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PERIODIC SOLUTIONS FOR NONLINEAR VOLTERRA–FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH ψ-CAPUTO FRACTIONAL DERIVATIVE

Abstract. In the present paper, by using the coincidence degree theory of Mawhin introduced in [15], we discuss the existence and uniqueness of periodic solutions to a large class of problems for a nonlinear Volterra–Fredholm integro-differential equation involving the ψ-Caputo fractional derivative. Two examples are given to substantiate the validity of our findings.

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რეზიუმე. ნაშრომში, მავინის დამთხვევის ხარისხის თეორიის გამოყენებით, რომელიც შემოღებულ იქნა [15]-ში, განვიხილავთ ამოცანათა ფართო კლასის პერიოდული ამონახსნების არსებობასა და ერთადერთობას არაწრფივი ვოლტერა-ფრედჰოლმის ინტეგრო-დიფერენციალური განტოლებისთვის ψ-კაპუტოს წილადური წარმოებულით. ჩვენი დასკვნების მართებულობის დასადასტურებლად მოყვანილია ორი მაგალითი.

1 Introduction

In this paper, we consider the following nonlinear Volterra–Fredholm integro-differential equation

$$ {}^{c}\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}(\tau) = \mathfrak{h}\bigg(\tau, \mathfrak{u}(\tau), \int_{\mathfrak{a}}^{\tau} \kappa_1(\tau, s, \mathfrak{u}(s))\, ds, \int_{\mathfrak{a}}^{\mathfrak{b}} \kappa_2(\tau, s, \mathfrak{u}(s))\, ds\bigg), \quad \tau \in \mathfrak{J}, \tag{1.1} $$

with the periodic conditions

$$ \mathfrak{u}(\mathfrak{a}) = \mathfrak{u}(\mathfrak{b}), \tag{1.2} $$

where $\mathfrak{J} := [\mathfrak{a}, \mathfrak{b}]$, $(-\infty < \mathfrak{a} < \mathfrak{b} < +\infty)$ and ${}^{c}\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}$ denote the ψ-Caputo derivative of fractional order $0 < \alpha \leqslant 1$, and

$$ \mathfrak{h} : \mathfrak{J} \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}, \quad \kappa_1 : \mathfrak{d} \times \mathfrak{R} \to \mathfrak{R} \ \text{and} \ \kappa_2 : \mathfrak{d}_0 \times \mathfrak{R} \to \mathfrak{R} $$

are continuous functions with $\mathfrak{d} = \{(\tau, s) : \ \mathfrak{a} \leqslant s \leqslant \tau \leqslant \mathfrak{b}\}$ and $\mathfrak{d}_0 = \mathfrak{J} \times \mathfrak{J}$. For the sake of brevity, we take

$$ \mathfrak{B}_1\mathfrak{u}(\tau) = \int_{\mathfrak{a}}^{\tau} \kappa_1(\tau, s, \mathfrak{u}(s))\, ds \ \text{and} \ \mathfrak{B}_2\mathfrak{u}(\tau) = \int_{\mathfrak{a}}^{\mathfrak{b}} \kappa_2(\tau, s, \mathfrak{u}(s))\, ds. \tag{1.3} $$

In recent years, fractional differential equations have been investigated extensively in numerous applications in various sciences such as physics, engineering, etc. For more details see [1–3, 17, 18, 22].

Beginning with the classical operators of Riemman–Liouville and Caputo, a large number of extensions of those operators have been presented by many authors (see [4, 5, 22]). Many researchers were interested in the subject of existence and uniqueness of solutions by using various methods such as the fixed point theory and the coincidence degree theory (see [8–11, 13, 14, 16]). The existence results of such problems with different fractional derivatives can be found in [6, 12, 24].

In [23], Tidke examined the following nonlinear mixed Volterra–Fredholm integro-differential problem:

$$ \mathfrak{u}'(t) = f\bigg(\mathfrak{u}(t), \int_0^t \kappa(t, s, \mathfrak{u}(s))\, ds, \int_0^{\mathfrak{b}} \mathfrak{h}(t, s, \mathfrak{u}(s))\, ds\bigg), \quad t \in [0, \mathfrak{b}], $$
$$ \mathfrak{u}(0) + g(\mathfrak{u}) = \mathfrak{u}_0. $$

He proved the existence of solutions by means of Leray–Schauder nonlinear alternative theorem. For the fractional mixed integro-differential equations with nonlocal integral initial conditions associated with the Caputo fractional derivative of order $\alpha \in]0, 1]$,

$$ \frac{d^{\alpha}\mathfrak{u}(t)}{dt^{\alpha}} = f\bigg(t, \mathfrak{u}(t), \int_0^t \kappa(t, s, \mathfrak{u}(s))\, ds, \int_0^1 \mathfrak{h}(t, s, \mathfrak{u}(s))\, ds\bigg), \quad t \in [0, 1], $$
$$ \mathfrak{u}(0) = \int_0^1 g(s)\mathfrak{u}(s)\, ds, $$

Anguraj *et al.* [7] proved the existence and uniqueness of solutions by applying some fixed-point theorems. In [19], Laadjal and Ma studied the existence and uniqueness of solutions for the following nonlinear Volterra–Fredholm integro-differential equation of Caputo fractional order with the boundary conditions:

$$ {}^{c}\mathfrak{D}^{\alpha}\mathfrak{u}(t) = g(t) + \lambda_1 \int_0^t \kappa_1(t, s) f_1(s, \mathfrak{u}(s))\, ds + \lambda_2 \int_0^t \kappa_2(t, s) f_2(s, \mathfrak{u}(s))\, ds, $$
$$ \mathfrak{a}\mathfrak{u}(0) + \mathfrak{b}\mathfrak{u}(1) = 0, $$

where $t \in [0,1]$, $\alpha \in]0,1]$ and $\mathfrak{a}, \mathfrak{b}$ are real constants with $\mathfrak{a}+\mathfrak{b} \neq 0$. The reasoning is mainly based upon the contraction mapping principle and Krasnosel'skii's fixed point theorem. However, if $\mathfrak{a}+\mathfrak{b}=0$, this method is not applicable to showing the existence of periodic solutions.

Motivated by the above works and using the technique of the coincidence degree theory of Mawhin [15, 20] for certain suitable operators, in this research, we prove some new existence and uniqueness results for the large class of nonlinear Volterra–Fredholm integro-differential equations (1.1), (1.2), involving the generalized ψ-Caputo fractional derivative.

For the organization of the rest of this research, in Section 2, we recall some definitions of the ψ-Caputo fractional operator and some results which will be used in the further stages. The study of the existence and uniqueness of periodic solutions of our problem (1.1), (1.2) is given in Section 3. Finally, we illustrate our mains results through two examples.

2 Basic concepts

We consider $C(\mathfrak{J},\mathfrak{R})$ and $C^{\mathfrak{m}}(\mathfrak{J},\mathfrak{R})$, the spaces of continuous and $\mathfrak{m}$ times continuously differentiable functions on $\mathfrak{J}$, respectively, with the norm $\|\cdot\|_\infty$ denoting the supremum norm on $C(\mathfrak{J},\mathfrak{R})$.

Definition 2.1 ([4])**.** Let $\mathfrak{J}=[\mathfrak{a},\mathfrak{b}]$ $(-\infty \leqslant \mathfrak{a} < \mathfrak{b} \leqslant \infty)$ be a finite or infinite interval and $\alpha > 0$, $\mathfrak{u}$ be an integrable function defined on $\mathfrak{J}$ and $\psi \in C^1(\mathfrak{J},\mathfrak{R})$ be an increasing and positive function such that $\psi'(\tau) \neq 0$ for all $\tau \in \mathfrak{J}$. Fractional integrals and fractional derivatives of a function $\mathfrak{u}$ with respect to another function ψ are defined as follows:

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) := \frac{1}{\Gamma(\alpha)}\int\limits_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\mathfrak{u}(s)\,ds$$

and

$$\begin{aligned}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) &:= \Big(\frac{1}{\psi'(\tau)}\,\frac{1}{d\tau}\Big)^n \mathfrak{I}_{\mathfrak{a}^+}^{n-\alpha;\psi}\mathfrak{u}(\tau)\\ &= \frac{1}{\Gamma(n-\alpha)}\Big(\frac{1}{\psi'(\tau)}\,\frac{1}{d\tau}\Big)^n \int\limits_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{n-\alpha-1}\mathfrak{u}(s)\,ds,\end{aligned}$$

respectively, where $n=[\alpha]+1$.

