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MULTIPLE POSITIVE SOLUTIONS FOR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

Abstract. In this paper, we study the existence of two or three positive solutions of a class of boundary value problems of nonlinear fractional differential equations with integral boundary conditions. To state our results, we use the functional-type cone expansion-compression fixed point theorem and the Leggett–Williams fixed point theorem, respectively. In addition, we show the effectiveness of the main result by using some examples.

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რეზიუმე. ნაშრომში შესწავლილია არაწრფივი წილადური დიფერენციალური განტოლებებისთვის დასმული სასაზღვრო ამოცანების ერთი კლასის ორი ან სამი დადებითი ამონახსნის არსებობა ინტეგრალური სასაზღვრო პირობებით. შედეგების დასაფიქსირებლად, შესაბამისად, გამოყენებულია ფუნქციური ტიპის კონუსის გაფართოება-შეკუმშვის უძრავი წერტილის თეორემა და ლეგეტ-უილიამსის უძრავი წერტილის თეორემა. გარდა ამისა, რამდენიმე მაგალითის მოყვანის საშუალებით ნაჩვენებია ძირითადი შედეგის ეფექტურობა.

1 Introduction

Fractional calculus has been applied to many fields of sciences and engineering, physics, chemistry, polymer rheology, electrical circuits, biology, control theory, etc. There are several books and monographs devoted to the development of fractional calculus (see, e.g.,e Kilbas et al. [13], Diethelm and Freed [7], Podlubny [16], [17], Miller and Ross [15], Samko et al. [18]). Recently, many results were obtained by proving the existence of solutions of different types of nonlinear fractional differential equation (see [2, 6, 12, 19–21]).

Integral boundary conditions have various applications in different real phenomena as blood flow problems, chemical engineering, underground water flow, thermo-elasticity. For more details, we refer the reader to [1, 3–5, 8, 10, 11, 22].

Motivated by the above-mentioned works, in this paper we investigate the existence of multiple solutions of the following nonlinear fractional differential equation with integral boundary conditions:

$$D^{\delta}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\, ds, \tag{1.2}$$

where $1 < \delta \leq 2$, $\lambda > 0$, $\lambda \neq \delta$, D^{δ}_{0+} is the Riemann–Liouville fractional derivative and f is a continuous function. Using some fixed point theorems on a cone, we will prove the existence of multiple positive solutions for the fractional boundary value problem (1.1), (1.2) under certain sufficient conditions.

The paper is organized as follows. In Section 2, we recall some definitions concerning the fractional integrals and the fractional derivatives. Our main results are given in Section 3. Some examples are introduced in the last Section 4.

2 Preliminaries

In this section, we give some necessary definitions from the fractional calculus theory. These definitions can be found in recent literature. For details see [13, 16, 18].

Definition 2.1. [16] The Riemann–Liouville fractional integral operator of order $\delta > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$I^{\delta}_{0+}f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([16])**.** The Riemann–Liouville fractional derivative operator of order $\delta > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D^{\delta}_{0+}f(t) = \frac{1}{\Gamma(n-\delta)} \Big(\frac{d}{dt}\Big)^{n} \int_0^t (t-s)^{n-\delta-1} f(s)\, ds,$$

where $n = [\delta] + 1$, $[\delta]$ denotes the integral part of the number δ, provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.1 ([16])**.** *Assume that $f \in \mathcal{C}(0,1) \cap \mathcal{L}(0,1)$ with a fractional derivative of order $\delta > 0$ that belongs to $\mathcal{C}(0,1) \cap \mathcal{L}(0,1)$. Then*

$$I^{\delta}_{0+}D^{\delta}_{0+}f(t) = f(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + \cdots + c_n t^{\delta-n},$$

where $c_i \in \mathbb{R}$; $i = 1, 2, \ldots, n$ with $n - 1 < \delta \leq n$.

