Memoirs on Differential Equations and Mathematical Physics

Volume 92, 2024, 101–116

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EXISTENCE OF SOLUTIONS FOR SOME ELLIPTIC SYSTEMS WITH PERTURBED GRADIENT

Abstract. In this paper, we study the existence of weak solutions for some quasilinear elliptic problems with perturbed gradients under homogeneous Dirichlet boundary conditions. Using the approximate Galerkin method and combining the convergence in terms of Young measure and the theory of Sobolev spaces, we can prove that there is at least one weak solution $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ to the problem treated.

2020 Mathematics Subject Classification. 35J60, 35D05, 76A05.

Key words and phrases. Quasilinear elliptic systems, approximate Galerkin method, Young measures.

რეზიუმე. ნაშრომში შესწავლილია სუსტი ამონახსნების არსებობა ზოგიერთი კვაზიწრფივი ელიფსური ამოცანისთვის შეშფოთებული გრადიენტებით ერთგვაროვან დირიხლეს სასაზღვრო პირობებში. გალიორკინის მიახლოებითი მეთოდის, იუნგის ზომით კრებადობისა და სობოლევის სივრცეთა თეორიის გამოყენებით დამტკიცებულია, რომ განსახილველი ამოცანისთვის არსებობს ერთი მაინც სუსტი ამონახსნი $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

1 Introduction

The main objective of this paper is to prove the existence of weak solutions for a class of quasilinear elliptic problems of the following form:

$$\begin{cases} -\operatorname{div}(a(x, Du - \Theta(u)) + \phi(u)) = v(x) + \operatorname{div} f(x, u) + h(x, u, Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where v belongs to $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, $h : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$, $f : \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ and $a : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ are the functions assumed to satisfy certain assumptions (see below).

In the last years, great attention has been devoted to the study of nonlinear problems, especially, quasilinear elliptic problems. In fact, from the physical pint of view, problem (1.1) simulates a number of natural phenomena that occur in the fields of oceanography, turbulent fluid flows, induction heating, and electrochemical issues. As an illustration, we provide the following parabolic model: fluid flow through porous media. This model is governed by the equation

$$\frac{\partial\theta}{\partial t} - \operatorname{div}\Big(\big|\nabla\varphi(\theta) - K(\theta)e\big|^{p-2}\big(\nabla\varphi(\theta) - K(\theta)e\big)\Big) = 0,$$

where θ is the volumetric content of moisture, $K(\theta)$ is the hydraulic conductivity, $\varphi(\theta)$ is the hydrostatic potential and e is the unit vector in the vertical direction. Then problem (1.1) is a generalization of the following nonlinear problem:

$$\begin{cases} -\operatorname{div}\Phi(Du - \Theta(u)) = h(x, u, Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Phi(A) = |A|^{p-2}A \quad \forall\, A \in \mathbb{M}^{m\times n}.$$

This problem has been studied in [6] by E. Azroul and F. Balaadich, who proved the existence of weak solutions by using Young's measures without any Leray–Lions type growth conditions. In [17], Hungerbühler considered the following quasilinear elliptic system

$$-\operatorname{div}\sigma(x, u, Du) = f \ \text{ in } \Omega \tag{1.2}$$

under certain natural conditions on the function σ and got some existence result by using the tool of Young's measures and weak monotonicity over σ. In [13], we showed that there is a weak solution for quasilinear elliptic system under regularity, growth and coercivity conditions by using Galerkin's approximation and the theory of Young measures. Many papers were written to investigate the existence of solutions to elliptic problems of type (1.2) by using classical monotone operator methods (see [8, 13, 14, 19, 20, 22] and the references therein). The goal of the present paper is to establish the existence of solutions to problem (1.1) and extend the result of [6] by considering a general source term. The Galerkin method is the main tool for developing approximation solutions and the theory of Young's measures is used to identify weak limits when approaching the limit.

2 Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$ with $\partial\Omega$ Lipschitz-continuous. For any Lebesgue-measurable function $p : \Omega \to [1, \infty)$, we define

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

and introduce the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \mid \ \rho_{p(\cdot)}(u) := \int\limits_\Omega |u(x)|^{p(x)}\, dx < \infty\Big\}.$$

Equipped with the Luxembourg norm

$$\|u\|_{p(\,\cdot\,)} := \inf\Big\{\lambda > 0 : \ \ \rho_{p(\,\cdot\,)}\Big(\frac{u}{\lambda}\Big) \le 1\Big\},$$

$L^{p(\,\cdot\,)}(\Omega)$ becomes a Banach space. If

$$1 < p_- \le p_+ < \infty, \tag{2.1}$$

$L^{p(\,\cdot\,)}(\Omega)$ is separable and reflexive. The dual space of $L^{p(\,\cdot\,)}(\Omega)$ is $L^{p'(\,\cdot\,)}(\Omega)$, where $p'(x)$ is the generalised Hölder conjugate of $p(x)$,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(\,\cdot\,)}(\Omega)$.

Proposition 2.1 (see [16]). *If (2.1) holds for $u \in L^{p(x)}(\Omega)$, then the following assertions hold:*

$$\min\big\{\|u\|_{p(\,\cdot\,)}^{p_-}, \|u\|_{p(\,\cdot\,)}^{p_+}\big\} \le \rho_{p(\,\cdot\,)}(u) \le \max\big\{\|u\|_{p(\,\cdot\,)}^{p_-}, \|u\|_{p(\,\cdot\,)}^{p_+}\big\},$$
$$\min\big\{\rho_{p(\,\cdot\,)}(u)^{\frac{1}{p_-}}, \rho_{p(\,\cdot\,)}(u)^{\frac{1}{p_+}}\big\} \le \|u\|_{p(\,\cdot\,)} \le \max\big\{\rho_{p(\,\cdot\,)}(u)^{\frac{1}{p_-}}, \rho_{p(\,\cdot\,)}(u)^{\frac{1}{p_+}}\big\},$$
$$\|u\|_{p(\,\cdot\,)}^{p_-} - 1 \le \rho_{p(\,\cdot\,)}(u) \le \|u\|_{p(\,\cdot\,)}^{p_+} + 1.$$

Proposition 2.2 (Generalised Hölder's inequality) (see [18]).