Lemma 2.1 ([4])**.** *Let $\alpha > 0$ and $\beta > 0$. Then we have*

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{I}_{\mathfrak{a}^+}^{\beta;\psi}\mathfrak{u}(\tau) = \mathfrak{I}_{\mathfrak{a}^+}^{\alpha+\beta;\psi}\mathfrak{u}(\tau) \ \text{ for all } \ \tau \in \mathfrak{J}.$$

Lemma 2.2 ([18])**.** *Let $\alpha > 0$, $\rho > 0$ and $\tau \in \mathfrak{J}$. If $\mathfrak{u}(\tau) = (\psi(\tau)-\psi(a))^{\rho-1}$, then*

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\delta)}\,(\psi(\tau)-\psi(\mathfrak{a}))^{\alpha+\rho-1}.$$

Definition 2.2 ([4])**.** Let $\mathfrak{n}-1 < \alpha < \mathfrak{n}$ with $\mathfrak{n} \in \mathbb{N}$ and $\mathfrak{u}, \psi \in C^n(\mathfrak{J},\mathfrak{R})$ be two functions such that ψ is increasing and positive where $\psi'(\tau) \neq 0$ for any $\tau \in \mathfrak{J}$. The left ψ-Caputo fractional derivative of $\mathfrak{u}$ of order α is given by

$${}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) := \mathfrak{I}_{\mathfrak{a}^+}^{\mathfrak{n}-\alpha;\psi}\Big(\frac{1}{\psi'(\tau)}\,\frac{d}{d\tau}\Big)^{\mathfrak{n}}\mathfrak{u}(\tau), \quad \tau \in \mathfrak{J}.$$

In particular, when $0 < \alpha < 1$, we have

$${}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = \frac{1}{\Gamma(1-\alpha)}\int\limits_{\mathfrak{a}}^{\tau} (\psi(\tau)-\psi(s))^{-\alpha}\mathfrak{u}'(s)\,ds, \quad \tau \in \mathfrak{J}.$$

Theorem 2.1 ([4])**.** *If* $\mathfrak{u} \in C^{\mathfrak{n}}(\mathfrak{J}, \mathfrak{R})$ *and* $\mathfrak{n} - 1 < \alpha < \mathfrak{n}$*, then*

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\, {}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = \mathfrak{u}(\tau) - \sum_{k=0}^{\mathfrak{n}-1} \frac{(\psi(\tau) - \psi(\mathfrak{a}))^k}{k!} \Big(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\Big)^k \mathfrak{u}(\mathfrak{a}).$$

In particular, when $0 < \alpha < 1$*, we have*

$$\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\, {}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = \mathfrak{u}(\tau) - \mathfrak{u}(\mathfrak{a}).$$

Theorem 2.2 ([4])**.** *Let* $\mathfrak{u} \in C^1(\mathfrak{J}, \mathfrak{R})$ *and* $\alpha > 0$*. Then*

$${}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = \mathfrak{u}(\tau).$$

Theorem 2.3 ([4])**.** *Let* $\mathfrak{u}, \upsilon \in C^{\mathfrak{n}}(\mathfrak{J}, \mathfrak{R})$ *and* $\alpha > 0$*. Then*

$${}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau) = {}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon(\tau) \Longleftrightarrow \mathfrak{u}(\tau) = \upsilon(\tau) + \sum_{k=0}^{\mathfrak{n}-1} c_k(\psi(\tau) - \psi(\mathfrak{a}))^k,$$

where

$$c_k = \frac{1}{k!}\Big(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\Big)^k(\mathfrak{u} - \upsilon)(\mathfrak{a}).$$

Remark 2.1. Let $\mathfrak{u} \in C^{\mathfrak{n}}(\mathfrak{J}, \mathfrak{R})$ and $\alpha > 0$. Then

$${}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha,\beta;\psi}\mathfrak{u}(\tau) = 0 \Longleftrightarrow \mathfrak{u}(\tau) = \sum_{k=0}^{\mathfrak{n}-1} c_k(\psi(\tau) - \psi(\mathfrak{a}))^k.$$

Below we present definitions and the coincidence degree theory that are essential in proofs of our results (see [15, 20]).

Definition 2.3. We consider the normed spaces $\mathcal{X}$ and $\mathcal{Y}$. A Fredholm operator of index zero is a linear operator $\mathfrak{L} : \mathrm{Dom}(\mathfrak{L}) \subset \mathcal{X} \to \mathcal{Y}$ such that

(a) $\mathfrak{dim}\ker\mathfrak{L} = \mathfrak{codim}\,\mathfrak{Img}\,\mathfrak{L} < +\infty$;

(b) $\mathfrak{Img}\,\mathfrak{L}$ is a closed subset of $\mathcal{Y}$.

By Definition 2.3, there exist continuous projections $\mathcal{Q} : \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ satisfying

$$\mathfrak{Img}\,\mathfrak{L} = \ker\mathcal{Q}, \quad \ker\mathfrak{L} = \mathfrak{Img}\,\mathcal{P}, \quad \mathcal{Y} = \mathfrak{Img}\,\mathcal{Q} \oplus \mathfrak{Img}\,\mathfrak{L}, \quad \mathcal{X} = \ker\mathcal{P} \oplus \ker\mathfrak{L}.$$

Thus the restriction of $\mathfrak{L}$ to $\mathrm{Dom}\,\mathfrak{L} \cap \ker\mathcal{P}$, denoted by $\mathfrak{L}_{\mathcal{P}}$, is an isomorphism onto its image.

Definition 2.4. Let $\Omega \subseteq \mathcal{X}$ be a bounded subset and $\mathfrak{L}$ be a Fredholm operator of index zero with $\mathrm{Dom}\,\mathfrak{L} \cap \Omega \neq \emptyset$. Then an operator $\mathcal{N} : \overline{\Omega} \to \mathcal{Y}$ is said to be $\mathfrak{L}$-compact in $\overline{\Omega}$ if

(a) the mapping $\mathcal{Q}\mathcal{N} : \overline{\Omega} \to \mathcal{Y}$ is continuous and $\mathcal{Q}\mathcal{N}(\overline{\Omega}) \subseteq \mathcal{Y}$ is bounded;

(b) the mapping $(\mathfrak{L}_{\mathcal{P}})^{-1}(id - \mathcal{Q})\mathcal{N} : \overline{\Omega} \to \mathcal{X}$ is completely continuous.

Lemma 2.3 ([21])**.** *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be the Banach spaces,* $\Omega \subset \mathcal{X}$ *be a bounded open and symmetric set with* $0 \in \Omega$*. Suppose that* $\mathfrak{L} : \mathrm{Dom}\,\mathfrak{L} \subset \mathcal{X} \to \mathcal{Y}$ *is a Fredholm operator of index zero with* $\mathrm{Dom}\,\mathfrak{L} \cap \overline{\Omega} \neq \varnothing$ *and* $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ *is an* $\mathfrak{L}$*-compact operator on* $\overline{\Omega}$*. Assume, moreover, that*

$$\mathfrak{L}x - \mathcal{N}x \neq -\zeta(\mathfrak{L}x + \mathcal{N}(-x))$$

for any $x \in \mathrm{Dom}\,\mathfrak{L} \cap \partial\Omega$ *and any* $\zeta \in (0, 1]$*, where* $\partial\Omega$ *is the boundary of* Ω *with respect to* $\mathcal{X}$*. Then there exists at least one solution of the equation* $\mathfrak{L}x = \mathcal{N}x$ *on* $\mathrm{Dom}\,\mathfrak{L} \cap \overline{\Omega}$*.*

3 Main results

Let the spaces

$$\mathcal{X} = \Big\{\mathfrak{u} \in C(\mathfrak{J}, \mathfrak{R}) : \ \mathfrak{u}(\tau) = \mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+} \upsilon(\tau) : \ \upsilon \in C(\mathfrak{J}, \mathfrak{R})\Big\} \ \text{ and } \ \mathcal{Y} = C(\mathfrak{J}, \mathfrak{R})$$

be endowed with the norms

$$\|\mathfrak{u}\|_{\mathcal{X}} = \|\mathfrak{u}\|_{\mathcal{Y}} = \|\mathfrak{u}\|_{\infty}.$$

