$

Hence J_λ satisfies the (PS)-condition. Finally, we prove that J_λ is unbounded from below. Owning to the assumption (A_7), we can find $\delta>0$ such that for every $M>0$, one has

$$|F(t,x)|>M|x|^\sigma \ \text{ for } \ 0<|x|\le\delta \ \text{ and almost every } \ t\in[0,T].$$

We choose a nonzero nonnegative function $v \in C_0^\infty([0,T])$ and take $\varepsilon > 0$ small enough. Then we obtain

$$\begin{aligned} J(\varepsilon v) &= \frac{1}{p}\|\varepsilon v\|^p - \lambda\int_0^T F(t,\varepsilon v(t))\,dt - \mu\int_0^T G(t,\varepsilon v(t))\,dt + \sum_{j=1}^m\sum_{i=1}^N \int_0^{\varepsilon v_i(t)(t_j)} I_{ij}(t)\,dt \\ &\le \frac{\varepsilon^p}{p}\|v\|^p - \lambda M\varepsilon^\sigma \int_0^T |v(t)|^\sigma\,dt - \sum_{j=1}^m\sum_{i=1}^N a_{ij}|u(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1} \\ &< \frac{\varepsilon^p}{p}\|v\|^p - \lambda M\varepsilon^\sigma \int_0^T |v(t)|^\sigma\,dt - \sum_{j=1}^m\sum_{i=1}^N a_{ij}|\varepsilon v_i(t)(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|\varepsilon v_i(t)(t_j)|^{\gamma_{ij}+1}. \end{aligned}$$

Since $\sigma > p$, this condition guarantees that J_λ is unbounded from below. Now, we have

$$\frac{\sup\limits_{u\in\phi^{-1}(r,+\infty)}\psi(u)}{r} \le \frac{\int_0^T \max\limits_{|u|<\theta}\left(F(t,u)+\frac{\mu}{\lambda}G(t,u)\right)dt}{\frac{(1-s)}{p}\left(\frac{\theta}{k}\right)^p} = \frac{pk^p}{(1-s)}\,\frac{\int_0^T \max\limits_{|u|<\theta}\left(F(t,u)+\frac{\mu}{\lambda}G(t,u)\right)dt}{\theta^p}.$$

Finally, for each $\lambda \in \left(0, \frac{r}{\sup_{u\in\phi^{-1}(r,+\infty)}\psi(u)}\right)$, problem (1.1) admits two distinct critical points. □

4 Applications

In this section, we point out some consequences and applications of the results previously obtained.

Theorem 4.1. *Assume that there exist two positive constants θ and d with*

$$\left(\frac{1+s}{1-s}\right)\overline{\lambda}Tk^pd^p < \theta^p$$

such that assumption (A_1) *in Theorem 3.1 holds. Furthermore, suppose that*

$$(\mathrm{A}_8)\quad \frac{\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt}{\theta^p} < \frac{\int_0^T F(t,d)\,dt}{\left(\frac{1+s}{1-s}\right)\overline{\lambda}Tk^pd^p}.$$

Then for each

$$\lambda \in \frac{(1-s)}{pk^p}\left(\frac{\left(\frac{1+s}{1-s}\right)\overline{\lambda}Tk^pd^p}{\int_0^T F(t,d)\,dt}, \frac{\theta^p}{\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt}\right),$$

the problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u), \quad a.e.\ t\in J, \\ \triangle\left(|u_i'(t_j)|^{p-2}u_i'(t_j)\right) = I_{ij}(u_i(t_j)), \quad i=1,2,\dots,N, \quad j=1,2,\dots,m, \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases}$$

admits at least one nontrivial weak solution.

Proof. The conclusion follows from Theorem 3.2, by taking $\theta_1 = 0$, $\theta_2 = \theta$ and $\mu = 0$. Indeed, owing to assumption (A_8), one has

$$a_\eta(\theta) = \frac{\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt - \int_0^T F(t,d)\,dt}{\theta^p - \left(\frac{1+s}{1-s}\right)\overline{\lambda}Tk^pd^p} < \frac{\left(1 - \frac{\left(\frac{1+s}{1-s}\right)\overline{\lambda}Tk^pd^p}{\theta^p}\right)\int_0^T \max\limits_{|v|<\theta}F(t,v)\,dt}{\theta^p - \left(\frac{1+s}{1-s}\right)\overline{\lambda}Tk^pd^p} = \frac{1}{\theta^p}\int_0^T \max_{|v|<\theta}F(t,v)\,dt.$$

On the other hand,

$$a_\eta(0) = \frac{\int_0^T F(t,d)\,dt}{(\frac{1+s}{1-s})\bar{\lambda} T k^p d^p}\,.$$

Hence, in view of (A_8), Theorem 3.2 ensures the conclusion. □

Now, we suppose that $\nabla F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function. We note the following lemma, which is useful to obtain the results on the existence of nonnegative solutions.