Lemma 2.2. *The fractional boundary value problem*

$$D^{\delta}_{0+}u(t)+y(t)=0,\quad 0<t<1, \tag{2.1}$$

$$u(0)=0,\quad u(1)=\lambda\int_0^1 u(s)\,ds, \tag{2.2}$$

where $\lambda\neq\delta$ *and* $y\in\mathcal{C}[0,1]$, *has a unique solution* $u\in\mathcal{C}[0,1]$ *given by*

$$u(t)=\int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s)=\begin{cases}\dfrac{t^{\delta-1}(1-s)^{\delta-1}(\delta+\lambda s-\lambda)-(\delta-\lambda)(t-s)^{\delta-1}}{(\delta-\lambda)\Gamma(\delta)}, & 0\le s\le t\le 1,\\[2mm] \dfrac{t^{\delta-1}(1-s)^{\delta-1}(\delta+\lambda s-\lambda)}{(\delta-\lambda)\Gamma(\delta)}, & 0\le t\le s\le 1,\end{cases} \tag{2.3}$$

and G *is called Green's function of the fractional boundary value problem.*

Proof. From Lemma 2.1, we have the solution of (2.1), (2.2) given by

$$u(t)=-\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}y(s)\,ds+c_1t^{\delta-1}+c_2t^{\delta-2}.$$

From $u(0)=0$, we have $c_2=0$, then

$$u(t)=-\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}y(s)\,ds+c_1t^{\delta-1}.$$

In particular, we obtain

$$c_1=u(1)+\frac{1}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}y(s)\,ds.$$

Therefore, we have

$$u(t)=-\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}y(s)\,ds+u(1)t^{\delta-1}+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}y(s)\,ds.$$

Let $\xi=\int_0^1 u(t)\,dt$. Then, from the previous equality, using that $u(1)=\lambda\xi$, we conclude that

$$\begin{aligned}\xi&=-\int_0^1\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\,y(s)\,ds\,dt+\lambda\int_0^1\int_0^1 t^{\delta-1}u(s)\,ds\,dt+\int_0^1\int_0^1 t^{\delta-1}\,\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\,y(s)\,ds\,dt\\ &=-\int_0^1\frac{(1-s)^{\delta}}{\delta\Gamma(\delta)}\,y(s)\,ds+\frac{\lambda}{\delta}\,\xi+\int_0^1\frac{(1-s)^{\delta-1}}{\delta\Gamma(\delta)}\,y(s)\,ds.\end{aligned}$$

Thus we have

$$\xi=\frac{\delta}{\delta-\lambda}\int_0^1\frac{s(1-s)^{\delta-1}}{\delta\Gamma(\delta)}\,y(s)\,ds,$$

which implies that

$$c_1 = \int_0^1 \frac{(1-s)^{\delta-1}(\delta+\lambda s-\lambda)}{(\delta-\lambda)\Gamma(\delta)}\, y(s)\, ds.$$

Finally,

$$u(t) = -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\, y(s)\, ds + t^{\delta-1}\int_0^1 \frac{(1-s)^{\delta-1}(\delta+\lambda s-\lambda)}{(\delta-\lambda)\Gamma(\delta)}\, y(s)\, ds = \int_0^1 G(t,s)y(s)\, ds. \qquad \square$$

Lemma 2.3. *Let G be the Green's function related to problem (1.1), (1.2), which is given by expression (2.3). Let $\omega \in (0,1)$ be fixed, then for all $1 < \delta \le 2$ and $0 < \lambda < \delta$, the following properties are fulfilled:*

(i) *$G(t,s) > 0$ for all $t, s \in (0,1)$.*

(ii) *There exists a positive function $L \in \mathcal{C}((0,1),(0,+\infty))$ such that*

$$\max_{0\le t\le 1} G(t,s) \le \frac{\delta(1-s)^{\delta-1}}{(\delta-\lambda)\Gamma(\delta)} = L(s) \ \ \textit{for all} \ \ s \in (0,1).$$

(iii)

$$\omega\,\frac{\lambda}{\delta}\, sL(s) \le \min_{\omega\le t\le 1} G(t,s) \ \ \textit{for all} \ \ s \in (0,1).$$

(iv)

$$G(t,s) \le \frac{\delta}{(\delta-\lambda)\Gamma(\delta)} \ \ \textit{for all} \ \ t,s \in [0,1].$$