– *For any functions $u \in L^{p(\,\cdot\,)}(\Omega)$ and $v \in L^{p'(\,\cdot\,)}(\Omega)$, we have*

$$\int\limits_{\Omega} uv\,dx \le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(\,\cdot\,)}\|v\|_{p'(\,\cdot\,)} \le 2\|u\|_{p(\,\cdot\,)}\|v\|_{p'(\,\cdot\,)}.$$

– *For all p satisfying (2.1), we have the following continuous embedding:*

$$L^{p(\,\cdot\,)}(\Omega) \hookrightarrow L^{r(\,\cdot\,)}(\Omega) \ \ \textit{whenever} \ \ p(x) \ge r(x) \ \ \textit{for a.e.} \ \ x \in \Omega.$$

In generalised Lebesgue spaces, a version of Young's inequality

$$|uv| \le \delta\frac{|u|^{p(x)}}{p(x)} + C(\delta)\,\frac{|v|^{p'(x)}}{p(x)}$$

holds for some positive constant $C(\delta)$ and any $\delta > 0$.

We define also the generalized Sobolev space by

$$W^{1,p(\,\cdot\,)}(\Omega) := \big\{u \in L^{p(\,\cdot\,)}(\Omega) : \ \ \nabla u \in L^{p(\,\cdot\,)}(\Omega)\big\},$$

which is a Banach space with the norm

$$\|u\|_{1,p(\,\cdot\,)} := \|u\|_{p(\,\cdot\,)} + \|\nabla u\|_{p(\,\cdot\,)}.$$

The space $W^{1,p(\,\cdot\,)}(\Omega)$ is separable and is reflexive when (2.1) is satisfied. We also have

$$W^{1,p(\,\cdot\,)}(\Omega) \hookrightarrow W^{1,r(\,\cdot\,)}(\Omega) \ \ \text{whenever} \ \ p(x) \ge r(x) \ \ \text{for a.e.} \ \ x \in \Omega.$$

Now, we introduce the function space

$$W_0^{1,p(\,\cdot\,)}(\Omega) := \big\{u \in \mathrm{W}_0^{1,1}(\Omega) : \ \ \nabla u \in L^{p(\,\cdot\,)}(\Omega)\big\},$$

endowed with the following norm:

$$\|u\|_{W_0^{1,p(\,\cdot\,)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{p(\,\cdot\,)}.$$

If $p \in C(\overline{\Omega})$, then the norm in $W_0^{1,p(\,\cdot\,)}(\Omega)$ is equivalent to $\|\nabla u\|_{p(\,\cdot\,)}$. When p is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\,\cdot\,)}(\Omega)$. Recall that a function $p(\,\cdot\,)$ is log-Hölder continuous in Ω if

$$\exists\, C>0: \quad |p(x)-p(y)| \le \frac{C}{\ln(\frac{1}{|x-y|})} \quad \forall\, x,y \in \Omega, \quad |x-y| < \frac{1}{2}\,. \tag{2.2}$$

If p is a measurable function in Ω satisfying $1 \le p_- \le p_+ < N$ and the Log-Hölder continuity property (2.2), then

$$\|u\|_{p^*(\,\cdot\,)} \le C\|\nabla u\|_{p(\,\cdot\,)} \quad \forall\, u \in W_0^{1,p(\,\cdot\,)}(\Omega),$$

for some positive constant C, where

$$p^*(x) := \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } \ p(x) < N, \\ \infty & \text{if } \ p(x) \ge N. \end{cases}$$

On the other hand, if p satisfies (2.2) and $p_- > N$, then

$$\|u\|_\infty \le C\|\nabla u\|_{p(\,\cdot\,)} \quad \forall\, u \in W_0^{1,p(\,\cdot\,)}(\Omega),$$

where C is another positive constant.

Weak convergence is a basic tool of modern nonlinear analysis because it has the same compactness properties as the convergence in finite-dimensional spaces. But this convergence sometimes does not behave as one desire with respect to nonlinear functionals and operators. To overcome this difficulty, one can use the technics of Young's measures.

In the ensuing, we denote by δ_c the Dirac measure on $\mathbb{R}^n$ $(n \in \mathbb{N})$ and $C_0(\mathbb{R}^m)$ denotes the closure of the space of continuous functions satisfying $\lim\limits_{|\lambda|\to\infty} \varphi(\lambda) = 0$. Its dual space can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with a finite mass. The related duality pairing is given by

$$\langle \nu, \varphi\rangle = \int\limits_{\mathbb{R}^m} \varphi(\lambda)\, d\nu(\lambda).$$

As in the introduction, the Young measure is the method we employ to show the intended result. We recall some fundamental conceptions and properties for the reader who would be unfamiliar with this notion (see [7] and [15]).

Lemma 2.1 ([15])**.** *Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence (denoted again by (z_k)) and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$ such that for each $\varphi \in C_0(\mathbb{R}^m)$, we have*

$$\varphi(z_k) \to^* \overline{\varphi} \ \text{ weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi\rangle$ for a.e. $x \in \Omega$.

Definition 2.1. We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated to (z_k). In [7], it is shown that if for all $R > 0$,

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \big|\big\{x \in \Omega \cap B_R(0): \ \ |z_k(x)| \ge L\big\}\big| = 0,$$

then the Young measure ν_x generated by z_k is a probability measure, i.e., $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$.

The following properties build the basic tools used in the sequel.

Lemma 2.2 ([1])**.** *If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence z_k then there holds:*

$$z_k \to z \ \text{ in measure} \iff \nu_x = \delta_{z(x)} \ \text{ for a.e. } \ x \in \Omega.$$

If we choose $z_k = Dw_k$ for $w_k : \Omega \to \mathbb{R}^m$, the above results remain valid.

