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A PREDATOR-PREY BIOLOGICAL MODEL WITH COMBINED BIRTH RATES, SELF-LIMITATION AND COMPETITION TERMS

Abstract. The purpose of this paper is to give sufficient conditions for the existence and uniqueness of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are super-sub solutions method, eigenvalues of operators, maximum principles, spectrum estimates, inverse function theory, and general elliptic theory. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of species of animals with predator-prey relation in many biological models.

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Key words and phrases. Predator-prey system, coexistence state, existence, uniqueness, perturbation.

რეზიუმე. ნაშრომის მიზანია საკმაოდ ზოგადი ტიპის ელიფსური სისტემის შემთხვევაში დავადგინოთ საკმარისი პირობები R^n-ის შემოსაზღვრულ Ω არეში დირიხლეს ამოცანის დადებითი ამონახსნის არსებობის და ერთადერთობისთვის. ასევე განიხილება შეშფოთებების გავლენა თანაარსებობის მდგომარეობასა და ერთადერთობაზე. გამოყენებული მეთოდები მოიცავს სუპერ-სუბ ამონახსნების მეთოდს, ოპერატორების საკუთარ მნიშვნელობებს, მაქსიმუმის პრინციპებს, სპექტრის შეფასებებს, შექცეულ ფუნქციათა თეორიას და ზოგად ელიფსურ თეორიას. არგუმენტები ასევე ეყრდნობა ლოგისტიკური განტოლებების ამონახსნის ზოგიერთ დეტალურ თვისებას. ეს შედეგები იძლევა ალგებრულად გამოთვლად კრიტერიუმს მტაცებელი-მსხვერპლი ტიპის ცხოველთა სახეობების პოზიტიური თანაარსებობისთვის ბევრ ბიოლოგიურ მოდელში.

1 Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the survival of two or more species of animals in the same environment. Especially, pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general predator-prey model to better understand the competitive interactions between two species. Specifically, we investigate the conditions needed for the coexistence of two species when the factors affecting them are fixed or perturbed.

2 Literature review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple predator-prey model, commonly known as the Lotka–Volterra predator-prey model. This system describes the predator-prey interaction of two species residing in the same environment in the following manner:

Suppose two species of animals with predator-prey interaction, rabbits and tigers for instance, are residing in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of preys and predators in the place x of Ω at time t, respectively. Then we have the dynamic predator-prey model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \alpha u(x,t) - au^2(x,t) - bu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + \beta v(x,t) - dv^2(x,t) + cu(x,t)v(x,t) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

where $\alpha, \beta > 0$ are growth rates, $a, d > 0$ are self-limitation rates, and $b, c > 0$ are competition rates. Here, we are interested in the time independent, positive solutions, i.e., the positive solutions $u(x)$, $v(x)$ of

$$\begin{cases} \Delta u(x) + u(x)(\alpha - au(x) - bv(x)) = 0 \\ \Delta v(x) + v(x)(\beta - dv(x) + cu(x)) = 0 \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \ \Omega, \tag{2.1}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so, its existence means that the two species of animals can live peacefully and forever.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (2.1) [7, 8, 10].

One of the initial important results for the time-independent Lotka–Volterra model was obtained by Korman and Leung. In 1986, they published the following sufficient conditions for the existence of a positive steady state solution to (2.1).

Theorem 2.1 ([7]). *If $ad > bc$, $\alpha > \frac{ad(\lambda_1 + \frac{b}{d}\beta)}{ad-bc}$ and $\beta > \lambda_1$, then* (2.1) *has a positive solution.*

Biologically, the conditions in Theorem 2.1 imply that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, in other words, if members of each species interact strongly among themselves and weakly with members of the other species, then there is a positive steady state solution to (2.1), that is, the two species within the same domain will coexist indefinitely at population densities.

Another important result was obtained by Zhengyuan and Mottoni. In 1992, they published the following characterization of non-negative solutions to (2.1) in terms of growth rates (α, β).

Theorem 2.2 ([10]). *There exist two functions $\gamma_0(\alpha), \mu_0(\beta)$ such that the set S of non-negative solutions to* (2.1) *is characterized as follows:*

(A) *If $\alpha \le \lambda_1, \beta \le \lambda_1$, then $S = \{(0,0)\}$.*

(B) *If $\alpha \le \lambda_1, \beta > \lambda_1$, then $S = \{(0,0), (0, \theta_{\frac{\beta}{d}})\}$.*

(C) *If* $\alpha > \lambda_1, \beta < \gamma_0(\alpha)$*, then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0)\}$.

(D) *If* $\lambda_1 < \alpha < \mu_0(\beta), \beta > \lambda_1$*, then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (0, \theta_{\frac{\beta}{d}})\}$.

(E) *If* $\alpha > \lambda_1, \gamma_0(\alpha) < \beta \leq \lambda_1$*, then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (u^+, v^+)\}$*, where* (u^+, v^+) *is a positive solution to* (2.1).

(F) *If* $\beta > \lambda_1, \alpha > \mu_0(\beta)$*, then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (0, \theta_{\frac{\beta}{d}}), (u^+, v^+)\}$.

These results provide insight into the predator-prey interactions of two species operating under the conditions described in the Lotka–Volterra model. However, their results are somewhat limited by a few key assumptions. In the Lotka–Volterra model that they studied, the rates of change of densities largely depend on constant rates of reproduction, self-limitation, and competition. The model also assumes a linear relationship of the terms affecting the rate of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple predator-prey model. Therefore, in the last decades, significant research has been focused on the existence and uniqueness of the positive steady state solution of the general predator-prey model for two species,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)g(u(x,t), v(x,t)) \\ v_t(x,t) = \Delta v(x,t) + v(x,t)h(u(x,t), v(x,t)) \\ u(x,t)\big|_{\partial\Omega} = v(x,t)\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega \times R^+,$$

or, equivalently, the positive solution to

$$\begin{cases} \Delta u(x) + u(x)g(u(x), v(x)) = 0 \\ \Delta v(x) + v(x)h(u(x), v(x)) = 0 \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{2.2}$$

where $g, h \in C^1$ designate reproduction, self-limitation and competition rates that satisfy certain growth conditions.

Because of its broader applicability, the general predator-prey model has become a more popular subject of research within the mathematical community over the past few years.

The functions g, h describe how species 1 (u) and 2 (v) interact among themselves and with each other.

The followings are the questions raised in the general model with nonlinear growth rates.

Problem 1: What are the sufficient conditions for the existence of positive solutions?

Problem 2: What are the sufficient conditions for the uniqueness of positive solutions?

Problem 3: What is the effect of perturbation for the existence and uniqueness?

In our analysis, we focus on the conditions required for the maintenance of the coexistence state of the model when interaction rates (g, h) are slightly perturbed. Biologically, our conclusion implies that two species may slightly relax ecologically and yet continue to coexist at unique densities.