(v)

$$\int_0^1 sL(s)\, ds > 0.$$

(vi) *$G(t,s)$ is a continuous function for all $t, s \in [0,1]$.*

Proof. If $0 \le s \le t \le 1$, then we have $t - s \le t(1-s)$. Thus

$$(t-s)^{\delta-1} \le t^{\delta-1}(1-s)^{\delta-1}.$$

(i) We can easily verify that for $0 < s \le t < 1$ the following inequalities hold:

$$G(t,s) = \frac{t^{\delta-1}(1-s)^{\delta-1}(\delta+\lambda s-\lambda) - (\delta-\lambda)(t-s)^{\delta-1}}{(\delta-\lambda)\Gamma(\delta)} > \frac{t^{\delta-1}(1-s)^{\delta-1}\lambda s}{(\delta-\lambda)\Gamma(\delta)} > 0.$$

Also, for $0 < t \le s < 1$,

$$G(t,s) = \frac{t^{\delta-1}(1-s)^{\delta-1}(\delta+\lambda s-\lambda)}{(\delta-\lambda)\Gamma(\delta)} > 0.$$

(ii) For all $s \in (0,1)$, we have

$$\max_{0\le t\le 1} G(t,s) \le \frac{(1-s)^{\delta-1}(\delta+\lambda s-\lambda)}{(\delta-\lambda)\Gamma(\delta)} \le \frac{\delta(1-s)^{\delta-1}}{(\delta-\lambda)\Gamma(\delta)} = L(s).$$

(iii) First, assume that $\omega \le t \le s < 1$, then we obtain

$$\frac{G(t,s)}{L(s)} = t^{\delta-1}\,\frac{\lambda s+\delta-\lambda}{\delta} \ge t^{\delta-1}\,\frac{\lambda}{\delta}\, s \ge \omega^{\delta-1}\,\frac{\lambda}{\delta}\, s \ge \omega\,\frac{\lambda}{\delta}\, s.$$

Secondly, we have

$$\frac{G(t,s)}{L(s)} = \frac{t^{\delta-1}(1-s)^{\delta-1}(\delta+\lambda s-\lambda)-(\delta-\lambda)(t-s)^{\delta-1}}{\delta(1-s)^{\delta-1}} \geq \frac{t^{\delta-1}(1-s)^{\delta-1}\lambda s}{\delta(1-s)^{\delta-1}} = t^{\delta-1}\,\frac{\lambda}{\delta}\,s.$$

As in the previous case, it is not difficult to verify that

$$G(t,s) \geq \omega\,\frac{\lambda}{\delta}\,sL(s) \ \ \forall\, t\in[\omega,1].$$

Then

$$\omega\,\frac{\lambda}{\delta}\,sL(s) \leq \min_{\omega\leq t\leq 1} G(t,s) \ \text{ for all } \ s\in(0,1).$$

(iv) If $t,s\in[0,1]$, then we have

$$G(t,s) \leq t^{\delta-1}L(s) \leq t^{\delta-1}\,\frac{\delta}{(\delta-\lambda)\Gamma(\delta)} \leq \frac{\delta}{(\delta-\lambda)\Gamma(\delta)}\,.$$

(v) It is clear that

$$\int_0^1 sL(s)\,ds = \frac{\delta}{(\delta-\lambda)\Gamma(\delta)}\int_0^1 s(1-s)^{\delta-1}\,ds > 0.$$

(vi) It is obvious from the definition of the function G. □

We will rely on the following fixed point theorems to demonstrate the main results.
First, we define the concept of a cone.

Definition 2.3 ([9])**.** Let E be a real Banach space. A nonempty closed convex set $P\subset E$ is called a cone if it satisfies the following two conditions:

(1) $u\in P$, $\lambda\geq 0$ implies $\lambda u\in P$;

(2) $u\in P$, $-u\in P$ implies $u=\theta$, where θ denotes the zero element of E.

Lemma 2.4 ([9])**.** *Let E be an ordered Banach space and let $P\subset E$ be a cone and suppose that Ω_1, Ω_2 and Ω_3 are bounded open subsets of E with $\theta\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, $\overline{\Omega}_2\subset\Omega_3$. Moreover, let $T: P\cap(\overline{\Omega}_3\setminus\Omega_1)\to P$ be a completely continuous operator such that:*

(A_1) $\|Tu\|\geq\|u\|$ $\forall\, u\in P\cap\partial\Omega_1$;

(A_2) $\|Tu\|\leq\|u\|$ $Tu\neq u$, $\forall\, u\in P\cap\partial\Omega_2$;

(A_3) $\|Tu\|\geq\|u\|$ $\forall\, u\in P\cap\partial\Omega_3$.