We consider the operator $\mathfrak{L} : \operatorname{Dom} \mathfrak{L} \subseteq \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathfrak{L}\mathfrak{u} := {}^{c}\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+} \mathfrak{u}, \tag{3.1}$$

where

$$\operatorname{Dom} \mathfrak{L} = \big\{\mathfrak{u} \in \mathcal{X} : \ {}^{c}\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+} \mathfrak{u} \in \mathcal{Y} : \mathfrak{u}(\mathfrak{a}) = \mathfrak{u}(\mathfrak{b})\big\}.$$

Lemma 3.1. *Using the definition of* $\mathfrak{L}$ *given in* (3.1)*, we have*

$$\ker \mathfrak{L} = \big\{\mathfrak{u} \in \mathcal{X} : \ \mathfrak{u}(\tau) = \mathfrak{u}(\mathfrak{a}), \ \tau \in \mathfrak{J}\big\}$$

and

$$\mathfrak{Img}\, \mathfrak{L} = \Bigg\{\upsilon \in \mathcal{Y} : \ \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \upsilon(s)\, ds = 0\Bigg\}.$$

Proof. By Remark 2.1, we find that for all $\mathfrak{u} \in \mathcal{X}$, the equation $\mathfrak{L}\mathfrak{u} = {}^{c}\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+} \mathfrak{u} = 0$ in $\mathfrak{J}$ has a solution of the form

$$\mathfrak{u}(\tau) = c_0 = \mathfrak{u}(\mathfrak{a}), \ \ \tau \in \mathfrak{J},$$

and then

$$\ker \mathfrak{L} = \big\{\mathfrak{u} \in \mathcal{X} : \ \mathfrak{u}(\tau) = \mathfrak{u}(\mathfrak{a}), \ \tau \in \mathfrak{J}\big\}.$$

For $\upsilon \in \mathfrak{Img}\, \mathfrak{L}$, there exists $\mathfrak{u} \in \operatorname{Dom} \mathfrak{L}$ such that $\upsilon = \mathfrak{L}\mathfrak{u} \in \mathcal{Y}$. Using Theorem 2.1, we obtain

$$\begin{aligned} \mathfrak{u}(\tau) &= \mathfrak{u}(\mathfrak{a}) + \mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+} \upsilon(\tau) \\ &= \mathfrak{u}(\mathfrak{a}) + \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1} \upsilon(s)\, ds \end{aligned}$$

for every $\tau \in \mathfrak{J}$. Since $\mathfrak{u} \in \operatorname{Dom} \mathfrak{L}$, we have $\mathfrak{u}(\mathfrak{a}) = \mathfrak{u}(\mathfrak{b})$. Thus

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \upsilon(s)\, ds = 0.$$

Furthermore, if $\upsilon \in \mathcal{Y}$ and satisfies

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \upsilon(s)\, ds = 0,$$

then for any $\mathfrak{u}(\tau) = \mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+} \upsilon(\tau)$, using Theorem 2.2, we get $\upsilon(\tau) = {}^{c}\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+} \mathfrak{u}(\tau)$. Therefore,

$$\mathfrak{u}(\mathfrak{b}) = \mathfrak{u}(\mathfrak{a}),$$

which implies that $\mathfrak{u} \in \operatorname{Dom} \mathfrak{L}$. So, $\upsilon \in \mathfrak{Img}\, \mathfrak{L}$. Thus

$$\mathfrak{Img}\, \mathfrak{L} = \Bigg\{\upsilon \in \mathcal{Y} : \ \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \upsilon(s)\, ds = 0\Bigg\},$$

which completes the proof. □

Lemma 3.2. *Let $\mathfrak{L}$ be the operator defined by* (3.1). *Then $\mathfrak{L}$ is a Fredholm operator of index zero, and the linear continuous projection operators $\mathcal{Q}:\mathcal{Y}\to\mathcal{Y}$ and $\mathcal{P}:\mathcal{X}\to\mathcal{X}$ can be written as*

$$\mathcal{Q}(\upsilon)=\frac{\alpha}{(\psi(\mathfrak{b})-\psi(a))^{\alpha}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\upsilon(s)\,ds$$

and

$$\mathcal{P}(\mathfrak{u})=\mathfrak{u}(\mathfrak{a}).$$

Furthermore, the operator $\mathfrak{L}_{\mathcal{P}}^{-1}:\mathfrak{Img}\,\mathfrak{L}\to\mathcal{X}\cap\ker\mathcal{P}$ can be written as

$$\mathfrak{L}_{\mathcal{P}}^{-1}(\upsilon)(\tau)=\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon(\tau),\quad \tau\in\mathfrak{J}.$$

Proof. Obviously, for each $\upsilon\in\mathcal{Y}$,

$$\mathcal{Q}^2\upsilon=\mathcal{Q}\upsilon\ \text{ and }\ \upsilon=\mathcal{Q}(\upsilon)+(\upsilon-\mathcal{Q}(\upsilon)),$$

where $(\upsilon-\mathcal{Q}(\upsilon))\in\ker\mathcal{Q}=\mathfrak{Img}\,\mathfrak{L}$.

Using the fact that $\mathfrak{Img}\,\mathfrak{L}=\ker\mathcal{Q}$ and $\mathcal{Q}^2=\mathcal{Q}$, it follows that $\mathfrak{Img}\,\mathcal{Q}\cap\mathfrak{Img}\,\mathfrak{L}=0$. So,

$$\mathcal{Y}=\mathfrak{Img}\,\mathfrak{L}\oplus\mathfrak{Img}\,\mathcal{Q}.$$

In the same way, we have $\mathfrak{Img}\,\mathcal{P}=\ker\mathfrak{L}$ and $\mathcal{P}^2=\mathcal{P}$. It follows for each $\mathfrak{u}\in\mathcal{X}$ that $\mathfrak{u}=(\mathfrak{u}-\mathcal{P}(\mathfrak{u}))+\mathcal{P}(\mathfrak{u})$, and then $\mathcal{X}=\ker\mathcal{P}+\ker\mathfrak{L}$. Clearly, we have $\ker\mathcal{P}\cap\ker\mathfrak{L}=0$. So,

$$\mathcal{X}=\ker\mathcal{P}\oplus\ker\mathfrak{L}.$$

Therefore,

$$\mathfrak{dim}\ker\mathfrak{L}=\mathfrak{dim}\,\mathfrak{Img}\,\mathcal{Q}=\mathfrak{codim}\,\mathfrak{Img}\,\mathfrak{L}.$$

Consequently, $\mathfrak{L}$ is a Fredholm operator of index zero.

Now, we show that the inverse of $\mathfrak{L}|_{\mathrm{Dom}\,\mathfrak{L}\cap\ker\mathcal{P}}$ is $\mathfrak{L}_{\mathcal{P}}^{-1}$. Effectively, for $\upsilon\in\mathfrak{Img}\,\mathfrak{L}$, by Theorem 2.2, we have

$$\mathfrak{L}\mathfrak{L}_{\mathcal{P}}^{-1}(\upsilon)={}^{c}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\big(\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\upsilon\big)=\upsilon. \tag{3.2}$$

Furthermore, for $\mathfrak{u}\in\mathrm{Dom}\,\mathfrak{L}\cap\ker\mathcal{P}$, we get

$$\mathfrak{L}_{\mathcal{P}}^{-1}(\mathfrak{L}(\mathfrak{u}(\tau)))=\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\big({}^{c}\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau)\big)=\mathfrak{u}(\tau)-\mathfrak{u}(\mathfrak{a}),\quad \tau\in\mathfrak{J}.$$

Using the fact that $\mathfrak{u}\in\mathrm{Dom}\,\mathfrak{L}\cap\ker\mathcal{P}$, we have

$$\mathfrak{u}(\mathfrak{a})=0.$$

Thus

$$\mathfrak{L}_{\mathcal{P}}^{-1}\mathfrak{L}(\mathfrak{u})=\mathfrak{u}. \tag{3.3}$$

Using (3.2) and (3.3) together, we get

$$\mathfrak{L}_{\mathcal{P}}^{-1}=(\mathfrak{L}|_{\mathrm{Dom}\,\mathfrak{L}\cap\ker\mathcal{P}})^{-1},$$

which completes the demonstration. □

Let us introduce the following hypotheses:

(A1) There exist positive constants γ, η_1, η_2 such that

$$\big|\mathfrak{h}(\tau,\mathfrak{u},\mathfrak{B}_1(\mathfrak{u}),\mathfrak{B}_2(\mathfrak{u}))-\mathfrak{h}(\tau,\overline{\mathfrak{u}},\mathfrak{B}_1(\overline{\mathfrak{u}}),\mathfrak{B}_2(\overline{\mathfrak{u}}))\big|\leqslant\gamma|\mathfrak{u}-\overline{\mathfrak{u}}|+\eta_1|\mathfrak{B}_1\mathfrak{u}-\mathfrak{B}_1\overline{\mathfrak{u}}|+\eta_2\big|\mathfrak{B}_2\mathfrak{u}-\mathfrak{B}_2\overline{\mathfrak{u}}\big|$$

for every $\tau\in\mathfrak{J}$ and $\mathfrak{u},\overline{\mathfrak{u}}\in\mathfrak{R}$.