In Section 4, we establish sufficient conditions for the existence and non-existence of positive solution of the system that generalizes Theorems 2.1 and 2.2. We also achieve solution estimates in Section 5 to prove the uniqueness and the invertibility of linearization in Sections 6, 7 and 8, where we investigate the effect of perturbation for existence and uniqueness.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the Lotka–Volterra model with constant rates (see [2, 3]) The research presented in this paper therefore begins the mathematical community's discussion on the stability of the steady state solution for the general predator-prey model.

3 Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition 3.1 (Super and Sub solutions)**.** The vector functions $(\overline{u}^1, \ldots, \overline{u}^N)$, $(\underline{u}^1, \ldots, \underline{u}^N)$ form a super/sub solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1, \ldots, u^N) = 0 & \text{in } \Omega, \\ u^i = 0 & \text{on } \partial\Omega, \end{cases}$$

if for $i = 1, \ldots, N$,

$$\begin{cases} \Delta \overline{u}^i + g^i(u^1, \ldots, u^{i-1}, \overline{u}^i, u^{i+1}, \ldots, u^N) \le 0 \\ \Delta \underline{u}^i + g^i(u^1, \ldots, u^{i-1}, \underline{u}^i, u^{i+1}, \ldots, u^N) \ge 0 \end{cases} \quad \text{in } \Omega \text{ for } \underline{u}^j \le u^j \le \overline{u}^j, \ j \ne i,$$

and

$$\begin{cases} \underline{u}^i \le \overline{u}^i & \text{on } \Omega, \\ \underline{u}^i \le 0 \le \overline{u}^i & \text{on } \partial\Omega. \end{cases}$$

Lemma 3.1. *If g^i in Definition 3.1 are in C^1 and the system admits a super/sub solution pair $(\underline{u}^1, \ldots, \underline{u}^N), (\overline{u}^1, \ldots, \overline{u}^N)$, respectively, then there is a solution $(u_1, \ldots, u_N)$ to the system in Definition 3.1 with $\underline{u}^i \le u^i \le \overline{u}^i$ in $\overline{\Omega}$. If*

$$\Delta \overline{u}^i + g^i(\overline{u}^1, \ldots, \overline{u}^N) \ne 0,$$
$$\Delta \underline{u}^i + g^i(\underline{u}^1, \ldots, \underline{u}^N) \ne 0$$

in Ω for $i = 1, \ldots, N$, then $\underline{u}^i < u^i < \overline{u}^i$ in Ω.

Lemma 3.2 (The first eigenvalue)**.** *Consider*

$$\begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain.

(A) *The first eigenvalue $\lambda_1(q)$ of (3.1), denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction ϕ_q.*

(B) *If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.*

(C) *(Variational Characterization of the first eigenvalue)*

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega),\ \phi \ne 0} \frac{\int_\Omega (|\nabla\phi|^2 + q\phi^2)\,dx}{\int_\Omega \phi^2\,dx}.$$

In our proof, we also employ the accepted conclusions concerning the solutions of the following logistic equations.

Lemma 3.3. *Consider*

$$\begin{cases} \Delta u + uf(u) = 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \le 0$ for $u \ge c_0$ and Ω is a bounded domain.

(A) *If $f(0) > \lambda_1$, then the above equation has a unique positive solution. We denote this unique positive solution by θ_f.*

(B) *If $f(0) \le \lambda_1$, then $u \equiv 0$ is the only nonnegative solution to the above equation.*

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $\alpha > \lambda_1$, we denote θ_α as the unique positive solution of

$$\begin{cases} \Delta u + u(\alpha - u) = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0. \end{cases}$$

Hence, θ_α is increasing as $\alpha > 0$ is increasing.

Having established these preliminaries, we now commence our investigation of the general predator-prey model.

4 Existence, nonexistence

A general type of elliptic interacting system of two functions with the homogeneous boundary condition is

$$\begin{cases} \Delta u + ug(u,v) = 0 \\ \Delta v + vh(u,v) = 0 \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \ \Omega, \tag{4.1}$$

where Ω is a bounded domain with a smooth boundary $\partial\Omega$, $g, h \in C^1$ are relative growth rates such that $g_u < 0$, $g_v < 0$, $h_u > 0$, $h_v < 0$.

We establish the following existence result.

Theorem 4.1. *If*

(A) $$\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u) > 0,$$

(B) $$g(0,0) > \frac{\sup(g_u)\sup(h_v)(\lambda_1 + \frac{\inf(g_v)}{\sup(h_v)}h(0,0))}{\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u)} \ \text{ and } \ h(0,0) > \lambda_1,$$

then (4.1) *has a positive solution.*

Proof. Let

$$\underline{u} = \gamma_1\omega, \quad \underline{v} = \gamma_2\omega, \quad \overline{u} = -\frac{g(0,0)}{\sup(g_u)}, \quad \overline{v} = -\frac{1}{\sup(h_v)}\Big(h(0,0) - \frac{g(0,0)\sup(h_u)}{\sup(g_u)}\Big),$$

where $\gamma_1 > 0$, $\gamma_2 > 0$ are the constants and ω is the eigenfunction of (3.1) with $q(x) \equiv 0$ corresponding to the first eigenvalue λ_1.

Then by the Mean Value Theorem, for all v such that $\underline{v} \le v \le \overline{v}$,

$$\begin{aligned} \Delta\overline{u} + \overline{u}g(\overline{u},v) &= \overline{u}\big[g(0,0) + g(\overline{u},v) - g(0,v) + g(0,v) - g(0,0)\big] \\ &\le \overline{u}\big[g(0,0) + \overline{u}\sup(g_u) + v\sup(g_v)\big] = \overline{u}v\sup(g_v) < 0, \end{aligned}$$

and for all u such that $\underline{u} \le u \le \overline{u}$,

$$\begin{aligned} \Delta\overline{v} + \overline{v}h(u,\overline{v}) &= \overline{v}\big[h(0,0) + h(u,\overline{v}) - h(0,\overline{v}) + h(0,\overline{v}) - h(0,0)\big] \\ &\le \overline{v}\big[h(0,0) + u\sup(h_u) + \overline{v}\sup(h_v)\big] \le \overline{v}\big[h(0,0) + \overline{u}\sup(h_u) + \overline{v}\sup(h_v)\big] = 0. \end{aligned}$$