Lemma 2.3 ([6]). *Assume that Dw_k is bounded in $L^p(\Omega;\mathbb{M}^{m\times n})$, then the Young measure ν_x generated by Dw_k satisfies:*

(1) *ν_x is a probability measure.*

(2) *The weak L^1-limit of Dw_k is given by $\langle \nu_x, id\rangle$.*

(3) *The identification $\langle \nu_x, id\rangle = Dw(x)$ holds for a.e. $x\in\Omega$.*

We conclude this section by recalling the following Fatou-type inequality.

Lemma 2.4 ([11]). *Let $\varphi:\mathbb{M}^{m\times n}\to\mathbb{R}$ be a continuous function and $w_k:\Omega\to\mathbb{R}^m$ be a sequence of measurable functions such that Dw_k generates the Young measure ν_x with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})}=1$. Then*

$$\liminf_{k\to\ \infty}\int_\Omega \varphi(Dw_k)\,dx \geq \int_\Omega\int_{\mathbb{M}^{m\times n}} \varphi(\lambda)\,d\nu_x(\lambda)\,dx$$

provided that the negative part of $\varphi(Dw_k)$ is equiintegrable.

In the sequel, we will need the following two technical lemmas.

Lemma 2.5 ([4]). *For $\xi,\eta\in\mathbb{R}^N$ and $1<p<\infty$, we have*

$$\frac{1}{p}\,|\xi|^p - \frac{1}{p}\,|\eta|^p \leq |\xi|^{p-2}\xi(\xi-\eta).$$

Lemma 2.6. *For $a\geq 0$, $b\geq 0$ and $1\leq p<\infty$, we have*

$$(a+b)^{p(x)} \leq 2^{p^+-1}(a^{p(x)}+b^{p(x)}).$$

3 Assumptions and main results

Let Ω be a bounded open domain of $\mathbb{R}^{n\geq 2}$ and $\mathbb{M}^{m\times n}$ be the set of $m\times n$ matrices with reduced $\mathbb{R}^{mn}$ topology, i.e., if $\delta\in\mathbb{M}^{m\times n}$, then $|\delta|$ is the norm of δ when regarded as a vector of $\mathbb{R}^{mn}$. We provide $\mathbb{M}^{m\times n}$ with the product

$$\delta:\eta = \sum_{i,j}\delta_{ij}\eta_{ij}.$$

Throughout this paper, we suppose that the following hypotheses are required to state the main result:

(A_0) $\phi:\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is linear and continuous and there exists a constant β_0 such that

$$|\phi(u)|\leq\beta_0.$$

(A_1) $\Theta:\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is continuous such that

$$\Theta(0)=0 \ \text{ and } \ |\Theta(x)-\Theta(y)|\leq C_\Theta|x-y| \quad \forall\, x,y\in\mathbb{R}^m,$$

where C_Θ is a positive constant related to the exponent p and the diameter of Ω by

$$C_\Theta < \frac{1}{\operatorname{diam}(\Omega)}\left(\frac{1}{2}\right)^{\frac{1}{p^-}}.$$

(A_2) $a:\Omega\times\mathbb{M}^{m\times n}\to\mathbb{M}^{m\times n}$ is a Carathéodory function, that is, $\eta\to a(x,\eta)$ is continuous for a.e. $x\in\Omega$, and $x\to a(x,\eta)$ is measurable for all $\eta\in\mathbb{M}^{m\times n}$.

(A_3) a is strictly monotone, that is,

$$\big(a(x,\eta-\Theta(s))-a(x,\xi-\Theta(s))\big)(\eta-\xi)>0 \ \text{ for all } \ \eta,\xi\in\mathbb{M}^{m\times n}, \ \ \eta\neq\xi.$$

(A_4) As well as the growth and the coercivity assumptions

$$|a(x, \eta - \Theta(s))| \leq b_0(x) + |\eta - \Theta(s)|^{p(x)-1},$$
$$a(x, \eta - \Theta(s)) : \eta \geq \alpha|\eta - \Theta(s)|^{p(x)} - b_1(x),$$

where $b_0 \in L^{p'}(\Omega)$, $b_1 \in L^1(\Omega)$ and α is a positive constant.

Moreover, we assume that h and g satisfy the following assumptions:

(H_0) $h : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ is a Carathéodory function (i.e., $x \mapsto h(x, s, \xi)$ is measurable for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(s, \xi) \mapsto h(x, s, \xi)$ is continuous for almost every $x \in \Omega$).

(H_1) h satisfies one of the following conditions:

(a) There exist $0 < \gamma(x) < p(x) - 1$, $0 \leq \mu(x) < p(x) - 1$, $d_0 \in L^{p'(x)}(\Omega)$ such that

$$|h(x, s, \xi)| \leq d_0(x) + |s|^{\gamma(x)} + |\xi|^{\mu(x)}$$

holds for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$.

(b) The function h is independent of the third variable, or, for almost $x \in \Omega$ and all $s \in \mathbb{R}^m$, the mapping $\xi \mapsto h(x, s, \xi)$ is linear.

(F_0) $f : \Omega \times \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

(F_1) There exist $d_1 \in L^{p'(x)}$ and $0 < q(x) < p(x) - 1$ such that

$$|f(x, u)| \leq d_1(x) + |u|^{q(x)}.$$

Remark. Hypothesis (A_3) can be replaced by one of the following hypotheses:

$(A_3)'$ For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $\xi \mapsto a(x, \xi - \Theta(u))$ is a C^1-function and is monotone, that is,

$$(a(x, \xi - \Theta(u)) - a(x, \eta - \Theta(u))) : (\xi - \eta) \geq 0 \quad \forall\, \xi, \eta \in \mathbb{M}^{m\times n}.$$

$(A_3)''$ There exists a function $L : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$xi - \Theta(u)) = \frac{\partial L}{\partial \xi}(x, \xi - \Theta(u)) := D_\xi L(x, \xi - \Theta(u)),$$

and $\xi \mapsto L(x, \xi - \Theta(u))$ is convex and C^1-function for all $x \in \Omega$ and $u \in \mathbb{R}^m$.