Then T has at least two fixed points u^* and u^{**} in $P\cap(\overline{\Omega}_3\setminus\Omega_1)$; moreover, $u^*\in P\cap(\Omega_2\setminus\Omega_1)$ and $u^{**}\in P\cap(\overline{\Omega}_3\setminus\overline{\Omega}_2)$.

Definition 2.4. We say that the map α is a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha: P\to[0,+\infty)$ is continuous and

$$\alpha(tu+(1-t)v)\geq t\alpha(u)+(1-t)\alpha(v),$$

for all $u,v\in P$ and $0\leq t\leq 1$.

Let

$$P(\alpha,b,d)=\big\{u\in P:\ \ b\leq\alpha(u),\ \ \|u\|\leq d\big\}.$$

Theorem 2.1 ((Leggett–Williams) [14])**.** *Let P be a cone in a real Banach space E, $\overline{P}_c=\{u\in P:\ \|u\|\leq c\}$, α be a nonnegative continuous concave positive functional on a cone P such that $\alpha(u)\leq\|u\|$ for all $u\in\overline{P}_c$. Suppose $T:\overline{P}_c\to\overline{P}_c$ is completely continuous and there exist the constants $0<a<b<d\leq c$ such that:*

(B_1) $\{u \in P(\alpha, b, d) : \ \alpha(u) > b\} \neq \varnothing$ *and* $\alpha(Tu) > b$ *for* $u \in P(\alpha, b, d)$*;*

(B_2) $\|Tu\| < a$ *for* $u \in \overline{P}_a$*;*

(B_3) $\alpha(Tu) > b$ *for* $u \in P(\alpha, b, c)$ *with* $\|Tu\| > d$.

Then T *has at least three fixed points* u_1, u_2 *and* u_3 *such that* $\|u_1\| < a$, $b < \alpha(u_2)$, $\|u_3\| > a$ *with* $\alpha(u_3) < b$.

3 Existence of two or three positive solutions

This section is devoted to proving the existence results for our problem (1.1), (1.2).

For the convenience, we introduce the following notations:

$$K = \frac{1}{\Gamma(\delta)} + \frac{1}{(\delta-\lambda)\Gamma(\delta-1)},$$

$$G = \left(\int_0^1 L(s)\,ds\right)^{-1}, \tag{3.1}$$

$$H = \left(\omega\frac{\lambda}{\delta}\int_\omega^1 sL(s)\,ds\right)^{-1}. \tag{3.2}$$

Let $E = (\mathcal{C}[0,1], \|\cdot\|)$ be the Banach space, where $\|u\| = \max\limits_{0\le t\le 1}|u(t)|$. Define the cone $P \subset E$ by

$$P = \big\{u \in E : \ \ u(t) \ge 0, \ \ t \in [0,1]\big\}.$$

Define now the operator $T : E \to E$ as

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s))\,ds, \tag{3.3}$$

with G defined in (2.3).

Note that the fixed points of the operator T are the solutions of problem (1.1), (1.2).

Lemma 3.1. *Assume that* $f \in \mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$*. The operator* $T : P \to P$ *defined by* (3.3) *is completely continuous.*

Proof. Since $G(t,s) \ge 0$, we haven $Tu(t) \ge 0$ for all $u \in P$.

The operator $T : P \to P$ is continuous in view of the continuity of $G(t,s)$ and $f(s,u(s))$. There exists a positive $R > 0$ such that $\mathcal{M} = \{u \in P : \ \|u\| \le R\}$. Let

$$N = \max_{0\le t\le 1,\ 0\le u\le R} |f(t,u)| + 1.$$

Then for $u \in \mathcal{M}$, using Lemma 2.3 part (iv), the following inequality holds:

$$|Tu(t)| = \left|\int_0^1 G(t,s) f(s,u(s))\,ds\right| \le N \int_0^1 G(t,s)\,ds \le \frac{N\delta}{(\delta-\lambda)\Gamma(\delta)} \quad \text{for all } \ t \in [0,1].$$

Hence $T(\mathcal{M})$ is bounded in E.