Lemma. *Let $\nabla F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a nonnegative function. Suppose that $u \in X$ is a weak solution of problem (1.1). Then u is nonnegative.*

Proof. Put $u^- = -\min\{u, 0\}$. Then $u^- \in X$. Taking into account that u is a weak solution and choosing $v = u^-$, we obtain

$$\begin{aligned} 0 &\le \lambda \int_0^T \big(\nabla F(t,u(t)), u^-(t)\big)\,dt + \mu \int_0^T \big(\nabla G(t,u(t)), u^-(t)\big)\,dt \\ &= \int_0^T \Big[\big(|u'(t)|^{p-2}u'(t), (u^-)'(t)\big) + \big(A(t)|u(t)|^{p-2}u(t), u^-(t)\big)\Big]\,dt + \sum_{j=1}^m \sum_{i=1}^N I_{ij}(u_i(t_j))u_i^-(t_j) \\ &= -\|u^-\|^p - \sum_{j=1}^m \sum_{i=1}^N I_{ij}(u_i(t_j))u_i^-(t_j). \end{aligned}$$

That is, $u^- = 0$ a.e. in $[0,T]$. Hence our claim is proved. □

Now, we point out a result when the nonlinear term has separable variables. To be precise, let $m : [0,T] \to \mathbb{R}$ be a function such that $m \in L^1([0,T])$, $m(t) \ge 0$ a.e. $t \in [0,T]$, $m \ne 0$, and let $\nabla H : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative and continuous function. Consider the following problem:

$$\begin{cases} -\big(|u'|^{p-2}u'\big)' + A(t)|u|^{p-2}u = \lambda m(t)\nabla H(u(t)), \quad \text{a.e. } t \in J, \\ \triangle\big(|u_i'(t_j)|^{p-2}u_i'(t_j)\big) = I_{ij}(u_i(t_j)), \quad i = 1,2,\dots,N, \quad j = 1,2,\dots,m, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{4.1}$$

Theorem 4.2. *Assume that (A_5) and (A_6) hold and there exist $\sigma > p$ and $M > 0$ such that*

$$0 < \sigma H(s) \le s\nabla H(s) \tag{4.2}$$

for all $s \in \mathbb{R}^N$ with $|s| \ge M$. Then for each $\lambda \in (0, \lambda^*)$, where

$$\lambda^* := \frac{(1-s)}{pk^p\|m\|_{L^1([0,T])}} \max_{\theta > 0} \frac{\theta^p}{H(\theta)}\,,$$

problem (4.1) has at least two nonnegative and non-zero weak solutions.

Corollary. *Let $\nabla F : \mathbb{R}^N \to \mathbb{R}$ be nonnegative and continuous function and assume (4.2) holds. Then for each that $\lambda \in (0, \lambda^{**})$, where*

$$\lambda^{**} := \frac{(1-s')}{pk^p} \max_{\theta > 0} \frac{\theta^p}{H(\theta)} \quad \text{and} \quad s' := k^p \sum_{j=1}^m I_j < 1,$$

the problem

$$\begin{cases} -\big(|u'|^{p-2}u'\big)' + |u|^{p-2}u = \lambda \nabla H(u(t)), \quad \text{a.e. } t \in J, \\ \triangle\big(|u'(t_j)|^{p-2}u'(t_j)\big) = I_j(u(t_j)), \quad j = 1,2,\dots,m, \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases}$$

has two nonnegative and non-zero classical solutions.

Proof. This is a consequence of Theorem 4.2 with $\mu = 0$, $A(t) = I$, where I is the identity matrix of order $p \times p$, and $m(t) = 1$ for all $t \in [0, T]$. □

Example 4.1. Consider $p = 4$ and the function $\nabla H(t) = 5t^4 + 1$ satisfying (4.2). We observe that $\max_{\theta>0} \frac{\theta^4}{H(\theta)} = \frac{\sqrt[4]{27}}{4}$, and for each $\lambda \in (0, 0.066)$,

$$\begin{cases} -\big(|u'|^2u'\big)' + |u|^2u = \lambda \nabla H(u(t)), \ \text{ a.e. } \ t \in (0,1), \\ \triangle\big(|u'(t)|^2u'(t)\big) = I(u(t)), \\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases}$$

admits at least two non-zero and nonnegative solutions.