$(A_3)'''$ The operator a is strictly quasimonotone, that is, there exists $c_0 > 0$ such that

$$\int_\Omega \big(a(x, Du - \Theta(u)) - a(x, Dv - \Theta(u))\big) : (Du - Dv)\, dx \geq c_0 \int_\Omega |Du - Dv|^{p(x)}\, dx.$$

Now, we give a definition of weak solutions for the elliptic problem (1.1) and state the main result.

Definition 3.1. A function $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is said to be a weak solution of (1.1) if

$$\int_\Omega \big(a(x, Du - \Theta(u)) : D\varphi + \phi(u) : D\varphi\big)\, dx = \langle v, \varphi\rangle - \int_\Omega f(x, u) : D\varphi\, dx + \int_\Omega h(x, u, Du) \cdot \varphi\, dx$$

holds for all $\varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$.

Theorem 3.1. *Assume that (A_0)–(A_4) and (H_0), (H_1), (F_0) and (F_1) hold. Then the Dirichlet problem (1.1) has a weak solution in the sense of Definition 3.1.*

4 Galerkin approximation

The object of this section is to create approximating solutions by using the well-known Galerkin approach. In order to construct the necessary estimates to support the desired results, the Hölder inequality and the consequent Poincaré inequality (see [17], Lemma 2.3) are important. There exists a positive constant α such that

$$\|v\|_{p(x)} \leq \frac{\alpha}{2}\,\|Dv\|_{p(x)} \quad \forall\, v \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m). \tag{4.1}$$

Now, consider the mapping

$$T : W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \to W^{-1,p(x)'}(\Omega;\mathbb{R}^m)$$

given for arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ and all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ by

$$\begin{aligned}\langle T(u),\varphi\rangle = &\int_\Omega \big(a(x,Du-\Theta(u)) : D\varphi + \phi(u) : D\varphi\big)\,dx - \langle v,\varphi\rangle \\ &+ \int_\Omega f(x,u) : D\varphi\,dx - \int_\Omega h(x,u,Du)\cdot\varphi\,dx.\end{aligned}$$

Lemma 4.1. *$T(u)$ is well defined, linear and bounded.*

Proof. For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear and (without loss of generality, we may assume that $\gamma(x) = \mu(x) = p(x)-1$) we can obtain,

$$\begin{aligned}|I_1| &\leq \int_\Omega |a(x,Du-\Theta(u))|\,|D\varphi|\,dx + \int_\Omega |\phi(u)|\,|D\varphi|\,dx \\ &\leq \int_\Omega b_0(x)|D\varphi|\,dx + \int_\Omega |Du-\Theta(u)|^{p(x)-1}|D\varphi|\,dx + \beta_0\|D\varphi\|_1 \\ &\leq \|b_0\|_{p'(x)}\|D\varphi\|_{p(x)} + \left(\int_\Omega |Du-\Theta(u)|^{p(x)}\,dx\right)^{\frac{1}{p'(x)}}\|D\varphi\|_{p(x)} + \beta_0 C_0\|D\varphi\|_{p(x)} \\ &\leq \|b_0\|_{p'(x)}\|D\varphi\|_{p(x)} + 2^{\frac{(p^+-1)^2}{p^-}}\left(\|Du\|_{p(x)}^{p(x)} + \|\Theta(u)\|_{p(x)}^{p(x)}\right)^{\frac{p(x)-1}{p(x)}}\|D\varphi\|_{p(x)} + \beta_0 C_0\|D\varphi\|_{p(x)} \\ &= \left(\|b_0\|_{p'(x)} + 2^{\frac{(p^+-1)^2}{p^-}}\left(\|Du\|_{p(x)}^{p(x)} + \|\Theta(u)\|_{p(x)}^{p(x)}\right)^{\frac{p(x)-1}{p(x)}}\right)\|D\varphi\|_{p(x)} + \beta_0 C_0\|D\varphi\|_{p(x)}.\end{aligned}$$

On the other hand, we have

$$|I_2| \leq \int_\Omega |h(x,u,Du)|\,|\varphi|\,dx \leq \left(\|d_0\|_{p'(x)} + \|u\|_{p(x)}^{p(x)-1} + \|Du\|_{p(x)}^{p(x)-1}\right)\|\varphi\|_{p(x)}.$$

By using the Hölder inequality, we have

$$|I_3| \leq \langle v,\varphi\rangle \leq \|v\|_{-1,p'(x)}\|\varphi\|_{p(x)}.$$

From the growth condition (F_1), we get

$$|I_4| := \int_\Omega |f(x,u)|\,|D\varphi|\,dx \leq \|d_1\|_{p'(x)}\|D\varphi\|_{p(x)} + \|u\|_{p(x)}^{p(x)-1}\|D\varphi\|_{p(x)}.$$

Since these expressions are finite by our assumptions, $T(u)$ is well defined. Finally, we have

$$|\langle T(u),\varphi\rangle| \leq |I_1| + |I_2| + |I_3| + |I_4| \leq C_1\|D\varphi\|_{p(x)} + C_2\|\varphi\|_{p(x)} \leq C_3\|D\varphi\|_{p(x)}.$$

Thus T is well defined and bounded. $\square$

Lemma 4.2. *The restriction of T to a finite linear subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is continuous.*