By the condition and the Mean Value Theorem again, for all v such that $\underline{v} \le v \le \overline{v}$,

$$\begin{aligned}
\Delta \underline{u} + \underline{u} g(\underline{u}, v) &= \Delta(\gamma_1 \omega) + \gamma_1 \omega g(\gamma_1 \omega, v) \\
&= \Delta(\gamma_1 \omega) + \gamma_1 \omega \big[g(0,0) + g(\gamma_1 \omega, v) - g(0,v) + g(0,v) - g(0,0) \big] \\
&\ge -\gamma_1 \lambda_1 \omega + \gamma_1 \omega \big[g(0,0) + \gamma_1 \omega \inf(g_u) + v \inf(g_v) \big] \\
&\ge -\gamma_1 \lambda_1 \omega + \gamma_1 \omega \Big[g(0,0) - \frac{1}{\sup(h_v)} \Big(h(0,0) - \frac{g(0,0)\sup(h_u)}{\sup(g_u)} \Big) \inf(g_v) + \gamma_1 \omega \inf(g_u) \Big] \\
&= -\gamma_1 \lambda_1 \omega + \gamma_1 \omega \Big[g(0,0) + \inf(g_v) \Big(-\frac{h(0,0)}{\sup(h_v)} + \frac{g(0,0)\sup(h_u)}{\sup(g_u)\sup(h_v)} \Big) + \gamma_1 \omega \inf(g_u) \Big] \\
&= -\gamma_1 \lambda_1 \omega + \gamma_1 \omega \Big[g(0,0) - \frac{\inf(g_v) h(0,0)}{\sup(h_v)} + \frac{g(0,0)\inf(g_v)\sup(h_u)}{\sup(g_u)\sup(h_v)} + \gamma_1 \omega \inf(g_u) \Big] \\
&= -\gamma_1 \lambda_1 \omega + \gamma_1 \omega \Big[g(0,0) \Big(1 + \frac{\inf(g_v)\sup(h_u)}{\sup(g_u)\sup(h_v)} \Big) - \frac{\inf(g_v) h(0,0)}{\sup(h_v)} + \gamma_1 \omega \inf(g_u) \Big] \\
&= \gamma_1 \omega \Big[-\lambda_1 + g(0,0) \Big(1 + \frac{\inf(g_v)\sup(h_u)}{\sup(g_u)\sup(h_v)} \Big) - \frac{\inf(g_v) h(0,0)}{\sup(h_v)} + \gamma_1 \omega \inf(g_u) \Big] > 0
\end{aligned}$$

with small enough $\gamma_1 > 0$, since

$$\begin{aligned}
&-\lambda_1 + g(0,0) \Big(1 + \frac{\inf(g_v)\sup(h_u)}{\sup(g_u)\sup(h_v)} \Big) - \frac{\inf(g_v) h(0,0)}{\sup(h_v)} \\
&\quad > -\lambda_1 + \frac{\sup(g_u)\sup(h_v)(\lambda_1 + \frac{\inf(g_v)}{\sup(h_v)} h(0,0))}{\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u)} \Big(\frac{\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u)}{\sup(g_u)\sup(h_v)} \Big) - \frac{\inf(g_v) h(0,0)}{\sup(h_v)} \\
&\quad = -\lambda_1 + \lambda_1 + \frac{\inf(g_v)}{\sup(h_v)} h(0,0) - \frac{\inf(g_v)}{\sup(h_v)} h(0,0) = 0
\end{aligned}$$

and for all u such that $\underline{u} \le u \le \overline{u}$,

$$\begin{aligned}
\Delta \underline{v} + \underline{v} h(u, \underline{v}) &= -\gamma_2 \lambda_1 \omega + \gamma_2 \omega h(u, \gamma_2 \omega) \\
&= -\gamma_2 \lambda_1 \omega + \gamma_2 \omega \big[h(0,0) + h(u, \gamma_2 \omega) - h(0, \gamma_2 \omega) + h(0, \gamma_2 \omega) - h(0,0) \big] \\
&\ge -\gamma_2 \lambda_1 \omega + \gamma_2 \omega \big[h(0,0) + \inf(h_u) u + \inf(h_v) \gamma_2 \omega \big] \\
&\ge \gamma_2 \omega (h(0,0) - \lambda_1) + \gamma_2 \omega \big[\inf(h_u) \gamma_1 \omega + \inf(h_v) \gamma_2 \omega \big] \\
&> \gamma_2 \omega (h(0,0) - \lambda_1) + \gamma_2^2 \omega^2 \inf(h_v) \\
&= \gamma_2 \omega \big[(h(0,0) - \lambda_1) + \gamma_2 \omega \inf(h_v) \big] > 0
\end{aligned}$$

with small enough $\gamma_2 > 0$. Furthermore,

$$\underline{u} = \underline{v} = \overline{u} = \overline{v} = 0 \text{ on } \partial\Omega$$

and

$$\underline{u} \le \overline{u}, \quad \underline{v} \le \overline{v}$$

with small enough $\lambda_1 > 0, \lambda_2 > 0$. Hence, by Lemma 3.1, there is a solution (u, v) to (4.1) with

$$\underline{u} \le u \le \overline{u}, \quad \underline{v} \le v \le \overline{v}. \qquad \square$$

We also establish the following nonexistence results.

Theorem 4.2. *Suppose* $g(0,0) \le \lambda_1$, $h(0,0) \le \lambda_1$. *Then* $u = v \equiv 0$ *is the only nonnegative solution to* (4.1).

Proof. Let (u, v) be a nonnegative solution to (4.1). By the Mean Value Theorem, there are $\widetilde{u}$, $\widetilde{v}$ such that

$$\begin{aligned}
g(u,v) - g(u,0) &= g_v(u, \widetilde{v}) v, \\
h(u,v) - h(0,v) &= h_u(\widetilde{u}, v) u.
\end{aligned}$$

Hence, (4.1) implies that

$$\begin{aligned}\Delta u + u\big(g(u,0) + g_v(u,\widetilde{v})v\big) &= \Delta u + u\big(g(u,0) + g(u,v) - g(u,0)\big)\\ &= \Delta u + ug(u,v) = 0 \ \text{ in } \ \Omega,\end{aligned}$$

and

$$\begin{aligned}\Delta v + v\big(h(0,v) + h_u(\widetilde{u},v)u\big) &= \Delta v + v\big(h(0,v) + h(u,v) - h(0,v)\big)\\ &= \Delta v + vh(u,v) = 0 \ \text{ in } \ \Omega.\end{aligned}$$

Thus

$$\begin{aligned}\Delta u + u(g(u,0) + \sup(g_v)v) \geq 0 \ \text{ in } \ \Omega,\\ \Delta v + v(h(0,v) + \sup(h_u)u) \geq 0 \ \text{ in } \ \Omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup(h_u)\phi_1\Delta u + \sup(h_u)\phi_1 u(g(u,0) + \sup(g_v)v) \geq 0 \ \text{ in } \ \Omega,\\ -\sup(g_v)\phi_1\Delta v - \sup(g_v)\phi_1 v(h(0,v) + \sup(h_u)u) \geq 0 \ \text{ in } \ \Omega.\end{aligned}$$

So,

$$\begin{aligned}\int_\Omega -\sup(h_u)\phi_1\Delta u\,dx &\leq \int_\Omega \big[g(u,0)\sup(h_u)u + \sup(g_v)\sup(h_u)uv\big]\phi_1\,dx,\\ \int_\Omega \sup(g_v)\phi_1\Delta v\,dx &\leq \int_\Omega \big[-h(0,v)\sup(g_v)v - \sup(g_v)\sup(h_u)uv\big]\phi_1\,dx.\end{aligned}$$