Example 4.2. Consider $p = 3$ and the function

$$h(t) = \begin{cases} \dfrac{3}{2}\sqrt{t} + 5t^4, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

We observe that it is enough to pick, for instance, $\mu = 4$ and that (4.2) holds. Moreover, $\max_{\theta>0} \frac{\theta^3}{H(\theta)} = \frac{2\sqrt[7]{54}}{7}$, and for each $\lambda \in (0, 0.08)$,

$$\begin{cases} -\big(|u'|u'\big)' + |u|u = \lambda \nabla H(u(t)), \ \text{ a.e. } \ t \in (0,1), \\ \triangle\big(|u'(t)|u'(t)\big) = I(u(t)), \\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases}$$

admits at least two non-zero and nonnegative solutions.

Acknowledgements

The authors would like to thank the referee for a very careful reading of this manuscript and for making good suggestions for improving the article.

References

[1] G. A. Afrouzi, M. Bohner, G. Caristi, S . Heidarkhani and Sh. Moradi, An existence result for impulsive multi-point boundary value systems using a local minimization principle. *J. Optim. Theory Appl.* **177** (2018), no. 1, 1–20.

[2] G. A. Afrouzi, A. Hadjian and V. D. Rădulescu, Variational approach to fourth-order impulsive differential equations with two control parameters. *Results Math.* **65** (2014), no. 3-4, 371–384.

[3] G. A. Afrouzi and S. Heidarkhani, Three solutions for a Dirichlet boundary value problem involving the p-Laplacian. *Nonlinear Anal.* **66** (2007), no. 10, 2281–2288.

[4] D. Averna and G. Bonanno, A three critical points theorem and its applications to the ordinary Dirichlet problem. *Topol. Methods Nonlinear Anal.* **22** (2003), no. 1, 93–103.

[5] G. Bonanno, Relations between the mountain pass theorem and local minima. *Adv. Nonlinear Anal.* **1** (2012), no. 3, 205–220.

[6] G. Bonanno, A critical point theorem via the Ekeland variational principle. *Nonlinear Anal.* **75** (2012), no. 5, 2992–3007.

[7] G. Bonanno, Existence of three solutions for a two point boundary value problem. *Appl. Math. Lett.* **13** (2000), no. 5, 53–57.

[8] G. Bonanno, Multiple critical points theorems without the Palais-Smale condition. *J. Math. Anal. Appl.* **299** (2004), no. 2, 600–614.

[9] G. Bonanno, Some remarks on a three critical points theorem. *Nonlinear Anal.* **54** (2003), no. 4, 651–665.

[10] G. Bonanno and P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian. *Arch. Math. (Basel)* **80** (2003), no. 4, 424–429.

[11] G. Bonanno and R. Livrea, Multiplicity theorems for the Dirichlet problem involving the p-Laplacian. *Nonlinear Anal.* **54** (2003), no. 1, 1–7.

[12] P. Candito, Existence of three solutions for a nonautonomous two point boundary value problem. *J. Math. Anal. Appl.* **252** (2000), no. 2, 532–537.

[13] G. Cordaro and G. Rao, Three periodic solutions for perturbed second order Hamiltonian systems. *J. Math. Anal. Appl.* **359** (2009), no. 2, 780–785.

[14] G. D'Aguì and G. Molica Bisci, Three non-zero solutions for elliptic Neumann problems. *Anal. Appl. (Singap.)* **9** (2011), no. 4, 383–394.

[15] A. d'Onofrio, On pulse vaccination strategy in the SIR epidemic model with vertical transmission. *Appl. Math. Lett.* **18** (2005), no. 7, 729–732.

[16] F. Faraci, Multiple periodic solutions for second order systems with changing sign potential. *J. Math. Anal. Appl.* **319** (2006), no. 2, 567–578.

[17] J. R. Graef, Sh. Heidarkhani and L. Kong, Infinitely many periodic solutions to a class of perturbed second-order impulsive Hamiltonian systems. *Differ. Equ. Appl.* **9** (2017), no. 2, 195–212.

[18] H. Gu and T. An, Existence of infinitely many periodic solutions for second-order Hamiltonian systems. *Electron. J. Differential Equations* **2013**, no. 251, 10 pp.

[19] X. He and W. Ge, Existence of three solutions for a quasilinear two-point boundary value problem. *Comput. Math. Appl.* **45** (2003), no. 4-5, 765–769.

[20] M. Izydorek and J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems. *J. Differential Equations* **219** (2005), no. 2, 375–389.

[21] P. Jebelean and G. Moroşanu, Ordinary p-Laplacian systems with nonlinear boundary conditions. *J. Math. Anal. Appl.* **313** (2006), no. 2, 738–753.