Proof. Let X be a finite subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ with dim $X=r$ and let $(x_i)_{i=1,\cdots,r}$ be a basis of X. We consider in X the sequence $(u_k=a_k^i x_i)$ which converges to $u=a^i x_i$ in X. Hence $u_k\to u$ and $Du_k\to Du$ almost everywhere for a subsequence still denoted by $(u_k)_k$. From the continuity of a, ϕ, h and f, one can obtain

$$a(x,Du_k-\Theta(u_k)):D\varphi\to a(x,Du-\Theta(u)):D\varphi,\quad \phi(u_k):D\varphi\to\phi(u):D\varphi,$$
$$h(x,u_k,Du_k)\varphi\to h(x,u,Du)\varphi \text{ and } f(x,u_k):D\varphi\to f(x,u):D\varphi$$

almost everywhere in Ω. Using the strong convergence of u_k to u in X and Lemma 2.6, we can infer that $\|u_k\|_{p(x)}$ and $\|Du_k\|_{p(x)}$ are bounded. Now, in order to apply the Vitali Theorem, we need to show the equi-integrability of the sequences $(a(x,Du_k-\Theta(u_k)):D\varphi)$, $(\phi(u_k):D\varphi)$, $(f(x,u_k):D\varphi)$ and $(h(x,u_k,Du_k)\cdot\varphi)$. To do this, let $E\subset\Omega$ be a measurable subset, then by the growth condition in (A_2), we have

$$\begin{aligned}
&\int_E \left|a(x,Du_k-\Theta(u_k)):\ D\varphi\right|dx\\
&\qquad\le\Big(\int_E |b_0(x)|^{p'(x)}+|Du_k-\Theta(u_k)|^{p(x)}\,dx\Big)^{\frac{1}{p'(x)}}\Big(\int_E|D\varphi|^{p(x)}\,dx\Big)^{\frac{1}{p(x)}}\\
&\qquad\le\Big(\|b_0(x)\|_{p'(x)}^{p'(x)}+2^{p^+-1}\Big(\underbrace{\|Du_k\|_{p(x)}^{p(x)}}_{\le C}+c^{p^+}\underbrace{\|u_k\|_{p(x)}^{p(x)}}_{\le C}\Big)\Big)^{\frac{1}{p'(x)}}\Big(\int_E|D\varphi|^{p(x)}\,dx\Big)^{\frac{1}{p(x)}}
\end{aligned}$$

and

$$\int_E\left|h(x,u_k,Du_k)\cdot\varphi\right|dx\le C\Big(\|d_0\|_{p'(x)}+\underbrace{\|u_k\|_{p(x)}^{p(x)-1}}_{\le C}+\underbrace{\|Du_k\|_p^{p(x)-1}}_{\le C}\Big)\Big(\int_E|D\varphi|^{p(x)}\,dx\Big)^{\frac{1}{p(x)}}.$$

From the growth condition (F_1), we deduce that

$$\int_E|g(x,u_k):\ D\varphi|\,dx\le\Big(\|d_1\|_{p'(x)}+\underbrace{\|u_k\|_{p(x)}^{p(x)-1}}_{\le C}\Big)\Big(\int_E|D\varphi|^{p(x)}\,dx\Big)^{\frac{1}{p(x)}}.$$

Since $\int_E|D\varphi|^{p(x)}\,dx$ is arbitrarily small if the measure of E is chosen small enough, we get the equi-integrability of $(a(x,Du_k-\Theta(u_k)):D\varphi)$, $(g(x,u_k):D\varphi)$ and $(h(x,u_k,Du_k)\cdot\varphi)$. The equi-integrability of $(\phi(u_k):D\varphi)$ follows from the assumption (A_0). From Vitali's Theorem, we conclude the continuity of mapping T. □

Lemma 4.3. *The operator T defined above is coercive.*

Proof. Taking $\varphi=u$ as a test function in the definition of T, we obtain

$$\begin{aligned}
\langle T(u),u\rangle=&\int_\Omega\big(a(x,Du-\Theta(u)):Du+\phi(u):Du\big)\,dx-\langle v,u\rangle\\
&+\int_\Omega f(x,u):Du\,dx-\int_\Omega h(x,u,Du)u\,dx.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{1}{2^{p^+-1}}|Du|^{p(x)}&=\frac{1}{2^{p^+-1}}|Du-\Theta(u)+\Theta(u)|^{p(x)}\\
&\le\frac{1}{2^{p^+-1}}\Big[2^{p^+-1}\big(|Du-\Theta(u)|^{p(x)}+|\Theta(u)|^{p(x)}\big)\Big]\\
&\le|Du-\Theta(u)|^{p(x)}+|\Theta(u)|^{p(x)}.
\end{aligned}$$

Then

$$
\begin{aligned}
I &\equiv \int_\Omega \big(a(x,,Du-\Theta(u)) : Du + \phi(u) : Du\big)\,dx \\
&\geq \alpha \int_\Omega |Du-\Theta(u)|^{p(x)}\,dx - \int_\Omega b_0(x)\,dx - \beta_0 \int_\Omega |Du|\,dx \\
&\geq \frac{\alpha}{2^{p^+-1}} \int_\Omega |Du|^{p(x)}\,dx - \alpha \int_\Omega |\Theta(u)|^{p(x)}\,dx - c - \beta_0 \int_\Omega |Du|\,dx.
\end{aligned}
$$

Next, the generalized Hölder inequality implies that

$$|II| \equiv |\langle v,u\rangle| \leqslant \|v\|_{-1,p'(x)}\|u\|_{1,p(x)} \leqslant C\|v\|_{-1,p'(x)}\|Du\|_{p(x)}.$$

Using the Hölder inequality, (4.1) and assumption (H_1), we deduce that

$$
\begin{aligned}
III &\equiv \int_\Omega h(x,u,Du)u\,dx \leq \int_\Omega d_0(x)|u|\,dx + \int_\Omega |u|^{\gamma(x)}|u|\,dx + \int_\Omega |Du|^{\mu(x)}|u|\,dx \\
&\leq \|d_0\|_{p'(x)}\|u\|_{p(x)} + \|u\|^{\gamma(x)}_{\gamma(x)p'(x)}\|u\|_{p(x)} + \|Du\|^{\mu(x)}_{\mu(x)p'(x)}\|u\|_{p(x)} \\
&\leq \frac{\alpha}{2}\|d_0\|_{p'(x)}\|Du\|_{p(x)} + \left(\frac{\alpha}{2}\right)^{\gamma^++1}\|Du\|^{\gamma(x)+1}_{p(x)} + \frac{\alpha}{2}\|Du\|^{\mu(x)+1}_{p(x)}.
\end{aligned}
$$