Hence, by Green's Identity, we have

$$\begin{aligned}\int_\Omega \sup(h_u)\lambda_1\phi_1 u\,dx &\leq \int_\Omega \big[g(u,0)\sup(h_u)u + \sup(g_v)\sup(h_u)uv\big]\phi_1\,dx,\\ \int_\Omega -\sup(g_v)\lambda_1\phi_1 v\,dx &\leq \int_\Omega \big[-h(0,v)\sup(g_v)v - \sup(g_v)\sup(h_u)uv\big]\phi_1\,dx.\end{aligned}$$

Therefore,

$$\int_\Omega \sup(h_u)(\lambda_1 - g(u,0))u\phi_1 - \sup(g_v)(\lambda_1 - h(0,v))v\phi_1\,dx \leq 0.$$

Since the left-hand side is nonnegative, from

$$\begin{aligned}g(u,0) \leq g(0,0) \leq \lambda_1,\\ h(0,v) \leq h(0,0) \leq \lambda_1,\end{aligned}$$

we conclude that $u = v \equiv 0$. □

Theorem 4.3. *Let $u \geq 0, v \geq 0$ be a solution to* (4.1). *If $g(0,0) \leq \lambda_1$, then $u \equiv 0$.*

Proof. Proceeding as in the proof of Theorem 4.2, we obtain

$$0 \leq \int_\Omega (\lambda_1 - g(u,0))u\phi_1\,dx \leq \int_\Omega \sup(g_v)vu\phi_1\,dx \leq 0,$$

and so, $u \equiv 0$. □

Theorem 4.3 implies that if $g(0,0) \leq \lambda_1$ and $h(0,0) > \lambda_1$, then all possible nonnegative solutions to (4.1) are $(0,0)$ and $(0, \theta_{h(0,\,\cdot\,)})$.

5 Solution estimate

In order to prove further results, we will need the following solution estimate.

For the rest of this section, we assume the following additional growth condition:

$$\lim_{x\to\infty} g(x,0) = \lim_{y\to\infty} h(0,y) = -\infty.$$

Lemma 5.1. *Let $u \geq 0, v \geq 0$ be a solution of the problem*

$$\begin{cases} -\Delta u = tug(u,v) & in \;\; \Omega, \\ -\Delta v = tvh(u,v) & in \;\; \Omega, \\ u_{\partial\Omega} = v_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where $t \in [0,1]$. Then

(A)

$$u \leq M_1, \quad v \leq M_2,$$

where

$$M_1 = -\frac{g(0,0)}{\sup(g_u)}, \quad M_2 = \frac{\sup(h_u)(M_1) + h(0,0)}{-\sup(h_v)}.$$

(B) *For $t = 1$, if $v > 0$,*

$$\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u) > 0,$$

$$g(0,0) + \inf(g_v)\Big[\frac{g(0,0)\sup(h_u) - h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big] > \lambda_1,$$

and $h(0,0) > \lambda_1$, then

$$\theta_{M(g,h)} \leq u \leq \theta_{g(\,\cdot\,,0)},$$

$$\theta_{h(0,\,\cdot\,)} \leq v \leq \theta_{h(0,\,\cdot\,) - \frac{g(0,0)\sup(h_u)}{\sup(g_u)}},$$

where

$$M(g,h) = g(\,\cdot\,,0) + \inf(g_v)\Big[\frac{g(0,0)\sup(h_u) - h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big].$$

Proof.

(A). Since (u,v) is a solution to (5.1), by the Mean Value Theorem,

$$\Delta u + u\big[g(0,0) + \sup(g_u)u + \sup(g_v)v\big]$$
$$\geq \Delta u + u\big[g(0,0) + g(u,v) - g(0,v) + g(0,v) - g(0,0)\big] = 0,$$

and so,

$$\Delta u + u\big[g(0,0) + \sup(g_u)u\big] \geq -\sup(g_v)uv \geq 0.$$

Hence, by the Maximum Principles,

$$g(0,0) + \sup(g_u)u \geq 0,$$

equivalently,

$$u \leq -\frac{g(0,0)}{\sup(g_u)}.$$

Since (u,v) is a solution to (5.1), by the Mean Value Theorem,

$$\Delta v + v\big[h(0,0) + \sup(h_u)u + \sup(h_v)v\big]$$
$$\geq \Delta v + v\big[h(0,0) + h(u,v) - h(0,v) + h(0,v) - h(0,0)\big] = 0,$$

and so,

$$\Delta v+v\Big[h(0,0)+\sup(h_v)v+\sup(h_u)\Big(-\frac{g(0,0)}{\sup(g_u)}\Big)\Big]$$
$$\geq -\sup(h_u)uv+\sup(h_u)\Big[-\frac{g(0,0)}{\sup(g_u)}\Big]v=\sup(h_u)v\Big[-u-\frac{g(0,0)}{\sup(g_u)}\Big]\geq 0.$$

Hence, by the Maximum Principles again,

$$h(0,0)+\sup(h_v)v+\sup(h_u)\Big[-\frac{g(0,0)}{\sup(g_u)}\Big]\geq 0,$$

equivalently,

$$v\leq\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}.$$

(B). By the monotonicity of g,

$$\Delta u+ug(u,0)=u\big[g(u,0)-g(u,v)\big]\geq 0,$$

and so, u is a subsolution to

$$\Delta Z+Zg(Z,0)=0 \text{ in } \Omega,$$
$$Z\big|_{\partial\Omega}=0.$$

But, since any sufficiently large positive constant is a supersolution to

$$\Delta Z+Zg(Z,0)=0 \text{ in } \Omega,$$
$$Z\big|_{\partial\Omega}=0,$$

by Lemmas 3.1 and 3.3, we conclude that

$$u\leq\theta_{g(\,\cdot\,,0)}. \tag{5.2}$$

Since

$$\Delta u+u\Big[g(u,0)+\inf(g_v)\Big(\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big)\Big]$$
$$=u\Big[-g(u,v)+g(u,0)+\inf(g_v)\Big(\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big)\Big]$$
$$\leq u\Big[-\inf(g_v)v+\inf(g_v)\,\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big]$$
$$=-\inf(g_v)u\Big[v-\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big]\leq 0,$$

by the Mean Value Theorem and (1), u is a supersolution to

$$\Delta Z+Z\Big[g(Z,0)+\inf(g_v)\,\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big]=0 \text{ in } \Omega,$$
$$Z\big|_{\partial\Omega}=0.$$

But, by the continuity of g and the condition, for a sufficiently small $\epsilon>0$,

$$\Delta\epsilon\phi_1+\epsilon\phi_1\Big[g(\epsilon\phi_1,0)+\inf(g_v)\,\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big]$$
$$=\epsilon\phi_1\Big[-\lambda_1+g(\epsilon\phi_1,0)+\inf(g_v)\,\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}]+h(0,0)}{-\sup(h_v)}\Big]>0,$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\Delta Z + Z\Big[g(Z,0) + \inf(g_v)\frac{\sup(h_u)[-\frac{g(0,0)}{\sup(g_u)}] + h(0,0)}{-\sup(h_v)}\Big] = 0 \text{ in } \Omega,$$
$$Z\big|_{\partial\Omega} = 0.$$