On the other hand, given $\epsilon > 0$, set

$$\eta = \frac{1}{2}\left(\frac{(\delta-1)\epsilon}{NK}\right)^{1/(\delta-1)}.$$

For any $u \in \mathcal{M}$, we have

$$|(Tu)'(t)| = \left| -\int_0^t \frac{(t-s)^{\delta-2}}{\Gamma(\delta-1)} f(s,u(s))\,ds + t^{\delta-2}\int_0^1 \frac{(1-s)^{\delta-1}(\delta+\lambda s-\lambda)}{(\delta-\lambda)\Gamma(\delta-1)} f(s,u(s))\,ds \right|$$

$$< \frac{N}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2}\,ds + \frac{Nt^{\delta-2}}{(\delta-\lambda)\Gamma(\delta-1)}\int_0^1 (1-s)^{\delta-1}(\delta+\lambda s-\lambda)\,ds$$

$$\le \frac{N}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2}\,ds + \frac{N\delta t^{\delta-2}}{(\delta-\lambda)\Gamma(\delta-1)}\int_0^1 (1-s)^{\delta-1}\,ds$$

$$\le \frac{Nt^{\delta-1}}{\Gamma(\delta)} + \frac{Nt^{\delta-2}}{(\delta-\lambda)\Gamma(\delta-1)} \le N\Big(\frac{1}{\Gamma(\delta)} + \frac{1}{(\delta-\lambda)\Gamma(\delta-1)}\Big)t^{\delta-2} \le NKt^{\delta-2}.$$

Now we prove that whenever $t_1, t_2 \in [0,1]$ and $0 < t_2 - t_1 < \eta$, then

$$|Tu(t_2) - Tu(t_1) < \epsilon.$$

In fact,

$$|Tu(t_2) - Tu(t_1)| \le \int_{t_1}^{t_2} |(Tu)'(s)|\,ds < NK\int_{t_1}^{t_2} s^{\delta-2}\,ds \le \frac{NK}{\delta-1}(t_2^{\delta-1} - t_1^{\delta-1}).$$

In order to estimate $t_2^{\delta-1} - t_1^{\delta-1}$, we can use a method applied in [2]. We have two cases.

Case 1. $\eta \le t_1 < t_2 < 1$. Then

$$|Tu(t_2) - Tu(t_1)| < \frac{NK}{\delta-1}(t_2^{\delta-1} - t_1^{\delta-1}) \le \frac{NK}{\delta-1}\frac{\delta-1}{\eta^{2-\delta}}(t_2 - t_1) \le NK\eta^{\delta-1} < \epsilon.$$

Case 2. $0 \le t_1 < \eta$, $t_2 < 2\eta$. Then

$$|Tu(t_2) - Tu(t_1)| < \frac{NK}{\delta-1}(t_2^{\delta-1} - t_1^{\delta-1}) \le \frac{NK}{\delta-1}t_2^{\delta-1} \le \frac{NK}{\delta-1}(2\eta)^{\delta-1} \le \epsilon.$$

Thus the set $T(\mathcal{M})$ is equicontinuous in E.

Now from the Arzelà–Ascoli Theorem we conclude that $\overline{T(\mathcal{M})}$ is compact, so the operator $T : P \to P$ is completely continuous. □

In the first result, we will show the existence of at least two positive solutions of problem (1.1), (1.2).

Theorem 3.1. *Assume that $f \in \mathcal{C}([0,1]\times[0,+\infty),[0,+\infty))$. There exist three positive constants $0 < \sigma_1 < \sigma_2 < \sigma_3$ such that:*

(C_1) *$f(t,u) \ge H\sigma_1$ for $(t,u) \in [0,1]\times[0,\sigma_1]$;*

(C_2) *$f(t,u) \le G\sigma_2$ for $(t,u) \in [0,1]\times[0,\sigma_2]$;*

(C_3) *$f(t,u) \ge H\sigma_3$ for $(t,u) \in [0,1]\times[0,\sigma_3]$.*

Then problem (1.1), (1.2) has at least two positive solutions u^*, $u^{**} \in P$ with

$$\sigma_1 \le \|u^*\| < \sigma_2 \ \text{ and } \ \sigma_2 < \|u^{**}\| \le \sigma_3.$$

Proof. We know that $T : P \to P$ is completely continuous by Lemma 3.1.