(A2) There exists a constant $\rho_1 > 0$ such that

$$\left|\kappa_1(\tau, s, \upsilon) - \kappa_1(\tau, s, \overline{\upsilon})\right| \leqslant \rho_1 |\upsilon - \overline{\upsilon}|$$

for every $(\tau, s) \in \mathfrak{d}$ and $\upsilon, \overline{\upsilon} \in \mathfrak{R}$.

(A3) There exists a constant $\rho_2 > 0$ such that

$$\left|\kappa_2(\tau, s, \upsilon) - \kappa_2(\tau, s, \overline{\upsilon})\right| \leqslant \rho_2 |\upsilon - \overline{\upsilon}|$$

for every $(\tau, s) \in \mathfrak{d}_0$ and $\upsilon, \overline{\upsilon} \in \mathfrak{R}$.

Define $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{N}\mathfrak{u}(\tau) := \mathfrak{h}\left(\tau, \mathfrak{u}(\tau), \int_{\mathfrak{a}}^{\tau} \kappa_1(\tau, s, \mathfrak{u}(s))\, ds, \int_{\mathfrak{a}}^{\mathfrak{b}} \kappa_2(\tau, s, \mathfrak{u}(s))\, ds\right), \quad \tau \in \mathfrak{J}.$$

The operator $\mathcal{N}$ is well defined because $\mathfrak{h}$, κ_1 and κ_2 are continuous functions.

We remark that problem (1.1)-(1.2) is equivalent to the problem $\mathfrak{L}\mathfrak{u} = \mathcal{N}\mathfrak{u}$.

Lemma 3.3. *Suppose that* (A1)*,* (A2) *and* (A3) *are satisfied. Then for any bounded open set* $\Omega \subset \mathcal{X}$*, the operator* $\mathcal{N}$ *is* $\mathfrak{L}$*-compact.*

Proof. We consider for $\mathcal{M} > 0$ the bounded open set $\Omega = \{\mathfrak{u} \in \mathcal{X} : \|\mathfrak{u}\|_{\mathcal{X}} < \mathcal{M}\}$. We divide the proof into three steps.

Step 1: $\mathcal{Q}\mathcal{N}$ is continuous.

Let $(\mathfrak{u}_{\mathfrak{n}})_{\mathfrak{n}\in\mathbb{N}}$ be a sequence such that $\mathfrak{u}_{\mathfrak{n}} \longrightarrow \mathfrak{u}$ in $\mathcal{Y}$. Then for each $\tau \in \mathfrak{J}$, we have

$$\left|\mathcal{Q}\mathcal{N}(\mathfrak{u}_{\mathfrak{n}})(\tau) - \mathcal{Q}\mathcal{N}(\mathfrak{u})(\tau)\right| \leqslant \frac{\alpha}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \left|\mathcal{N}(\mathfrak{u}_{\mathfrak{n}})(s) - \mathcal{N}(\mathfrak{u})(s)\right| ds.$$

By (A1), we have

$$\begin{aligned}
&\left|\mathcal{Q}\mathcal{N}(\mathfrak{u}_{\mathfrak{n}})(\tau) - \mathcal{Q}\mathcal{N}(\mathfrak{u})(\tau)\right| \\
&\leqslant \frac{\alpha\gamma}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} |\mathfrak{u}_{\mathfrak{n}}(s) - \mathfrak{u}(s)|\, ds \\
&\quad + \frac{\alpha\eta_1}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \left|\mathfrak{B}_1(\mathfrak{u}_{\mathfrak{n}})(s) - \mathfrak{B}_1(\mathfrak{u})(s)\right| ds \\
&\quad + \frac{\alpha\eta_2}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \left|\mathfrak{B}_2(\mathfrak{u}_{\mathfrak{n}})(s) - \mathfrak{B}_2(\mathfrak{u})(s)\right| ds \\
&\leqslant \frac{\alpha\gamma\|\mathfrak{u}_{\mathfrak{n}} - \mathfrak{u}\|_{\mathcal{Y}}}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1}\, ds \\
&\quad + \frac{\alpha\eta_1}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \left|\mathfrak{B}_1(\mathfrak{u}_{\mathfrak{n}})(s) - \mathfrak{B}_1(\mathfrak{u})(s)\right| ds \\
&\quad + \frac{\alpha\eta_2}{(\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha}} \int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \left|\mathfrak{B}_2(\mathfrak{u}_{\mathfrak{n}})(s) - \mathfrak{B}_2(\mathfrak{u})(s)\right| ds.
\end{aligned}$$

Using (A2) and (A3), we get

$$\begin{aligned}\big|\mathcal{QN}(\mathfrak{u}_\mathfrak{n})(\tau)-\mathcal{QN}(\mathfrak{u})(\tau)\big| &\leqslant \frac{\alpha\gamma\|\mathfrak{u}_\mathfrak{n}-\mathfrak{u}\|_\mathcal{Y}}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\,ds\\ &\quad+\frac{\alpha(b-a)(\eta_1\rho_1+\eta_2\rho_2)\|\mathfrak{u}_\mathfrak{n}-\mathfrak{u}\|_\mathcal{Y}}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\,ds\\ &\leqslant \big[\gamma+(b-a)(\eta_1\rho_1+\eta_2\rho_2)\big]\|\mathfrak{u}_\mathfrak{n}-\mathfrak{u}\|_\mathcal{Y}.\end{aligned}$$

Thus, for each $\tau\in\mathfrak{J}$, we obtain

$$\big|\mathcal{QN}(\mathfrak{u}_\mathfrak{n})(\tau)-\mathcal{QN}(\mathfrak{u})(\tau)\big|\longrightarrow 0 \ \text{ as } \ n\to+\infty,$$

and, therefore,

$$\big\|\mathcal{QN}(\mathfrak{u}_\mathfrak{n})-\mathcal{QN}(\mathfrak{u})\big\|_\mathcal{Y}\longrightarrow 0 \ \text{ as } \ n\to+\infty.$$

We conclude that $\mathcal{QN}$ is continuous.

Step 2: $\mathcal{QN}(\overline{\Omega})$ is bounded.

For $\tau\in\mathfrak{J}$ and $\mathfrak{u}\in\overline{\Omega}$, we have

$$\begin{aligned}|\mathcal{QN}(\mathfrak{u})(\tau)| &\leqslant \frac{\alpha}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathcal{N}(\mathfrak{u})(s)|\,ds\\ &\leqslant \frac{\alpha}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\big|\mathfrak{h}(s,\mathfrak{u}(s),\mathfrak{B}_1(\mathfrak{u})(s),\mathfrak{B}_2(\mathfrak{u})(s))-\mathfrak{h}(s,0,0,0)\big|\,ds\\ &\quad+\frac{\alpha}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds\\ &\leqslant \mathfrak{h}^*+\frac{\alpha\gamma}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,ds\\ &\quad+\frac{\alpha\eta_1}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_1(\mathfrak{u})(s)-\mathfrak{B}_1(0)(s)|\,ds\\ &\quad+\frac{\alpha\eta_1}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_1(0)(s)|\,ds\\ &\quad+\frac{\alpha\eta_2}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_2(\mathfrak{u})(s)-\mathfrak{B}_2(0)(s)|\,ds\\ &\quad+\frac{\alpha\eta_2}{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_\mathfrak{a}^\mathfrak{b}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_2(0)(s)|\,ds\\ &\leqslant \mathfrak{h}^*+\gamma\mathcal{M}+(\mathfrak{b}-\mathfrak{a})\big[(\rho_1\eta_1+\rho_2\eta_2)\mathcal{M}+\kappa_1^*\eta_1+\kappa_2^*\eta_2\big],\end{aligned}$$

where

$$\mathfrak{h}^*=\|\mathfrak{h}(\,\cdot\,,0,0,0)\|_\infty,\quad \kappa_1^*=\sup_{(\tau,s)\in\mathfrak{d}}|\kappa(\tau,s,0,0)|$$

and

$$\kappa_2^*=\sup_{(\tau,s)\in\mathfrak{d}_0}|\kappa(\tau,s,0,0)|.$$

Thus

$$\|\mathcal{Q}\mathcal{N}(\mathfrak{u})\|_{\mathcal{Y}} \leqslant \mathfrak{h}^* + \gamma\mathcal{M} + (\mathfrak{b}-\mathfrak{a})\big[(\rho_1\eta_1 + \rho_2\eta_2)\mathcal{M} + \kappa_1^*\eta_1 + \kappa_2^*\eta_2\big].$$

So, $\mathcal{Q}\mathcal{N}(\overline{\Omega})$ is a bounded set in $\mathcal{Y}$.