Therefore, by Lemmas 3.1 and 3.3, we have

$$\theta_{g(\,\cdot\,,0)+\inf(g_v)\,[\frac{g(0,0)\sup(h_u)-h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}]} \le u. \tag{5.3}$$

By the monotonicity of h,

$$\Delta v + vh(0,v) = v\big[h(0,v) - h(u,v)\big] < 0,$$

and so, v is a supersolution to

$$\Delta Z + Zh(0,Z) = 0 \text{ in } \Omega,$$
$$Z\big|_{\partial\Omega} = 0.$$

But, by the continuity of h and the condition, for a sufficiently small $\epsilon > 0$,

$$\epsilon\phi_1 + \epsilon\phi_1 h(0,\epsilon\phi_1) = \epsilon\phi_1\big[-\lambda_1 + h(0,\epsilon\phi_1)\big] > 0,$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\Delta Z + Zh(0,Z) = 0 \text{ in } \Omega,$$
$$Z\big|_{\partial\Omega} = 0.$$

Hence, by Lemmas 3.1 and 3.3, we have

$$\theta_{h(0,\,\cdot\,)} \le v. \tag{5.4}$$

Since (u,v) is a solution to (5.1), by the Mean Value Theorem and (A),

$$\begin{aligned}\Delta v + v\Big[h(0,v) - \frac{\sup(h_u)g(0,0)}{\sup(g_u)}\Big] &= v\Big[-h(u,v) + h(0,v) - \frac{\sup(h_u)g(0,0)}{\sup(g_u)}\Big] \\ &\ge v\Big[-\sup(h_u)u - \frac{\sup(h_u)g(0,0)}{\sup(g_u)}\Big] = \sup(h_u)v\Big[-u - \frac{g(0,0)}{\sup(g_u)}\Big] \ge 0.\end{aligned}$$

Therefore, v is a subsolution to

$$\Delta v + v\Big[h(0,v) - \frac{\sup(h_u)g(0,0)}{\sup(g_u)}\Big] = 0 \text{ in } \Omega,$$
$$v\big|_{\partial\Omega} = 0.$$

But, since any sufficiently large constant is a supersolution to

$$\Delta v + v\Big[h(0,v) - \frac{\sup(h_u)g(0,0)}{\sup(g_u)}\Big] = 0 \text{ in } \Omega,$$
$$v\big|_{\partial\Omega} = 0,$$

by Lemmas 3.1 and 3.3, we have

$$v \le \theta_{h(0,\,\cdot\,)-\frac{g(0,0)\sup(h_u)}{\sup(g_u)}}. \tag{5.5}$$

By (5.2), (5.3), (5.4) and (5.5), we establish the desired inequalities.

6 Uniqueness

In this section, we prove the uniqueness of a positive solution to (4.1) with the following additional growth condition:

$$\lim_{x\to\infty} g(x,0) = \lim_{y\to\infty} h(0,y) = -\infty.$$

We have the following uniqueness result.

Theorem 6.1. *In addition to Theorem 4.1, if*

(A)

$$g(0,0) + \inf(g_v)\Big[\frac{g(0,0)\sup(h_u) - h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big] > \lambda_1$$

and

(B)

$$4\sup(g_u)\sup(h_v) > \big[\inf(g_v)\big]^2 \frac{\theta_{g(\,\cdot\,,0)}}{\theta_{h(0,\,\cdot\,)}} + \big[\sup(h_u)\big]^2 \frac{\theta_{L(g,h)}}{\theta_{M(g,h)}},$$

where

$$L(g,h) = h(0,\,\cdot\,) - \frac{g(0,0)\sup(h_u)}{\sup(g_u)},$$

$$M(g,h) = g(\,\cdot\,,0) - \inf(g_v)\Big[\frac{h(0,0)\sup(g_u) - g(0,0)\sup(h_u)}{\sup(g_u)\sup(h_v)}\Big],$$

then (4.1) *has a unique positive solution.*

The conditions imply that species 1 interact strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1.

Proof. The existence has been already proved in the last section. We prove the uniqueness.

Let (u_1, v_1), (u_2, v_2) be positive solutions to (4.1) and let $p = u_1 - u_2$, $q = v_1 - v_2$. We want to show that $p \equiv q \equiv 0$.

Since (u_1, v_1), (u_2, v_2) are the solutions to (4.1),

$$\Delta(p) + g(u_1,v_1)p + g(u_1,v_1)u_2 - g(u_2,v_2)u_2 = 0,$$
$$\Delta(q) + h(u_2,v_2)q + h(u_1,v_1)v_1 - h(u_2,v_2)v_1 = 0.$$

So,

$$\Delta(p) + g(u_1,v_1)p + \big[g(u_1,v_1) - g(u_2,v_1)\big]u_2 + \big[g(u_2,v_1) - g(u_2,v_2)\big]u_2 = 0,$$
$$\Delta(q) + h(u_2,v_2)q + \big[h(u_1,v_1) - h(u_2,v_1)\big]v_1 + \big[h(u_2,v_1) - h(u_2,v_2)\big]v_1 = 0.$$

By the Mean Value Theorem, there are $\overline{u}$, $\overline{v}$, and $\widetilde{u}$, $\widetilde{v}$ such that $\overline{u}$ and $\widetilde{u}$ are between u_1 and u_2, $\overline{v}$ and $\widetilde{v}$ are between v_1 and v_2, and

$$g(u_1,v_1) - g(u_2,v_1) = g_u(\overline{u},v_1)p, \quad g(u_2,v_1) - g(u_2,v_2) = g_v(u_2,\overline{v})q,$$
$$h(u_1,v_1) - h(u_2,v_1) = h_u(\widetilde{u},v_1)p, \quad h(u_2,v_1) - h(u_2,v_2) = h_v(u_2,\widetilde{v})q.$$

Therefore,

$$\Delta(p) + g(u_1,v_1)p + g_u(\overline{u},v_1)u_2p + g_v(u_2,\overline{v})u_2q = 0,$$
$$\Delta(q) + h(u_2,v_2)q + h_u(\widetilde{u},v_1)v_1p + h_v(u_2,\widetilde{v})v_1q = 0.$$

Since

$$\Delta(u_1) + g(u_1,v_1)u_1 = 0,$$
$$\Delta(v_2) + h(u_2,v_2)v_2 = 0,$$

by Lemma 3.2, we have

$$\int_\Omega -p\Delta(p) - g(u_1, v_1)p^2\,dx \geq 0,$$
$$\int_\Omega -q\Delta(q) - h(u_2, v_2)q^2\,dx \geq 0,$$

and so,

$$\int_\Omega -g_u(\overline{u}, v_1)u_2p^2 - \big[g_v(u_2, \overline{v})u_2 + h_u(\widetilde{u}, v_1)v_1\big]pq - h_v(u_2, \widetilde{v})v_1q^2\,dx \leq 0.$$

Hence $p \equiv q \equiv 0$ if the integrand is positive definite, in other words,

$$\big[g_v(u_2, \overline{v})u_2 + h_u(\widetilde{u}, v_1)v_1\big]^2 < 4g_u(\overline{u}, v_1)h_v(u_2, \widetilde{v})u_2v_1,$$

which is true if

$$\big[\inf(g_v)\big]^2u_2^2 + \big[\sup(h_u)\big]^2v_1^2 < 4\sup(g_u)\sup(h_v)u_2v_1,$$

which is true if

$$4\sup(g_u)\sup(h_v) > \big[\inf(g_v)\big]^2\,\frac{u_2}{v_1} + \big[\sup(h_u)\big]^2\,\frac{v_1}{u_2}\,,$$

which is true if the condition is satisfied by the solution estimates in Theorem 4.3. □

7 Uniqueness with perturbation

We consider the model

$$\begin{cases} \Delta u + ug(u,v) = 0 \\ \Delta v + vh(u,v) = 0 \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{7.1}$$

where Ω is a smooth, bounded domain and the C^1 functions g, h are such that $g_u < 0$, $g_v < 0$, $h_u > 0$, $h_v < 0$.