Now, we divide the proof into three steps.

Step 1. Let $\Omega_1 = \{u \in P : \ \|u\| < \sigma_1\}$. For any $u \in P \cap \partial\Omega_1$, we have $\|u\| = \sigma_1$ and $0 \le u(t) \le \sigma_1$ for all $t \in [0, 1]$. It follows from condition (C_1) and Lemma 2.3 part (iii) that

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right|\\ &\ge \max_{\omega\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \ge H\sigma_1\left[\omega\frac{\lambda}{\delta}\int_\omega^1 sL(s)\,ds\right] = \sigma_1 = \|u\|,\end{aligned}$$

which implies that $\|Tu\| \ge \|u\|\ \forall u \in P \cap \partial\Omega_1$.

Step 2. Let $\Omega_2 = \{u \in P : \ \|u\| < \sigma_2\}$. For any $u \in P \cap \partial\Omega_2$, we have $0 \le u(t) \le \sigma_2$ for all $t \in [0, 1]$. It follows from condition (C_2) and Lemma 2.3 part (ii) that for $t \in [0, 1]$,

$$\|Tu\| = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \le \int_0^1 L(s)f(s,u(s))\,ds \le G\sigma_2\left[\int_0^1 L(s)\,ds\right] = \sigma_2 = \|u\|,$$

so, $\|Tu\| \le \|u\|\ \forall u \in P \cap \partial\Omega_2$.

Step 3. Let $\Omega_3 = \{u \in P : \ \|u\| < \sigma_3\}$. For any $u \in P \cap \partial\Omega_3$, we have $\|u\| = \sigma_3$, then $0 \le u(t) \le \sigma_3$ for all $t \in [0, 1]$. By condition (C_3), we have

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right|\\ &\ge \max_{\omega\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \ge H\sigma_3\left[\omega\,\frac{\lambda}{\delta}\int_\omega^1 sL(s)\,ds\right] = \sigma_3 = \|u\|,\end{aligned}$$

which implies that $\|Tu\| \ge \|u\|\ \forall u \in P \cap \partial\Omega_3$.

By Lemma 2.4, T has at least two fixed points u^* and u^{**} in $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$, then problem (1.1), (1.2) has at least two positive solutions u^*, $u^{**} \in P$ such that

$$\sigma_1 \le \|u^*\| < \sigma_2 \ \text{ and } \ \sigma_2 < \|u^{**}\| \le \sigma_3. \qquad \square$$

Secondly, we prove the existence of at least three positive solutions of problem (1.1), (1.2).

Let the nonnegative continuous concave positive functional α on the cone P be defined by

$$\alpha(u) = \min_{\omega\le t\le 1}|u(t)|.$$

It is obvious that, $\forall u \in P\ \ \alpha(u) \le \|u\|$.

Theorem 3.2. *Assume that $f \in \mathcal{C}([0, 1] \times [0, +\infty), [0, +\infty))$. Let $\omega \in (0, 1)$ and there exist the constants $0 < a < b < c$ such that:*

(H_1) *$f(t, u) < Ga$ for $(t, u) \in [0, 1] \times [0, a]$;*

(H_2) *$f(t, u) \ge Hb$ for $(t, u) \in [\omega, 1] \times [b, c]$;*

(H_3) *$f(t, u) \le Gc$ for $(t, u) \in [0, 1] \times [0, c]$.*

Then problem (1.1), (1.2) has at least three positive solutions u_1, u_2 and u_3 with

$$\max_{0\le t\le 1}|u_1(t)| < a, \quad b < \min_{\omega\le t\le 1}|u_2(t)| < \max_{0\le t\le 1}|u_2(t)| \le c,$$
$$a < \max_{0\le t\le 1}|u_3(t)| \le c, \quad \min_{\omega\le t\le 1}|u_3(t)| < b.$$

Proof. We show that all conditions of Theorem 2.1 are satisfied. Let $u \in \overline{P}_c$ then $\|u\| \le c$ and by (H_3) with Eq. (3.1), we have

$$\|Tu\| = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \le \int_0^1 L(s)f(s,u(s))\,ds \le \int_0^1 L(s)Gc\,ds = c.$$

Hence $T : \overline{P}_c \to \overline{P}_c$ and, by Lemma 3.1, it is completely continuous.