Step 3: $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N} : \overline{\Omega} \to \mathcal{X}$ ***is completely continuous.***

We use the Arzelà–Ascoli theorem, so we have to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega}) \subset \mathcal{X}$ is equicontinuous and bounded. First, for any $\mathfrak{u} \in \overline{\Omega}$ and $\tau \in \mathfrak{J}$, we get

$$\begin{aligned}
\mathfrak{L}_{\mathcal{P}}^{-1}(\mathcal{N}\mathfrak{u}(\tau) - \mathcal{Q}\mathcal{N}\mathfrak{u}(\tau)) &= \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\Bigg[\mathfrak{h}\big(\tau,\mathfrak{u}(\tau),\mathfrak{B}_1\mathfrak{u}(\tau),\mathfrak{B}_2\mathfrak{u}(\tau)\big) \\
&\quad - \frac{\alpha}{(\psi(\mathfrak{b})-\psi(a))^{\alpha}}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)\,ds\Bigg] \\
&= \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\mathfrak{h}(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s))\,ds \\
&\quad - \frac{(\psi(\tau)-\psi(a))^{\alpha}}{\Gamma(\alpha)(\psi(\mathfrak{b})-\psi(a))^{\alpha}}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\mathfrak{h}(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s))\,ds.
\end{aligned}$$

Next, for all $\mathfrak{u} \in \overline{\Omega}$ and $\tau \in \mathfrak{J}$, we get

$$\begin{aligned}
&\big|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\mathfrak{u}(\tau)\big| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big) - \mathfrak{h}(s,0,0,0)\big|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big) - \mathfrak{h}(s,0,0,0)\big|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds, \\
&\leqslant \frac{2\mathfrak{h}^*}{\alpha\Gamma(\alpha)}(\psi(\mathfrak{b})-\psi(a))^{\alpha} + \frac{\gamma}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,ds \\
&\quad + \frac{\eta_1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{B}_1(\mathfrak{u})(s) - \mathfrak{B}_1(0)(s)|\,ds \\
&\quad + \frac{\eta_1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{B}_1(0)(s)|\,ds \\
&\quad + \frac{\eta_2}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{B}_2(\mathfrak{u})(s) - \mathfrak{B}_2(0)(s)|\,ds \\
&\quad + \frac{\eta_2}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{B}_2(0)(s)|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,ds\\
&+\frac{\eta_1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_1(\mathfrak{u})(s)-\mathfrak{B}_1(0)(s)|\,ds\\
&+\frac{\eta_1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_1(0)(s)|\,ds\\
&+\frac{\eta_2}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_2(\mathfrak{u})(s)-\mathfrak{B}_2(0)(s)|\,ds\\
&+\frac{\eta_2}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{B}_2(0)(s)|\,ds\\
\leqslant\ &\frac{2(\psi(\mathfrak{b})-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Big[\mathfrak{h}^*+\gamma\mathcal{M}+(\mathfrak{b}-\mathfrak{a})\big[\kappa_1^*\eta_1+\kappa_2^*\eta_2+(\eta_1\rho_1+\eta_2\rho_2)\mathcal{M}\big]\Big].
\end{aligned}
$$

Therefore,

$$
\|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\mathfrak{u}\|_{\mathcal{X}}\leqslant\frac{2(\psi(\mathfrak{b})-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Big[\mathfrak{h}^*+\gamma\mathcal{M}+(\mathfrak{b}-\mathfrak{a})\big[\kappa_1^*\eta_1+\kappa_2^*\eta_2+(\eta_1\rho_1+\eta_2\rho_2)\mathcal{M}\big]\Big].
$$

This means that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is uniformly bounded in $\mathcal{X}$.

It remains to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is equicontinuous.

For $\mathfrak{a}\leqslant\tau_1<\tau_2\leqslant\mathfrak{b},\mathfrak{u}\in\overline{\Omega}$, we have

$$
\begin{aligned}
&\big|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\mathfrak{u}(\tau_2)-\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\mathfrak{u}(\tau_1)\big|\\
&\leqslant\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau_1}\Big[\psi'(s)\big|(\psi(\tau_2)-\psi(s)\big)^{\alpha-1}-(\psi(\tau_1)-\psi(s))^{\alpha-1}\big|\,\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)\big|\Big]\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)\big|\,ds\\
&+\frac{[(\psi(\tau_2)-\psi(\mathfrak{a}))^{\alpha}-(\psi(\tau_1)-\psi(\mathfrak{a}))^{\alpha}]}{\Gamma(\alpha)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)\big|\,ds\\
&\leqslant\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau_1}\psi'(s)(\psi(\tau_1)-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)-\mathfrak{h}(s,0,0,0)\big|\,ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau_1}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)-\mathfrak{h}(s,0,0,0)\big|\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau_1}\psi'(s)\big[(\psi(\tau_1)-\psi(s))^{\alpha-1}-(\psi(\tau_2)-\psi(s))^{\alpha-1}\big]|\mathfrak{h}(s,0,0,0)|\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)-\mathfrak{h}(s,0,0,0)\big|\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds
\end{aligned}
$$

$$+\frac{[(\psi(\tau_2)-\psi(\mathfrak{a}))^\alpha-(\psi(\tau_1)-\psi(\mathfrak{a}))^\alpha]}{\Gamma(\alpha)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}$$
$$\times\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}\left|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)-\mathfrak{h}(s,0,0,0)\right|ds$$
$$+\frac{[(\psi(\tau_2)-\psi(\mathfrak{a}))^\alpha-(\psi(\tau_1)-\psi(\mathfrak{a}))^\alpha]}{\Gamma(\alpha)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}\int_{\mathfrak{a}}^{\mathfrak{b}}\psi'(s)(\psi(\mathfrak{b})-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds$$
$$\leqslant 2\Lambda(\psi(\tau_2)-\psi(\tau_1))^\alpha+\Lambda\big[(\psi(\tau_1)-\psi(a))^\alpha-(\psi(\tau_2)-\psi(a))^\alpha\big]$$
$$+\Lambda\big[(\psi(\tau_2)-\psi(a))^\alpha-(\psi(\tau_1)-\psi(a))^\alpha\big]=2\Lambda(\psi(\tau_2)-\psi(\tau_1))^\alpha,$$

where

$$\Lambda:=\frac{1}{\Gamma(\alpha+1)}\Big[\mathfrak{h}^*+\gamma\mathcal{M}+(\mathfrak{b}-\mathfrak{a})\big[\kappa_1^*\eta_1+\kappa_2^*\eta_2+(\eta_1\rho_1+\eta_2\rho_2)\mathcal{M}\big]\Big].$$

The operator $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is equicontinuous in $\mathcal{X}$ because the right-hand side of the above inequality tends to zero as $\tau_1\to\tau_2$ and the limit is independent of $\mathfrak{u}$. The Arzelà–Ascoli theorem implies that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is relatively compact in $\mathcal{X}$. As a consequence of Steps 1 to 3, we get that $\mathcal{N}$ is $\mathfrak{L}$-compact in $\overline{\Omega}$, which completes the demonstration. □

Lemma 3.4. *Assume that* (A1), (A2) *and* (A3) *are fulfilled. If the condition*

$$\frac{\gamma+(\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a})}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha<\frac{1}{2}\tag{3.4}$$

is satisfied, then there exists $\mathcal{A}>0$ independent of ζ such that

$$\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})=-\zeta\big[\mathfrak{L}(\mathfrak{u})+\mathcal{N}(-\mathfrak{u})\big]\Longrightarrow\|\mathfrak{u}\|_{\mathcal{X}}\leqslant\mathcal{A},\quad\zeta\in(0,1].$$