Define

$$B = \Big\{(\alpha,\beta) \in [C^1]^2 \mid \lim_{x\to\infty} \alpha(x,0) = \lim_{y\to\infty} \beta(0,y) = -\infty\Big\}$$

with

$$\|(\alpha,\beta)\|_B = |\alpha(0,0)| + \sup|\alpha_u| + \sup|\alpha_v| + |\beta(0,0)| + \sup|\beta_u| + \sup|\beta_v| \quad \text{for all } (\alpha,\beta) \in B.$$

Then by the functional analysis theory, $(B, \|\cdot\|_B)$ is a Banach space. □

The following theorem is our main result about the perturbation of the uniqueness.

Theorem 7.1. *Suppose $(g,h) \in B$ is such that*

(A) $\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u) > 0$, $h(0,0) > \lambda_1$,

(B) $g(0,0) + \inf(g_v)\Big[\dfrac{g(0,0)\sup(h_u) - h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big] > \lambda_1$, $\lambda_1\big[-g(0,\theta_{h(0,\,\cdot\,)})\big] < 0$,

(C) (7.1) *has a unique coexistence state* (u,v),

(D) *the Frechet derivative of* (7.1) *at* (u,v) *is invertible.*

Then there is a neighborhood V of (g,h) in B such that if $(\overline{g},\overline{h}) \in V$, then (7.1) *with $(\overline{g},\overline{h})$ has a unique positive solution.*

Biologically, the first condition in Theorem 7.1 indicates that the rates of reproduction are relatively large. Especially, the condition

$$g(0,0)+\inf(g_v)\Big[\frac{g(0,0)\sup(h_u)-h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big]>\lambda_1$$

is equivalent to

$$g(0,0)\Big[1+\frac{\inf(g_v)\sup(h_u)}{\sup(g_u)\sup(h_v)}\Big]>\lambda_1+\frac{h(0,0)\inf(g_v)}{\sup(h_v)}\,,$$

which implies that the birth rate of the prey($g(0,0)$) must be larger than that of predator($h(0,0)$). Furthermore, comparing the two conditions $\lambda_1[-g(0,\theta_{h(0,\,\cdot\,)})]<0$ and $h(0,0)>\lambda_1$, it also implies that the birth capacity of prey($g(0,0)$) is stronger than that of predator($h(0,0)$). Similarly, the fourth condition, which requires the invertibility of the Frechet derivative, signifies that the rates of self-limitation are relatively larger than the rates of competition, a relationship that will be established in Lemma 7.1. When these conditions are fulfilled, the conclusion of our theorem asserts that small perturbations of the rates do not affect the existence and uniqueness of the positive steady state. That is, the two species implied can continue to coexist even if the factors determining the population densities vary slightly.

Now, at first glance, Theorem 7.1 may appear to be a consequence of the Implicit Function Theorem. However, the Implicit Function Theorem guarantees only the local uniqueness. In contrast, our result in Theorem 7.1 guarantees global uniqueness. The techniques we will use in the proof of Theorem 7.1 include the Implicit Function Theorem and a priori estimates on solutions of (7.1).

Proof. Since the Frechet derivative of (7.1) at (u,v) is invertible, by the Implicit Function Theorem, there is a neighborhood V of (g,h) in B and a neighborhood W of (u,v) in $[C_0^{2,\alpha}(\overline{\Omega})]^2$ such that for all $(\overline{g},\overline{h})\in V$, there is a unique positive solution $(\overline{u},\overline{v})\in W$ of (7.1) with $(\overline{g},\overline{h})$. Thus the local uniqueness of the solution is guaranteed.

To prove the global uniqueness, suppose that the conclusion of Theorem 7.1 is false. Then there are the sequences (g_n,h_n,u_n,v_n) and (g_n,h_n,u_n^*,v_n^*) in $V\times[C_0^{2,\alpha}(\overline{\Omega})]^2$ such that (u_n,v_n) and (u_n^*,v_n^*) are positive solutions of (7.1) with $(g_n,h_n),(u_n,v_n)\neq(u_n^*,v_n^*)$ and $(g_n,h_n)\to(g,h)$. By Schauder's estimate in the elliptic theory and the solution estimate in the Lemma 5.1, there are uniformly convergent subsequences of u_n and v_n, which again will be denoted by u_n and v_n.

Thus, let

$$(u_n,v_n)\to(\overline{u},\overline{v}),$$
$$(u_n^*,v_n^*)\to(u^*,v^*).$$

Then $(\overline{u},\overline{v}),(u^*,v^*)\in(C^{2,\alpha})^2$ are also the solutions to (7.1) with (g,h). We claim that $\overline{u}>0$, $\overline{v}>0$, $u^*>0$, $v^*>0$. By the Maximum Principles, it suffices to claim that $\overline{u}$, $\overline{v}$, u^*, v^* are not identically zero.

Suppose that it is not true. Then by the Maximum Principles again, either one of the followings will hold:

(1) $\overline{u}\equiv0$, $\overline{v}>0$;

(2) $\overline{u}>0$, $\overline{v}\equiv0$;

(3) $\overline{u}\equiv\overline{v}\equiv0$.

First, suppose $\overline{u}\equiv0$.

Let

$$\widetilde{u}_n=\frac{u_n}{\|u_n\|_\infty}\ \text{ and }\ \widetilde{v}_n=v_n.$$

Then

$$\Delta\widetilde{u}_n+\widetilde{u}_ng_n(u_n,\widetilde{v}_n)=0,$$
$$\Delta\widetilde{v}_n+\widetilde{v}_nh_n(u_n,\widetilde{v}_n)=0.$$

By the elliptic theory again, there is $\widetilde{u}$ such that $\widetilde{u}_n \to \widetilde{u}$, and so,

$$\Delta\widetilde{u} + \widetilde{u}g(0,\overline{v}) = 0,$$
$$\Delta\overline{v} + \overline{v}h(0,\overline{v}) = 0.$$

Hence $\lambda_1[-g(0,\overline{v})] = 0$.

If $\overline{v} \equiv 0$, then $g(0,0) = \lambda_1$, which is a contradiction to our assumption. If $\overline{v}$ is not identically zero, then $\lambda_1[-g(0,\theta_{h(0,\,\cdot\,)})] = 0$, which is also a contradiction.