Let d be a fixed constant such that $b < d \le c$. We can easily see that $u(t) = d \in P(\alpha, b, d)$, $\alpha(u) = \alpha(d) > b$, consequently, $\{u \in P(\alpha, b, d) : \ \alpha(u) > b\} \neq \varnothing$. Let $u \in P(\alpha, b, c)$, then $b \le u(t) \le c$ for $t \in [\omega, 1]$. With assumption (H_2) and Eq. (3.2), we obtain

$$\alpha(Tu) = \min_{\omega\le t\le 1}\int_0^1 G(t,s)f(s,u(s))\,ds \ge \omega\,\frac{\lambda}{\delta}\int_0^1 sL(s)f(s,u(s))\,ds > \omega\,\frac{\lambda}{\delta}\int_\omega^1 sL(s)Hb\,ds.$$

Therefore, we have $\alpha(Tu) > b \ \ \forall u \in P(\alpha, b, c)$. Hence, condition (B_1) from Theorem 2.1 holds.

Now, we show that condition (B_2) of Theorem 2.1 is satisfied. If $u \in \overline{P}_a \Longrightarrow \|u\| \le a$, assumption (H_1) implies that $f(t, u) < Ga$ for $t \in [0, 1]$. Thus

$$\|Tu\| = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \le \int_0^1 L(s)f(s,u(s))\,ds < \int_0^1 L(s)Ga\,ds = a.$$

This implies that condition (B_2) from Theorem 2.1 is satisfied.

Finally, we prove that condition (B_3) of Theorem 2.1 is satisfied. If $u \in P(\alpha, b, c)$ with $\|Tu\| > d$, then $b \le u(t) \le c$ for $\omega \le t \le 1$. From assumption (H_2), we have

$$\alpha(Tu) = \min_{\omega\le t\le 1}\int_0^1 G(t,s)f(s,u(s))\,ds \ge \omega\,\frac{\lambda}{\delta}\int_0^1 sL(s)f(s,u(s))\,ds > \omega\,\frac{\lambda}{\delta}\int_\omega^1 sL(s)Hb\,ds = b.$$

Thus condition (B_3) from Theorem 2.1 is also satisfied.

By Theorem 2.1, problem (1.1), (1.2) has at least three positive solutions u_1, u_2 and u_3 with

$$\max_{0\le t\le 1}|u_1(t)| < a, \quad b < \min_{\omega\le t\le 1}|u_2(t)| < \max_{0\le t\le 1}|u_2(t)| \le c,$$
$$a < \max_{0\le t\le 1}|u_3(t)| \le c, \quad \min_{\omega\le t\le 1}|u_3(t)| < b. \qquad \square$$

4 Examples

In this section, we present some examples to illustrate the previous results.

Example 4.1. Let $\delta = \frac{3}{2}$ and $\lambda = 1$. Consider the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{3/2}u(t) + f(t,u(t)) = 0, \ \ 0 < t < 1, \\ u(0) = 0, \ \ u(1) = \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \frac{u^3}{9} + \frac{e^t}{10}.$$

Let $\omega = \frac{1}{2}$. We have

$$G \approx 0.4431 \ \text{ and } \ H \approx 5.3713.$$

Choosing $\sigma_1 = \frac{1}{70}$, $\sigma_2 = 1$ and $\sigma_3 = 8$, we get

$(C_1)'$ $f(t,u) = \dfrac{u^3}{9} + \dfrac{e^t}{10} > 0.1 \geq H\sigma_1 \approx 0.076$ for $t \in [0,1]$ and $\|u\| = \dfrac{1}{70}$;

$(C_2)'$ $f(t,u) = \dfrac{u^3}{9} + \dfrac{e^t}{10} \leq 0.383 \leq G\sigma_2 \approx 0.4431$ for $t \in [0,1]$ and $\|u\| = 1$;

$(C_3)'$ $f(t,u) = \dfrac{u^3}{9} + \dfrac{e^t}{10} \geq 56.9 \geq H\sigma_3 \approx 42.97$ for $t \in [0,1]$ and $\|u\| = 8$.