Proof. Let $\mathfrak{u}\in\mathcal{X}$ satisfy

$$\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})=-\zeta\mathfrak{L}(\mathfrak{u})-\zeta\mathcal{N}(-\mathfrak{u}),$$

then

$$\mathfrak{L}(\mathfrak{u})=\frac{1}{1+\zeta}\mathcal{N}(\mathfrak{u})-\frac{\zeta}{1+\zeta}\mathcal{N}(-\mathfrak{u}).$$

So, from the expressions for $\mathfrak{L}$ and $\mathcal{N}$, for any $\tau\in\mathfrak{J}$ we get

$$\mathfrak{L}\mathfrak{u}(\tau)={}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{u}(\tau)=\frac{1}{1+\zeta}\mathfrak{h}\big(\tau,\mathfrak{u}(\tau),\mathfrak{B}_1\mathfrak{u}(\tau),\mathfrak{B}_2\mathfrak{u}(\tau)\big)-\frac{\zeta}{1+\zeta}\mathfrak{h}\big(\tau,-\mathfrak{u}(\tau),\mathfrak{B}_1(-\mathfrak{u})(\tau),\mathfrak{B}_2(-\mathfrak{u})(\tau)\big).$$

By Theorem 2.1, we have

$$\mathfrak{u}(\tau)=\mathfrak{u}(\mathfrak{a})+\frac{1}{\zeta+1}\Big[\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\big(\mathfrak{h}(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s))\big)(\tau)$$
$$-\zeta\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}\big(\mathfrak{h}(s,-\mathfrak{u}(s),\mathfrak{B}_1(-\mathfrak{u})(s),\mathfrak{B}_2(-\mathfrak{u})(s))\big)(\tau)\Big].$$

Thus, for every $\tau\in\mathfrak{J}$, we obtain

$$|\mathfrak{u}(\tau)|\leqslant|\mathfrak{u}(\mathfrak{a})|+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\left|\mathfrak{h}(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s))\right|ds$$
$$+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\left|\mathfrak{h}\big(s,-\mathfrak{u}(s),\mathfrak{B}_1(-\mathfrak{u})(s),\mathfrak{B}_2(-\mathfrak{u})(s)\big)\right|ds$$
$$\leqslant|\mathfrak{u}(\mathfrak{a})|+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\left|\mathfrak{h}\big(s,\mathfrak{u}(s),\mathfrak{B}_1\mathfrak{u}(s),\mathfrak{B}_2\mathfrak{u}(s)\big)-\mathfrak{h}(s,0,0,0)\right|ds$$

$$+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int\limits_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds$$

$$+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int\limits_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\big|\mathfrak{h}\big(s,-\mathfrak{u}(s),\mathfrak{B}_1(-\mathfrak{u})(s),\mathfrak{B}_2(-\mathfrak{u})(s)\big)-\mathfrak{h}(s,0,0,0)\big|\,ds$$

$$+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int\limits_{\mathfrak{a}}^{\tau}\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{h}(s,0,0,0)|\,ds$$

$$\leqslant|\mathfrak{u}(\mathfrak{a})|+\frac{2(\mathfrak{h}^*+(\eta_1\kappa_1^*+\eta_2\kappa_2^*)(\mathfrak{b}-\mathfrak{a}))}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}$$
$$+\frac{2(\gamma+(\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a}))}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}\|\mathfrak{u}\|_{\mathcal{X}},$$

hence

$$\|\mathfrak{u}\|_{\mathcal{X}}\leqslant|\mathfrak{u}(\mathfrak{a})|+\frac{2(\mathfrak{h}^*+(\eta_1\kappa_1^*+\eta_2\kappa_2^*)(\mathfrak{b}-\mathfrak{a}))}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}$$
$$+\frac{2(\gamma+(\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a}))}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}\|\mathfrak{u}\|_{\mathcal{X}}.$$

We deduce that

$$\|\mathfrak{u}\|_{\mathcal{X}}\leqslant\frac{|\mathfrak{u}(\mathfrak{a})|+\frac{2(\mathfrak{h}^*+(\eta_1\kappa_1^*+\eta_2\kappa_2^*)(\mathfrak{b}-\mathfrak{a}))}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}}{\big[1-\frac{2(\gamma+(\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a}))}{\Gamma(\alpha+1)}(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}\big]}:=\mathcal{A}.$$

The demonstration is completed. □

Lemma 3.5. *If conditions* (A1), (A2), (A3) *and* (3.4) *are satisfied, then there exists a bounded open set* $\Omega\subset\mathcal{X}$ *with*

$$\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})\neq-\zeta[\mathfrak{L}(\mathfrak{u})+\mathcal{N}(-\mathfrak{u})] \tag{3.5}$$

for any $\mathfrak{u}\in\partial\Omega$ *and any* $\zeta\in(0,1]$.

Proof. Using Lemma 3.4, there exists a positive constant $\mathcal{A}$ independent of ζ such that if $\mathfrak{u}$ satisfies

$$\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})=-\zeta\big[\mathfrak{L}(\mathfrak{u})+\mathcal{N}(-\mathfrak{u})\big],\ \ \zeta\in(0,1],$$

then $\|\mathfrak{u}\|_{\mathcal{X}}\leqslant\mathcal{A}$. So, if

$$\Omega=\big\{\mathfrak{u}\in\mathcal{X};\ \|\mathfrak{u}\|_{\mathcal{X}}<\vartheta\big\} \tag{3.6}$$

such that $\vartheta>\mathcal{A}$, we deduce that

$$\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})\neq-\zeta\big[\mathfrak{L}(\mathfrak{u})-\mathcal{N}(-\mathfrak{u})\big]$$

for all $\mathfrak{u}\in\partial\Omega=\{\mathfrak{u}\in\mathcal{X};\|\mathfrak{u}\|_{\mathcal{X}}=\vartheta\}$ and $\zeta\in(0,1]$. □

Theorem 3.1. *Assume* (A1), (A2), (A3) *and* (3.4) *are satisfied. Then there exists at least one solution of problem* (1.1), (1.2).

Proof. It is clear that the set Ω defined in (3.6) is symmetric, $0\in\Omega$ and $\mathcal{X}\cap\overline{\Omega}=\overline{\Omega}\neq\varnothing$. In addition, by assuming (A1), (A2), (A3) and (3.4), Lemma 3.5 implies

$$\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})\neq-\zeta\big[\mathfrak{L}(\mathfrak{u})-\mathcal{N}(-\mathfrak{u})\big]$$

for each $\mathfrak{u}\in\mathcal{X}\cap\partial\Omega=\partial\Omega$ and each $\zeta\in(0,1]$. By Lemma 2.3, problem (1.1), (1.2) has at least one solution on $\operatorname{Dom}\mathfrak{L}\cap\overline{\Omega}$, which completes the proof. □

Now, we investigate the existence and uniqueness of periodic solutions for our problem (1.1), (1.2).

Theorem 3.2. *Let* (A1), (A2) *and* (A3) *be satisfied. Moreover, assume that*

(A4) *there exist the constants* $\overline{\gamma} > 0$ *and* $\overline{\eta}_1, \overline{\eta}_2 \geqslant 0$ *such that*

$$\left|\mathfrak{h}(\tau, \mathfrak{u}, \mathfrak{B}_1(\mathfrak{u}), \mathfrak{B}_2(\mathfrak{u})) - \mathfrak{h}(\tau, \overline{\mathfrak{u}}, \mathfrak{B}_1(\overline{\mathfrak{u}}), \mathfrak{B}_2(\overline{\mathfrak{u}}))\right| \geqslant \overline{\gamma}|\mathfrak{u} - \overline{\mathfrak{u}}| - \overline{\eta}_1|\mathfrak{B}_1\mathfrak{u} - \mathfrak{B}_1\overline{\mathfrak{u}}| - \overline{\eta}_2|\mathfrak{B}_2\mathfrak{u} - \mathfrak{B}_2\overline{\mathfrak{u}}|,$$

for every $\tau \in \mathfrak{J}$ *and* $\mathfrak{u}, \overline{\mathfrak{u}} \in \mathfrak{R}$.