Next, suppose that $\overline{u} > 0$ and $\overline{v} \equiv 0$.

Let

$$\widetilde{u}_n = u_n \text{ and } \widetilde{v}_n = \frac{v_n}{\|v_n\|_\infty}.$$

Then

$$\Delta\widetilde{u}_n + \widetilde{u}_n g_n(\widetilde{u}_n, v_n) = 0,$$
$$\Delta\widetilde{v}_n + \widetilde{v}_n h_n(\widetilde{u}_n, v_n) = 0.$$

By the elliptic theory again, there is $\widetilde{v}$ such that $\widetilde{v}_n \to \widetilde{v}$, and so,

$$\Delta\overline{u} + \overline{u}g(\overline{u},0) = 0,$$
$$\Delta\widetilde{v} + \widetilde{v}h(\overline{u},0) = 0.$$

Therefore,

$$\lambda_1 - h(0,0) = \lambda_1[-h(0,0)] > \lambda_1[-h(\overline{u},0)] = 0,$$

and so, $h(0,0) < \lambda_1$, which is a contradiction.

Consequently, $(\overline{u},\overline{v})$ and (u^*,v^*) are the positive solutions to (7.1) with (g,h), and so $(\overline{u},\overline{v}) = (u^*,v^*) = (u,v)$ by the uniqueness condition. But this is a contradiction to the Implicit Function Theorem, since $(u_n,v_n) \neq (u_n^*,v_n^*)$. □

In biological terms, the proof of our theorem indicates that if one of two species living in the same domain becomes extinct, that is, if one species is excluded by the other, then the reproduction rates of both must be small. In other words, the region condition of reproduction rates (A) is reasonable.

Now, condition (C) in Theorem 7.1 requiring the invertibility of the Frechet derivative is too artificial to have any direct biological implications. We therefore turn our attention to more applicable conditions that will guarantee the invertibility of the Frechet derivative. We then obtain the following relationship.

We consider the model

$$\begin{cases} \Delta u + ug(u,v) = 0 \\ \Delta v + vh(u,v) = 0 \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{7.2}$$

where Ω is a smooth, bounded domain and the C^1 functions g and h are such that $g_u < 0$, $g_v < 0$, $h_u > 0$, $h_v < 0$.

Lemma 7.1. *Suppose* (u,v) *is a positive solution to* (7.2)*. If*

$$4\sup(g_u)\sup(h_v)uv > \big[\inf(g_v)\big]^2u^2 + \big[\sup(h_u)\big]^2v^2,$$

then the Frechet derivative of (7.2) *at* (u,v) *is invertible.*

Proof. The Frechet derivative of (7.2) at (u,v) is

$$A = \begin{pmatrix} \Delta + g(u,v) + ug_u(u,v) & ug_v(u,v) \\ vh_u(u,v) & \Delta + h(u,v) + vh_v(u,v) \end{pmatrix}.$$

We need to show that $N(A) = \{0\}$ by the Fredholm alternative, where $N(A)$ is the null space of A.

If

$$\Delta\phi + \big[g(u,v) + ug_u(u,v)\big]\phi + g_v(u,v)u\psi = 0,$$
$$\Delta\psi + h_u(u,v)v\phi + \big[h(u,v) + vh_v(u,v)\big]\psi = 0,$$

then

$$\int_\Omega |\nabla\phi|^2 - \big[g(u,v) + ug_u(u,v)\big]\phi^2 - g_v(u,v)u\phi\psi\,dx = 0,$$
$$\int_\Omega |\nabla\psi|^2 - h_u(u,v)v\phi\psi - \big[h(u,v) + vh_v(u,v)\big]\psi^2\,dx = 0.$$

Since (u,v) is a positive solution to (7.2), by Lemma 3.2, we have

$$\int_\Omega |\nabla\phi|^2 - g(u,v)\phi^2\,dx \geq 0,$$
$$\int_\Omega |\nabla\psi|^2 - h(u,v)\psi^2\,dx \geq 0.$$

Hence

$$\int_\Omega -ug_u(u,v)\phi^2 - g_v(u,v)u\phi\psi\,dx \leq 0,$$
$$\int_\Omega -h_u(u,v)v\phi\psi - h_v(u,v)v\psi^2\,dx \leq 0.$$

Therefore,

$$\int_\Omega -ug_u(u,v)\phi^2 - \big[g_v(u,v)u + h_u(u,v)v\big]\phi\psi - h_v(u,v)v\psi^2\,dx \leq 0.$$

Hence $(\phi,\psi) = (0,0)$ if the integrand is positive definite, which is true if the condition is satisfied. □

Combining Lemma 5.1, Theorem 6.1, Theorem 7.1 and Lemma 7.1, we obtain the following corollary.

Corollary 7.1. *If $(g,h) \in B$ is such that*

(A) $\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u) > 0$,

(B) $g(0,0) > \dfrac{\sup(g_u)\sup(h_v)(\lambda_1 + \frac{\inf(g_v)}{\sup(h_v)}h(0,0))}{\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u)}$,

(C) $g(0,0) + \inf(g_v)\Big[\dfrac{g(0,0)\sup(h_u) - h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big] > \lambda_1$,

(D) $h(0,0) > \lambda_1, \quad \lambda_1\big[-g(0,\theta_{h(0,\,\cdot\,)})\big] < 0$,

(E) $4\sup(g_u)\sup(h_v) > \big[\inf(g_v)\big]^2\dfrac{\theta_{g(\,\cdot\,,0)}}{\theta_{h(0,\,\cdot\,)}} + \big[\sup(h_u)\big]^2\dfrac{\theta_{L(g,h)}}{\theta_{M(g,h)}}$, *where*

$$L(g,h) = h(0,\,\cdot\,) - \frac{g(0,0)\sup(h_u)}{\sup(g_u)},$$
$$M(g,h) = g(\,\cdot\,,0) - \inf(g_v)\Big[\frac{h(0,0)\sup(g_u) - g(0,0)\sup(h_u)}{\sup(g_u)\sup(h_v)}\Big],$$

then there is a neighborhood V of (g,h) such that if $(\overline{g},\overline{h}) \in V$, then (7.1) with $(\overline{g},\overline{h})$ has a unique positive solution.

In biological terms, the result obtained in Corollary 7.1 confirms that under certain conditions, two species that relax ecologically can continue to coexist at fixed rates. The requirements given in (A) and (B) simply state that each species must interact strongly with itself and weakly with the other species.

8 Uniqueness with perturbation of region

We consider the model

$$\begin{cases} \Delta u + ug(u,v) = 0 \\ \Delta v + vh(u,v) = 0 \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{8.1}$$

where $g,h \in C^1$ are such that $g_u < 0$, $g_v < 0$, $h_u > 0$, $h_v < 0$, and Ω is a smooth, bounded domain.

The following Theorem is the main result.