With the use of Theorem 3.1, problem (4.1) has at least two positive solutions u^* and u^{**} such that

$$\frac{1}{70} \leq \|u^*\| < 1 \ \text{ and } \ 1 < \|u^{**}\| \leq 8.$$

Example 4.2. Let $\delta = \frac{3}{2}$ and $\lambda = 1$. Now, we consider the following problem:

$$\begin{cases} D_{0^+}^{3/2} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\, ds, \end{cases} \tag{4.2}$$

where

$$f(t,u) = \begin{cases} \dfrac{1}{15} t^2 + 6u^4, & u < 1, \\ \dfrac{47}{8} + \dfrac{1}{15} t^2 + \dfrac{u}{8}, & u \geq 1. \end{cases}$$

Let $\omega = \frac{1}{2}$. We have

$$G \approx 0.4431 \ \text{ and } \ H \approx 5.3713.$$

Choosing $a = \frac{1}{4}$, $b = 1$ and $c = 20$, we have:

$(H_1)'$ $f(t,u) = \dfrac{1}{15} t^2 + 6u^4 \leq 0.091 < Ga \approx 0.1107$ for $(t,u) \in [0,1] \times [0, 1/4]$;

$(H_2)'$ $f(t,u) = \dfrac{47}{8} + \dfrac{1}{15} t^2 + \dfrac{u}{8} \geq 6.016 > Hb \approx 5.3713$ for $(t,u) \in [1/2, 1] \times [1, 20]$;

$(H_3)'$ $f(t,u) = \dfrac{47}{8} + \dfrac{1}{15} t^2 + \dfrac{u}{8} \leq 8.4417 < Gc \approx 8.8622$ for $(t,u) \in [0,1] \times [0, 20]$.

With the use of Theorem 3.2, problem (4.2) has at least three positive solutions u_1, u_2 and u_3 with

$$\max_{0 \leq t \leq 1} |u_1(t)| < \frac{1}{4}, \quad 1 < \min_{1/2 \leq t \leq 1} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq 20,$$
$$\frac{1}{4} < \max_{0 \leq t \leq 1} |u_3(t)| \leq 20, \qquad \min_{1/2 \leq t \leq 1} |u_3(t)| < 1.$$

Example 4.3. Let $\delta = \frac{3}{2}$ and $\lambda = \frac{1}{2}$. Now, we have the boundary value problem:

$$\begin{cases} D_{0^+}^{3/2} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \dfrac{1}{2} \displaystyle\int_0^1 u(s)\, ds, \end{cases} \tag{4.3}$$

where

$$f(t,u) = \begin{cases} \dfrac{1}{20} t + 25u^3, & u < 1, \\ \dfrac{99}{4} + \dfrac{1}{20} t + \dfrac{u}{4}, & u \geq 1. \end{cases}$$

Let $\omega = \frac{1}{3}$. By a simple calculation, we obtain

$$G \approx 0.8862 \text{ and } H \approx 24.4215.$$

Choosing $a = \frac{1}{10}$, $b = 1$ and $c = 40$, we have:

$(H_1)''$ $f(t,u) = \dfrac{1}{20}t + 25u^3 \leq 0.075 < Ga \approx 0.088$ for $(t,u) \in [0,1] \times [0,1/10]$;

$(H_2)''$ $f(t,u) = \dfrac{99}{4} + \dfrac{1}{20}t + \dfrac{u}{4} \geq 25.0166 > Hb \approx 24.4215$ for $(t,u) \in [1/3,1] \times [1,40]$;

$(H_3)''$ $f(t,u) = \dfrac{99}{4} + \dfrac{1}{20}t + \dfrac{u}{4} \leq 34.8 < Gc \approx 35.448$ for $(t,u) \in [0,1] \times [0,40]$.

Then all conditions of Theorem 3.2 hold.

Thus by Theorem 3.2, problem (4.3) has at least three positive solutions u_1, u_2 and u_3 with

$$\max_{0\leq t\leq 1} |u_1(t)| < \frac{1}{10}, \quad 1 < \min_{1/3\leq t\leq 1} |u_2(t)| < \max_{0\leq t\leq 1} |u_2(t)| \leq 40,$$
$$\frac{1}{10} < \max_{0\leq t\leq 1} |u_3(t)| \leq 40, \qquad \min_{1/3\leq t\leq 1} |u_3(t)| < 1.$$

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