If

$$\frac{(\overline{\eta}_1\rho_1 + \overline{\eta}_2\rho_2)(\mathfrak{b} - \mathfrak{a})}{\overline{\gamma}} + \frac{2[\gamma + (\eta_1\rho_1 + \eta_2\rho_2)(\mathfrak{b} - \mathfrak{a})]}{\Gamma(\alpha + 1)} (\psi(\mathfrak{b}) - \psi(\mathfrak{a}))^{\alpha} < 1, \tag{3.7}$$

then problem (1.1), (1.2) *has a unique solution in* $\operatorname{Dom}\mathfrak{L} \cap \overline{\Omega}$.

Proof. Note that condition (3.7) is stronger than condition (3.4). Then, by Theorem 3.1, we find that problem (1.1), (1.2) has at least one solution in $\operatorname{Dom}\mathfrak{L} \cap \overline{\Omega}$.

Now, we prove the uniqueness result. Suppose that problem (1.1), (1.2) has two solutions $\mathfrak{u}_1, \mathfrak{u}_2 \in \operatorname{Dom}\mathfrak{L} \cap \overline{\Omega}$. Then for each $\tau \in \mathfrak{J}$, we have

$$\begin{aligned}
{}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}_1(\tau) &= \mathfrak{h}\big(\tau, \mathfrak{u}_1(\tau), \mathfrak{B}_1(\mathfrak{u}_1)(\tau), \mathfrak{B}_2(\mathfrak{u}_1)(\tau)\big),\\
{}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}_2(\tau) &= \mathfrak{h}\big(\tau, \mathfrak{u}_2(\tau), \mathfrak{B}_1(\mathfrak{u}_2)(\tau), \mathfrak{B}_2(\mathfrak{u}_2)(\tau)\big),
\end{aligned}$$

where $\mathfrak{B}_1$, $\mathfrak{B}_2$ are defined as in (1.3) and

$$\mathfrak{u}_1(\mathfrak{a}) = \mathfrak{u}_1(\mathfrak{b}), \quad \mathfrak{u}_2(\mathfrak{a}) = \mathfrak{u}_2(\mathfrak{b}).$$

Let

$$\mathfrak{U}(\tau) = \mathfrak{u}_1(\tau) - \mathfrak{u}_2(\tau) \ \text{ for all } \ \tau \in \mathfrak{J}.$$

Then

$$\begin{aligned}
\mathfrak{L}\mathfrak{U}(\tau) = {}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{U}(\tau) &= {}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}_1(\tau) - {}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{u}_2(\tau)\\
&= \mathfrak{h}\big(\tau, \mathfrak{u}_1(\tau), \mathfrak{B}_1(\mathfrak{u}_1)(\tau), \mathfrak{B}_2(\mathfrak{u}_1)(\tau)\big) - \mathfrak{h}\big(\tau, \mathfrak{u}_2(\tau), \mathfrak{B}_1(\mathfrak{u}_2)(\tau), \mathfrak{B}_2(\mathfrak{u}_2)(\tau)\big).
\end{aligned} \tag{3.8}$$

Using the fact that $\mathfrak{Img}\,\mathfrak{L} = \ker\mathcal{Q}$, we have

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \psi'(s)(\psi(\mathfrak{b}) - \psi(s))^{\alpha-1} \times \Big[\mathfrak{h}\big(s, \mathfrak{u}_1(s), \mathfrak{B}_1(\mathfrak{u}_1)(s), \mathfrak{B}_2(\mathfrak{u}_1)(s)\big) - \mathfrak{h}\big(s, \mathfrak{u}_2(s), \mathfrak{B}_1(\mathfrak{u}_2)(s), \mathfrak{B}_2(\mathfrak{u}_2)(s)\big)\Big]\, ds = 0.$$

Since $\mathfrak{h}$ is a continuous function, there exists $\tau_0 \in [\mathfrak{a}, \mathfrak{b}]$ such that

$$\mathfrak{h}\big(\tau_0, \mathfrak{u}_1(\tau_0), \mathfrak{B}_1(\mathfrak{u}_1)(\tau_0), \mathfrak{B}_2(\mathfrak{u}_1)(\tau_0)\big) - \mathfrak{h}\big(\tau, \mathfrak{u}_2(\tau_0), \mathfrak{B}_1(\mathfrak{u}_2)(\tau_0), \mathfrak{B}_2(\mathfrak{u}_2)(\tau_0)\big) = 0.$$

In view of (A4), we have

$$\big|\mathfrak{u}_1(\tau_0) - \mathfrak{u}_2(\tau_0)\big| \leqslant \frac{(\mathfrak{b} - \mathfrak{a})(\overline{\eta}_1\rho_1 + \overline{\eta}_2\rho_2)}{\overline{\gamma}} \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathcal{X}},$$

then

$$|\mathfrak{U}(\tau_0)| \leqslant \frac{(\mathfrak{b} - \mathfrak{a})(\overline{\eta}_1\rho_1 + \overline{\eta}_2\rho_2)}{\overline{\gamma}} \|\mathfrak{U}\|_{\mathcal{X}}. \tag{3.9}$$

On the other hand, by Theorem 2.1, we have

$$\mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+}{}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{U}(\tau) = \mathfrak{U}(\tau) - \mathfrak{U}(\mathfrak{a}),$$

which implies that

$$\mathfrak{U}(\mathfrak{a}) = \mathfrak{U}(\tau_0) - \mathfrak{I}^{\alpha;\psi}_{\mathfrak{a}^+}{}^c\mathfrak{D}^{\alpha;\psi}_{\mathfrak{a}^+}\mathfrak{U}(\tau_0)$$

and, therefore,

$$\mathfrak{U}(\tau) = \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}(\tau) + \mathfrak{U}(\tau_0) - \mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}(\tau_0).$$

Using (3.9), for every $\tau \in \mathfrak{J}$ we obtain

$$\begin{aligned}|\mathfrak{U}(\tau)| &\leqslant |\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}(\tau)| + |\mathfrak{U}(\tau_0)| + |\mathfrak{I}_{\mathfrak{a}^+}^{\alpha;\psi}{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}(\tau_0)| \\ &\leqslant \frac{(\mathfrak{b}-\mathfrak{a})(\overline{\eta}_1\rho_1 + \overline{\eta}_2\rho_2)}{\overline{\gamma}}\,\|\mathfrak{U}\|_{\mathcal{X}} + \frac{2(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha}{\Gamma(\alpha+1)}\,\|{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}\|_{\mathcal{X}}.\end{aligned} \tag{3.10}$$

By (A1), (A2), (A3) and (3.8), we find that

$$\begin{aligned}\left|{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}(\tau)\right| &= \Big|\mathfrak{h}\big(\tau,\mathfrak{u}_1(\tau),\mathfrak{B}_1(\mathfrak{u}_1)(\tau),\mathfrak{B}_2(\mathfrak{u}_1)(\tau)\big) - \mathfrak{h}\big(\tau,\mathfrak{u}_2(\tau),\mathfrak{B}_1(\mathfrak{u}_2)(\tau),\mathfrak{B}_2(\mathfrak{u}_2)(\tau)\big)\Big| \\ &\leqslant \big[\gamma + (\mathfrak{b}-\mathfrak{a})(\eta_1\rho_1+\eta_2\rho_2)\big]\,\|\mathfrak{U}\|_{\mathcal{X}}.\end{aligned}$$

Then

$$\|{}^c\mathfrak{D}_{\mathfrak{a}^+}^{\alpha;\psi}\mathfrak{U}\|_{\mathcal{X}} \leqslant \big[\gamma + (\mathfrak{b}-\mathfrak{a})(\eta_1\rho_1+\eta_2\rho_2)\big]\,\|\mathfrak{U}\|_{\mathcal{X}}. \tag{3.11}$$

Substituting (3.11) in the right-hand side of (3.10), for every $\tau \in \mathfrak{J}$ we get

$$|\mathfrak{U}(\tau)| \leqslant \Big[\frac{(\overline{\eta}_1\rho_1 + \overline{\eta}_2\rho_2)(\mathfrak{b}-\mathfrak{a})}{\overline{\gamma}} + \frac{2[\gamma + (\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a})]}{\Gamma(\alpha+1)}\,(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha\Big]\,\|\mathfrak{U}\|_{\mathcal{X}}.$$

Therefore,

$$\|\mathfrak{U}\|_{\mathcal{X}} \leqslant \Big[\frac{(\overline{\eta}_1\rho_1 + \overline{\eta}_2\rho_2)(\mathfrak{b}-\mathfrak{a})}{\overline{\gamma}} + \frac{2[\gamma + (\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a})]}{\Gamma(\alpha+1)}\,(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^\alpha\Big]\,\|\mathfrak{U}\|_{\mathcal{X}}.$$

Hence, by (3.7), we conclude that

$$\|\mathfrak{U}\|_{\mathcal{X}} = 0.$$

As a result, for any $\tau \in \mathfrak{J}$, we get

$$\mathfrak{U}(\tau) = 0 \Longrightarrow \mathfrak{u}_1(\tau) = \mathfrak{u}_2(\tau).$$

This completes the proof. □

4 Examples

In this section, we illustrate the applicability of Theorems 3.1 and 3.2 through two examples.