From (4.1) and assumption (F_1), we get

$$
\begin{aligned}
|IV| &\equiv \left|\int_\Omega f(x,u) : Du\,dx\right| \leq \int_\Omega d_1(x)|Du|\,dx + \int_\Omega |u|^{q(x)}|Du|\,dx \\
&\leq \|d_1\|_{p'(x)}\|Du\|_{p(x)} + C_0^{q^+}\|Du\|^{q(x)+1}_{p(x)}.
\end{aligned}
$$

From the above inequalities, (4.1) and the choice of the constant C_Θ in the assumption on Θ, we obtain

$$\langle T(u),u\rangle = I - II - III + IV \to \infty \ \text{ as } \ \|u\|_{1,p(x)} \to \infty,$$

since

$$p^+ > \max\big\{1, q^+ + 1, \gamma^+ + 1, \mu^+ + 1\big\}. \qquad \square$$

Let us fix some k and assume that X_k has the dimension r and $e_1,\dots,e_r$ is a basis of X_k. We define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^r \end{pmatrix} \longmapsto \begin{pmatrix} \langle T(\beta^i e_i), e_1\rangle \\ \langle T(\beta^i e_i), e_2\rangle \\ \vdots \\ \langle T(\beta^i e_i), e_r\rangle \end{pmatrix}.$$

Lemma 4.4. *G is continuous and $G(\beta)\cdot\beta \to \infty$ as $\|\beta\|_{\mathbb{R}^r} \to \infty$, where $\beta = (\beta^1,\dots,\beta^r)^t$ and the dot is the inner product of two vectors of $\mathbb{R}^r$.*

Proof. Let $u_j = \beta_i^j e_i \in X_k$, $u_0 = \beta_i^0 e_i \in X_k$. Then $\|\beta^j\|_{\mathbb{R}^r}$ is equivalent to $\|u_j\|_{1,p(x)}$ and $\|\beta^0\|_{\mathbb{R}^r}$ is equivalent to $\|u_0\|_{1,p(x)}$ and

$$G(\beta)\cdot\beta = \langle T(u),u\rangle.$$

From Lemma 4.3, we get $G(\beta)\cdot\beta \to \infty$ as $\|\beta\|_{\mathbb{R}^r} \to \infty$. $\square$

Lemma 4.5. *For all $k \in \mathbb{N}$, there exists $u_k \in X_k$ such that*

$$\langle T(u_k),\varphi\rangle = 0 \ \text{ for all } \ \varphi \in X_k, \tag{4.2}$$

and there is a constant $R > 0$ such that

$$\|u_k\|_{1,p(x)} \leq R \ \text{ for all } \ k \in \mathbb{N}.$$

Proof. From Lemma 4.4 follows the existence of a constant $R>0$ such that for any $\beta \in \partial B_R(0) \subset \mathbb{R}^r$, we have $G(\beta)\cdot\beta > 0$ and the topological argument [21] gives that $G(x)=0$ has a solution $x \in B_R(0)$. Therefore, for each $k \in \mathbb{N}$, there exists $u_k \in X_k$ such that (4.2) holds. □

5 Convergence result for functions a

Assertion 5.1. *The sequence (u_k) is uniformly bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, thus a subsequence converges weakly in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ to an element denoted by u.*

Proof. We have
$$\langle T(u),u\rangle \to \infty \ \text{ as } \ \|u\|_{1,p(x)} \to \infty.$$
Hence there exists $R>0$ with the property that $\langle T(u),u\rangle > 1$ whenever $\|u\|_{1,p(x)} > R$. Consequently, for the sequence of Galerkin approximations $u_k \in X_k$ which satisfy (4.2) with φ replaced by u_k, we get that (u_k) is uniformly bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. □

Assertion 5.2. *The sequence a_k defined by $a_k := a(x, Du_k - \Theta(u_k))$ is uniformly bounded in $L^{p'(x)}(\Omega;\mathbb{R}^m)$ and therefore equi-integrable on Ω.*

Proof. Using the growth assumption (A_4), we get
$$\int_\Omega \left|a(x,Du_k-\Theta(u_k))\right|^{p'(x)} dx \le \int_\Omega b_0(x)\,dx + \int_\Omega |Du_k-\Theta(u_k)|^{p(x)}\,dx < \infty,$$
by the boundedness of $(u_k)_k$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

Hence $a_k(x)$ is uniformly bounded in $L^{p'(x)}(\Omega;\mathbb{R}^m)$. □

Assertion 5.3. *The sequence $(a_k(x):Du_k)^-$ is equi-integrable on Ω. Moreover, there exists a sequence (v_k) such that $v_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ and*
$$\int_\Omega a_k(x) : (Du_k - Dv_k)\,dx \to 0 \ \ \textit{as} \ \ k\to\infty.$$

Proof. For any measurable subset E of Ω and by the coercivity assumption, we have
$$\int_\Omega \left|\min(a(x,Du_k-\Theta(u_k)) : Du_k, 0)\right| dx \le \frac{\alpha}{2^{p^- -1}}\int_E |Du_k|^{p(x)}\,dx + \alpha\int_E |\Theta(u_k)|^{p(x)}\,dx + \int_E |b_0(x)|\,dx < \infty.$$
Then $(a_k(x):Du_k)^-$ is equi-integrable.

We choose a subsequence v_k which belongs to the same finite dimensional space X_k as u_k such that $v_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

Taking $u_k - v_k$ as a test function in (4.2), we deduce that
$$\begin{aligned}\int_\Omega a(x,Du_k-\Theta(u_k)) : (Du_k-Dv_k)\,dx &= \langle v, u_k - v_k\rangle - \int_\Omega f(x,u_k) : (Du_k - Dv_k)\,dx \\ &\quad + \int_\Omega h(x,u_k,Du_k)(u_k-v_k)\,dx - \int_\Omega \phi(u_k) : (Du_k-Dv_k)\,dx. \qquad (5.1)\end{aligned}$$