Theorem 8.1. *Suppose $\Gamma \subseteq B$ is a closed, bounded, convex region in B such that*

(A) *for all $(g,h) \in \Gamma$, $\sup(g_u)\sup(h_v) + \inf(g_v)\sup(h_u) > 0$,*

(B) *for all $(g,h) \in \Gamma$,*

$$g(0,0) + \inf(g_v)\Big[\frac{g(0,0)\sup(h_u) - h(0,0)\sup(g_u)}{\sup(g_u)\sup(h_v)}\Big] > \lambda_1,$$
$$\lambda_1\big[-g(0,\theta_{h(0,\,\cdot\,)})\big] < 0, \quad h(0,0) > \lambda_1,$$

(C) *for all $(g,h) \in \partial_L\Gamma$, (8.1) has a unique positive solution, where*

$$\partial_L\Gamma = \Big\{(\lambda_h,h) \in \Gamma \mid \text{ for any fixed } h,\ \|\lambda_h\| = \inf\big\{\|g\| \mid (g,h) \in \Gamma\big\}\Big\},$$

(D) *for all $(g,h) \in \Gamma$, the Frechet derivative of (8.1) at every positive solution (u,v) is invertible.*

Then for all $(g,h) \in \Gamma$, (8.1) has a unique positive solution. Furthermore, there is an open set W in B such that $\Gamma \subseteq W$ and for every $(g,h) \in W$, (8.1) has a unique positive solution.

Theorem 8.1 goes even further than Theorem 7.1 which states the uniqueness in the whole region of (g,h) whenever we have the uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed h, consider $(g,h) \in \partial_L\Gamma$ and $(\overline{g},h) \in \Gamma$. We need to show that for all $0 \le t \le 1$, (8.1) with $(1-t)(g,h) + t(\overline{g},h)$ has a unique positive solution. Since (8.1) with (g,h) has a unique positive solution (u,v) and the Frechet derivative of (8.1) at (u,v) is invertible, Theorem 7.1 implies that there is an open neighborhood V of (g,h) in B such that if $(g_0,h_0) \in V$, then (8.1) with (g_0,h_0) has a unique positive solution. Let

$$\lambda_s = \sup\Big\{0 \le \lambda \le 1 \mid \text{ (8.1) with } (1-t)(g,h) + t(\overline{g},h) \text{ has a unique coexistence state for } 0 \le t \le \lambda\Big\}.$$

We need to show that $\lambda_s = 1$. Suppose $\lambda_s < 1$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and there is a sequence (u_n,v_n) of the unique positive solutions of (8.1) with $(1-\lambda_n)(g,h) + \lambda_n(\overline{g},h)$. Then by the elliptic theory, there is (u_0,v_0) such that (u_n,v_n) converges to (u_0,v_0) uniformly and (u_0,v_0) is a solution of (8.1) with $(1-\lambda_s)(g,h) + \lambda_s(\overline{g},h)$.

But by the same proof as in Section 7, $u_0 > 0, v_0 > 0$.

We claim that (8.1) has a unique coexistence state with $(1-\lambda_s)(g,h)+\lambda_s(\overline{g},h)$. In fact, if not, assume that $(\overline{u}_0,\overline{v}_0)\neq(u_0,v_0)$ is another coexistence state. By the Implicit Function Theorem, there exists $0<\widetilde{a}<\lambda_s$, which is very close to λ_s, such that (8.1) with $(1-\widetilde{a})(g,h)+\widetilde{a}(\overline{g},h)$ has a coexistence state very close to $(\overline{u}_0,\overline{v}_0)$, which means that (8.1) with $(1-\widetilde{a})(g,h)+\widetilde{a}(\overline{g},h)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But since (8.1) with $(1-\lambda_s)(g,h)+\lambda_s(\overline{g},h)$ has a unique coexistence state and the Frechet derivative is invertible, Theorem 7.1 implies that λ_s can not be as defined. Therefore, for each $(g,h)\in\Gamma$, (8.1) with (g,h) has a unique coexistence state (u,v). Furthermore, by the assumption, for each $(g,h)\in\Gamma$, the Frechet derivative of (8.1) with (g,h) at the unique solution (u,v) is invertible. Hence Theorem 7.1 concludes that for each $(g,h)\in\Gamma$, there is an open neighborhood $V_{(g,h)}$ of (g,h) in B such that if $(\overline{g},\overline{h})\in V_{(g,h)}$, then (8.1) with $(\overline{g},\overline{h})$ has a unique coexistence state. Let $W=\bigcup_{(g,h)\in\Gamma}V_{(g,h)}$. Then W is an open set in B such that $\Gamma\subseteq W$ and for each $(\overline{g},\overline{h})\in W$, (8.1) with $(\overline{g},\overline{h})$ has a unique coexistence state. □

Apparently, Theorem 8.1 generalizes Theorem 7.1.

9 Conclusions

In this paper, our investigation of the effects of perturbations on the general predator-prey model resulted in the development and proof of Theorem 7.1, Lemma 7.1 and Corollary 7.1 as detailed above. The three together assert that given the existence of a unique solution (u,v) to system (7.1), the perturbations of the birth rates (g,h) within a specified neighborhood, will maintain the existence and uniqueness of the positive steady state. Indeed, our results specifically outline the conditions, sufficient to maintain the positive, steady state solution, when the general predator-prey model is perturbed within some region.

Applying this mathematical result to real world situations, our results establish that the species residing in the same environment can vary their interactions within certain bounds and continue to survive together indefinitely at unique densities. The conditions necessary for the coexistence, as described in the theorem, simply require that members of each species interact strongly with themselves and weakly with members of the other species.

The research presented in this paper has a number of strengths, which confirm both the validity and the applicability of the project. First, the mathematical conditions required in Corollary 7.1 are identical to those required in Theorem 6.1. However, in Theorem 6.1, we have used these conditions to prove the existence and uniqueness of the positive steady state solution for the general predator-prey model. In contrast, the Corollary 7.1 employs the same conditions to establish that the existence and uniqueness of this solution is maintained when the model is perturbed within some neighborhood. Thus, our findings extend and improve the established mathematical theory.

Secondly, perturbations of the general model render its implications more applicable both mathematically and biologically. Because our theorem extends the steady state to any value within some neighborhood of (g,h), the results for the general model pertain to a far wider variety of values. Biologically, perturbations extend the model's description to species affected by factors that vary slightly yet erratically. Thus the description of competitive interactions given by the model becomes a closer approximation of real-world population dynamics.

While our research therefore represents a progression in the field, the results obtained have an important limitation. Theorem 7.1, Lemma 7.1, and Corollary 7.1 establish that a region of perturbation exists within which the coexistence state is maintained for the general predator-prey model. However, the exact extent of that region remains unknown. Therefore, the results presented in this paper may serve as a platform for research of the question given above. Mathematicians should now attempt to establish the exact extent of the perturbation region in which coexistence is maintained for the general model. Such information would prove very useful not only mathematically but also biologically. Specifically, the knowledge of the extent of the region would imply exactly how far the species can relax and yet continue to coexist. Thus the results achieved through our research will enable the field to continue the development of the theory on predator-prey interaction of populations.

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