[22] V. Lakshmikantham, D. D. Baĭnov and P. S. Simeonov, *Theory of Impulsive Differential Equations.* Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[23] F.-fang Liao and J. Sun, Variational approach to impulsive problems: a survey of recent results. *Abstr. Appl. Anal.* **2014**, Art. ID 382970, 11 pp.

[24] C. Li and C.-L. Tang, Three solutions for a class of quasilinear elliptic systems involving the (p, q)-Laplacian. *Nonlinear Anal.* **69** (2008), no. 10, 3322–3329.

[25] C. Li, Z.-Q. Ou and C.-L. Tang, Three periodic solutions for p-Hamiltonian systems. *Nonlinear Anal.* **74** (2011), no. 5, 1596–1606.

[26] S. Ma and Y. Zhang, Existence of infinitely many periodic solutions for ordinary p-Laplacian systems. *J. Math. Anal. Appl.* **351** (2009), no. 1, 469–479.

[27] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems.* Applied Mathematical Sciences, 74. Springer-Verlag, New York, 1989.

[28] Q. Meng, Three periodic solutions for a class of ordinary p-Hamiltonian systems. *Bound. Value Probl.* **2014**, 2014:150, 6 pp.

[29] G. Molica Bisci, Fractional equations with bounded primitive. *Appl. Math. Lett.* **27** (2014), 53–58.

[30] G. Molica Bisci and D. Repovš, Multiple solutions for elliptic equations involving a general operator in divergence form. *Ann. Acad. Sci. Fenn. Math.* **39** (2014), no. 1, 259–273.

[31] S. I. Nenov, Impulsive controllability and optimization problems in population dynamics. *Nonlinear Anal.* **36** (1999), no. 7, Ser. A: Theory Methods, 881–890.

[32] P. H. Rabinowitz, Variational methods for Hamiltonian systems. *Handbook of dynamical systems, Vol. 1A*, 1091–1127, *North-Holland, Amsterdam*, 2002.

[33] B. Ricceri, A three critical points theorem revisited. *Nonlinear Anal.* **70** (2009), no. 9, 3084–3089.

[34] Z.-L. Tao and C.-L. Tang, Periodic and subharmonic solutions of second-order Hamiltonian systems. *J. Math. Anal. Appl.* **293** (2004), no. 2, 435–445.

[35] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*. Translated from the Russian by Y. Chapovsky. World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 14. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[36] X. Shang and J. Zhang, Three solutions for a perturbed Dirichlet boundary value problem involving the p-Laplacian. *Nonlinear Anal.* **72** (2010), no. 3-4, 1417–1422.

[37] T. Shen and W. Liu, Infinitely many rotating periodic solutions for suplinear second-order impulsive Hamiltonian systems. *Appl. Math. Lett.* **88** (2019), 164–170.

[38] J. Sun, H. Chen and J. J. Nieto, Infinitely many solutions for second-order Hamiltonian system with impulsive effects. *Math. Comput. Modelling* **54** (2011), no. 1-2, 544–555.

[39] Z. Wang and J. Zhang, Periodic solutions of a class of second order non-autonomous Hamiltonian systems. *Nonlinear Anal.* **72** (2010), no. 12, 4480–4487.

[40] B. Xu and C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems. *J. Math. Anal. Appl.* **333** (2007), no. 2, 1228–1236.

[41] Q. Zhang and C. Liu, Infinitely many periodic solutions for second order Hamiltonian systems. *J. Differential Equations* **251** (2011), no. 4-5, 816–833.

[42] A. Zhang and X. H. Tang, New existence of periodic solutions for second order non-autonomous Hamiltonian systems. *J. Math. Anal. Appl.* **369** (2010), no. 1, 357–367.

[43] X. Zhang and Y. Zhou, Periodic solutions of non-autonomous second order Hamiltonian systems. *J. Math. Anal. Appl.* **345** (2008), no. 2, 929–933.

[44] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, III. *Variational Methods and Optimization*. Translated from the German by Leo F. Boron. Springer-Verlag, New York, 1985.

[45] J. Zhou and Y. Li, Existence of solutions for a class of second-order Hamiltonian systems with impulsive effects. *Nonlinear Anal.* **72** (2010), no. 3-4, 1594–1603.

[46] W. Zou and S. Li, Infinitely many solutions for Hamiltonian systems. *J. Differential Equations* **186** (2002), no. 1, 141–164.

(Received 30.01.2024; revised 22.03.2024; accepted 03.04.2024)

Authors' addresses:

Arezou Soleimani Nia
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail: `a.soleimani03@umail.umz.ac.ir`

Ghasem A. Afrouzi
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail: `afrouzi@umz.ac.ir`

Hadi Haghshenas
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail: `haghshenas60@gmail.com`