Since $u_k - v_k \rightharpoonup 0$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, the first term on the right-hand side of (5.1) converges to zero. From (F_1), we have that $|f(x,u_k)|^{p'(x)}$ is bounded by an integrable function. Hence (F_0) and the Dominated Convergence Theorem imply that

$$\begin{aligned}\int_\Omega & f(x,u_k) : (Du_k - Dv_k)\,dx \\ &\le \|f(u_k) - f(u)\|_{p'(x)}\|Dv_k - Du_k\|_{p(x)} + \left|\int_\Omega f(x,u) : (Dv_k - Du_k)\,dx\right| \to 0\end{aligned}$$

as k tends to $+\infty$. For the third term on the right-hand side of (5.1), we have

$$\begin{aligned}\int_\Omega & f(x,u_k,Du_k)(u_k - v_k)\,dx \\ &\le \|h(x,u_k,Du_k)\|_{p'(x)}\|u_k - v_k\|_{p(x)} \le C\|u_k - v_k\|_{p(x)} \to 0 \text{ as } k \to +\infty.\end{aligned}$$

For the last term in (5.1), notice that since ϕ is linear and continuous and (u_k) is bounded, $\phi(u_k)$ is bounded. Then

$$\int_\Omega \phi(u_k) : (Du_k - Dv_k)\,dx \le C\|Dv_k - Du_k\|_1 \to 0 \;\text{ as }\; k \to +\infty.$$

It follows that

$$\int_\Omega a(x, Du_k - \Theta(u_k)) : (Du_k - Dv_k)\,dx \to 0 \;\text{ as }\; k \to \infty. \qquad \square$$

Assertion 5.4. *The following* div-curl *inequality holds:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(a(x,\lambda - \Theta(u)) - a(x, Du - \Theta(u))\big) : (\lambda - Du)\,d\nu_x(\lambda)\,dx \le 0. \tag{5.2}$$

Proof. We define the sequence

$$\begin{aligned}J_k &:= \big(a(x, Du_k - \Theta(u)) - a(x, Du - \Theta(u))\big) : (Du_k - Du) \\ &= a(x, Du_k - \Theta(u)) : (Du_k - Du) - a(x, Du - \Theta(u)) : (Du_k - Du) \\ &=: J_{k,1} + J_{k,2}.\end{aligned}$$

Using the growth condition in (A_3), (A_2) and the Poincaré inequality, we get

$$\int_\Omega |a(x, Du - \Theta(u))|^{p'(x)}\,dx \le C + C'\int_\Omega |Du|^{p(x)}\,dx < \infty$$

for arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ hence $a(x, Du - \Theta(u)) \in L^{p'(x)}(\Omega;\mathbb{M}^{m\times n})$. According to the weak convergence described in Lemma 2.3, one can obtain

$$\liminf_{k\to\infty}\int_\Omega J_{k,2}\,dx = \int_\Omega a(x, Du - \Theta(u)) : \left(\int_{\mathbb{M}^{m\times n}} \lambda\,d\nu_x(\lambda) - Du\right)dx = 0.$$

Next, from Assertion 5.1, there exits a subsequence u_k such that $u_k \to u$ in measure. Since Θ is continuous, $\Theta(u_k) \to \Theta(u)$ almost everywhere in Ω. In view of Lemma 2.4, one can conclude that

$$J := \liminf_{k\to\infty}\int_\Omega J_k\,dx = \liminf_{k\to\infty}\int_\Omega J_{k,1}\,dx \ge \int_\Omega\int_{\mathbb{M}^{m\times n}} a(x, \lambda - \Theta(u)) : (\lambda - Du)\,d\nu_x(\lambda)\,dx.$$

Show (5.2) is equivalent to proving that $J \leq 0$. By virtue of Assertion 5.3, we deduce that

$$\begin{aligned} A &= \liminf_{k\to\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : (Du_k - Du)\, dx \\ &= \liminf_{k\to\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : (Du_k - Dv_k)\, dx \\ &\qquad + \liminf_{k\to\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : (Dv_k - Du)\, dx \\ &= \liminf_{k\to\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : (Dv_k - Du)\, dx \\ &\leq \liminf_{k\to\infty} \big\| |a(x, Du_k - \Theta(u_k))| \big\|_{p'(x)} \, \|v_k - u\|_{1,p(x)} = 0. \end{aligned}$$

It follows that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(a(x, \lambda - \Theta(u)) - a(x, Du - \Theta(u))\big) : (\lambda - Du)\, d\nu_x(\lambda)\, dx \leq 0.$$

Moreover, the monotonicity of the function a implies that the above integral must vanish with respect to the product measure $d\nu_x(\lambda) \otimes dx$, hence

$$\big(a(x, \lambda - \Theta(u)) - a(x, Du - \Theta(u))\big) : (\lambda - Du) = 0 \;\text{ on }\; \operatorname{supp} \nu_x. \qquad \square$$

Assertion 5.5. *The sequence a_k converges weakly in the space $L^1(\Omega; \mathbb{M}^{m\times n})$ as $k \to +\infty$ to the weak limit $\bar{a}$ given by*

$$\overline{a}(x) = a(x, Du - \Theta(u))$$

and Du_k converges to Du in measure on Ω as $k \to +\infty$.

Proof. Using (5.2) and the strict monotonicity assumption (F_0), we deduce that

$$\big(a(x, \lambda - \Theta(u)) - a(x, Du - \Theta(u))\big) : (\lambda - Du) = 0 \;\text{ a.e. }\; x \in \Omega, \;\; \lambda \in \mathbb{R}^N.$$

Then $\lambda = Du(x)$ a.e. $x \in \Omega$ with respect to the measure ν_x on $\mathbb{R}^N$. Therefore, the measure ν_x reduces to the Dirac measure $\delta_{Du(x)}$. By virtue of Lemma 2.2, we deduce that $Du_k \to Du$ in measure, then $u_k \to u$ and $Du_k \to Du$ almost everywhere (up to a subsequence) in Ω. From the continuity of Θ and a, one can deduce that

$$a(x, Du_k - \Theta(u_k)) \to a(x, Du - \Theta(u)) \;\text{ a.e. }\; x \in \Omega.$$

From Assertion 5.2, a_k is equi-integrable, then one can apply Vitali's Theorem to get

$$a(x, Du_k - \Theta(u_k)) \to a(x, Du - \Theta(u)) \;\text{ in }\; L^1(\Omega; \mathbb{M}^{m\times n}). \qquad \square$$

Lemma 5.1. *The function u is a weak solution to problem (1.1).*

At this point, we have everything we need to achieve the limit and demonstrate the main result. From Assertion 5.5, we have

$$\lim_{k\to+\infty} \int_\Omega a(x, Du_k - \Theta(u_k)) : D\varphi\, dx = \int_\Omega a(x, Du - \Theta(u)) : D\varphi\, dx \quad \forall\, \varphi \bigcup_{k\in\mathbb{N}} X_k.$$

Since $u_k \to u$ in measure when $k \to +\infty$, we may extract a suitable subsequence (if necessary) for which

$$u_k \to u \;\text{ almost everywhere for }\; k \to +\infty,$$

and

$$Du_k \to Du \text{ almost everywhere for } k \to +\infty.$$

Therefore, $f(x, u_k) \to f(x, u)$ and $\phi(u_k) \to \phi(u)$ almost everywhere by using the continuity of g and ϕ. Since $(f(x, u_k) : D\varphi)$ and $(\phi(u_k) : D\varphi)$ are equi-integrable, by the Vitali convergence Theorem we get $f(x, u_k) : D\varphi \to f(x, u) : D\varphi$ and $\phi(u_k) : D\varphi \to \phi(u) : D\varphi$ in $L^1(\Omega)$. This implies that

$$\lim_{k\to+\infty} \int_\Omega f(x, u_k) : D\varphi \, dx = \int_\Omega f(x, u) : D\varphi \, dx \quad \forall \varphi \bigcup_{k\in\mathbb{N}} X_k$$

and

$$\lim_{k\to+\infty} \int_\Omega \phi(u_k) : D\varphi \, dx = \int_\Omega \phi(u) : D\varphi \, dx \quad \forall \varphi \bigcup_{k\in\mathbb{N}} X_k.$$

Let us start with the case $(H_1)(a)$. The continuity of f permits to deduce that

$$h(x, u_k, Du_k) \cdot \varphi \to h(x, u, Du) \cdot \varphi$$

for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$. From the growth condition in $(H_1)(a)$, we deduce the equi-integrability of $(h(x, u_k, Du_k) \cdot \varphi(x))$, which implies by Vitali Convergence Theorem that $h(x, u_k, Du_k) \cdot \varphi(x) \to h(x, u, Du) \cdot \varphi(x)$ in $L^1(\Omega)$. Therefore,

$$\lim_{k\to\infty} \int_\Omega h(x, u_k, Du_k) \cdot \varphi(x) \, dx = \int_\Omega h(x, u, Du) \cdot \varphi(x) \, dx \quad \forall \varphi \in \bigcup_{k\in\mathbb{N}} X_k.$$

Next, we consider the case $(H_1)(b)$. If the function h is independent of the third variable, then we can obtain

$$h(x, u_k) \rightharpoonup h(x, u) \text{ in } L^{p'(x)}(\Omega).$$

On the other hand, we assume that the mapping $A \mapsto h(x, u, A)$ is linear for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$. Since $h(x, u_k, Du_k)$ is equi-integrable, we deduce that

$$h(x, u_k, Du_k) \to \langle \nu_x, h(x, u, \cdot)\rangle = \int_{\mathbb{M}^{m\times n}} h(x, u, \lambda) \, d\nu_x(\lambda) = h(x, u, \cdot) \circ \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda \, d\nu_x(\lambda)}_{=:Du(x)} = h(x, u, Du)$$

by the linearity of h.

It remains to show that $\langle T(u), \varphi\rangle = 0$ for any $\varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ to complete the proof of Theorem 3.1.

Let $\varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, the density of $\bigcup_{k\in\mathbb{N}} X_k$ in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ implies the existence of a sequence $\{\varphi_k\} \subset \bigcup_{k\in\mathbb{N}} X_k$ such that $\varphi_k \to \varphi$ in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ as k tends to $+\infty$. We conclude that

$$\begin{aligned}
&\langle T(u_k), \varphi_k\rangle - \langle T(u), \varphi\rangle \\
&\qquad = \int_\Omega a(x, Du_k - \Theta(u_k)) : D\varphi_k \, dx - \int_\Omega a(x, Du - \Theta(u)) : D\varphi \, dx \\
&\qquad\quad + \int_\Omega \phi(u_k) : D\varphi_k \, dx - \int_\Omega \phi(u) : D\varphi \, dx - \int_\Omega f(x, u_k) : D\varphi_k \, dx \\
&\qquad\quad + \int_\Omega f(x, u) : D\varphi \, dx - \int_\Omega h(x, u_k, Du_k) \cdot \varphi_k \, dx + \int_\Omega h(x, u, Du) \cdot \varphi \, dx \\
&= \int_\Omega a(x, Du_k - \Theta(u_k)) : (D\varphi_k - D\varphi) \, dx + \int_\Omega \big(a(x, Du_k - \Theta(u_k)) - a(x, Du - \Theta(u))\big) : D\varphi \, dx
\end{aligned}$$

$$+\int_\Omega \phi(u_k):(D\varphi_k - D\varphi)\,dx + \int_\Omega (\phi(u_k)-\phi(u)):D\varphi\,dx - \int_\Omega f(x,u_k):(D\varphi_k - D\varphi)$$

$$-\int_\Omega (f(x,u_k)-f(x,u)):D\varphi\,dx - \int_\Omega h(x,u_k,Du_k)\cdot(\varphi_k-\varphi)\,dx - \int_\Omega (h(x,u_k,Du_k)-h(x,u,Du))\cdot\varphi\,dx.$$

We take the limit as k tends to $+\infty$, it follows that

$$\lim_{k\to+\infty}\langle T(u_k),\varphi_k\rangle = \langle T(u),\varphi\rangle.$$

From Lemma 4.5, we deduce that $\langle T(u),\varphi\rangle = 0$ for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

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(Received 28.11.2022; revised 10.02.2023; accepted 13.02.2023)

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