Example 4.1. Consider the following problem for a nonlinear Volterra–Fredholm integro-differential fractional equation:

$${}^c\mathfrak{D}_{0^+}^{\frac{1}{2};2^\tau}\mathfrak{u}(\tau) = \mathfrak{h}\big(\tau,\mathfrak{u}(\tau),\mathfrak{B}_1\mathfrak{u}(\tau),\mathfrak{B}_2\mathfrak{u}(\tau)\big), \quad \tau \in [0,1], \tag{4.1}$$

$$\mathfrak{u}(0) = \mathfrak{u}(1), \tag{4.2}$$

where

$$\mathfrak{h}\big(\tau,\mathfrak{u}(\tau),\mathfrak{B}_1\mathfrak{u}(\tau),\mathfrak{B}_2\mathfrak{u}(\tau)\big) = e^{\tau+1} + \frac{\tau}{3\sqrt{\pi}}\,\ln\big(|\mathfrak{u}(\tau)|+2\big) + \frac{1}{17}\,\mathfrak{B}_1\mathfrak{u}(\tau) + \frac{1}{27}\,\mathfrak{B}_2\mathfrak{u}(\tau),$$

with

$$\mathfrak{B}_1\mathfrak{u}(\tau) = \int_0^\tau \kappa_1(\tau,s,\mathfrak{u}(s))\,ds = \int_0^\tau \frac{1}{\tau^2+5e^2}\,\sin\mathfrak{u}(s)\,ds, \quad \tau \in [0,1],$$

and

$$\mathfrak{B}_2\mathfrak{u}(\tau) = \int_0^1 \kappa_2(\tau,s,\mathfrak{u}(s))\,ds = \int_0^1 \frac{\mathfrak{u}(s)}{3e^{\tau+3}(1+\mathfrak{u}(s))}\,ds, \quad \tau \in [0,1].$$

We have
$$\alpha = \frac{1}{2} \text{ and } \psi(\tau) = 2^\tau.$$
It is clear that the function $\mathfrak{h} \in C([0,1] \times \mathfrak{R}^3, \mathfrak{R})$.

Then for all $\tau \in [0,1]$ and $\mathfrak{u}, \overline{\mathfrak{u}} \in \mathfrak{R}$, we obtain
$$\left|\mathfrak{h}\big(\tau, \mathfrak{u}, \mathfrak{B}_1(\mathfrak{u}), \mathfrak{B}_2(\mathfrak{u})\big) - \mathfrak{h}\big(\tau, \overline{\mathfrak{u}}, \mathfrak{B}_1(\overline{\mathfrak{u}}), \mathfrak{B}_2(\overline{\mathfrak{u}})\big)\right| \leqslant \gamma|\mathfrak{u} - \overline{\mathfrak{u}}| + \eta_1|\mathfrak{B}_1\mathfrak{u} - \mathfrak{B}_1\overline{\mathfrak{u}}| + \eta_2|\mathfrak{B}_2\mathfrak{u} - \mathfrak{B}_2\overline{\mathfrak{u}}|$$
and
$$\begin{aligned}\left|\kappa_1(\tau, s, \mathfrak{u}) - \kappa_1(\tau, s, \overline{\mathfrak{u}})\right| &\leqslant \rho_1|\mathfrak{u} - \overline{\mathfrak{u}}|, \quad (\tau, s) \in \mathfrak{d},\\ \left|\kappa_2(\tau, s, \mathfrak{u}) - \kappa_2(\tau, s, \overline{\mathfrak{u}})\right| &\leqslant \rho_2|\mathfrak{u} - \overline{\mathfrak{u}}|, \quad (\tau, s) \in \mathfrak{d}_0,\end{aligned}$$
with $\mathfrak{d} = \{(\tau, s) :\ 0 \leqslant s \leqslant \tau \leqslant 1\}$ and $\mathfrak{d}_0 = [0,1] \times [0,1]$, which imply that (A1), (A2) and (A3) are satisfied with
$$\gamma = \frac{1}{6\sqrt{\pi}}\,, \quad \eta_1 = \frac{1}{17}\,, \quad \eta_2 = \frac{1}{27}\,, \quad \rho_1 = \frac{1}{5e^2} \text{ and } \rho_2 = \frac{1}{3e^3}\,.$$
Further, by some simple calculations, we see that
$$\frac{\gamma + (\eta_1\rho_1 + \eta_2\rho_2)(\mathfrak{b} - \mathfrak{a})}{\Gamma(\alpha + 1)} \left(\psi(\mathfrak{b}) - \psi(\mathfrak{a})\right)^\alpha \approx 0.10859 < \frac{1}{2}\,.$$
Theorem 3.1 implies that problem (4.1), (4.2) has at least one solution.

Example 4.2. Consider the following problem for a nonlinear Volterra–Fredholm integro-differential fractional equation:
$$\begin{gathered}{}^c\mathfrak{D}_{1^+}^{\frac{1}{2};\ln\tau}\mathfrak{u}(\tau) = \mathfrak{h}\big(\tau, \mathfrak{u}(\tau), \mathfrak{B}_1\mathfrak{u}(\tau), \mathfrak{B}_2\mathfrak{u}(\tau)\big), \quad \tau \in \mathfrak{J} := [1, e],\\ \mathfrak{u}(1) = \mathfrak{u}(e),\end{gathered}$$
where
$$\mathfrak{h}\big(\tau, \mathfrak{u}(\tau), \mathfrak{B}_1\mathfrak{u}(\tau), \mathfrak{B}_2\mathfrak{u}(\tau)\big) = \frac{1}{\tau^2 + 2} + \frac{1}{3e^2}\left(\cos\mathfrak{u}(\tau) + \frac{3}{2}\mathfrak{u}(\tau)\right) + \frac{1}{7\sqrt{\pi}}\,\mathfrak{B}_1\mathfrak{u}(\tau) + \frac{1}{23}\,\mathfrak{B}_2\mathfrak{u}(\tau),$$
with
$$\mathfrak{B}_1\mathfrak{u}(\tau) = \int_1^\tau \kappa_1(\tau, s, \mathfrak{u}(s))\, ds = \int_1^\tau \tau^3 e^{-6-\tau^2} \sin(\mathfrak{u}(s))\, ds, \quad \tau \in [1, e],$$
and
$$\mathfrak{B}_2\mathfrak{u}(\tau) = \int_1^e \kappa_2(\tau, s, \mathfrak{u}(s))\, ds = \int_1^e \frac{e^{-11-\tau}}{37(1 + \mathfrak{u}(s))}\, ds, \quad \tau \in [1, e].$$
Here, $\alpha = \frac{1}{2}$, and $\psi(\tau) = \ln\tau$.

It is easy to see that $\mathfrak{h} \in C([1,e] \times \mathfrak{R}^3, \mathfrak{R})$. Then, for all $\tau \in [1, e]$ and $\mathfrak{u}, \overline{\mathfrak{u}} \in \mathfrak{R}$, we obtain
$$\begin{aligned}\left|\mathfrak{h}\big(\tau, \mathfrak{u}, \mathfrak{B}_1(\mathfrak{u}), \mathfrak{B}_2(\mathfrak{u})\big) - \mathfrak{h}\big(\tau, \overline{\mathfrak{u}}, \mathfrak{B}_1(\overline{\mathfrak{u}}), \mathfrak{B}_2(\overline{\mathfrak{u}})\big)\right| &\leqslant \gamma|\mathfrak{u} - \overline{\mathfrak{u}}| + \eta_1|\mathfrak{B}_1\mathfrak{u} - \mathfrak{B}_1\overline{\mathfrak{u}}| + \eta_2|\mathfrak{B}_2\mathfrak{u} - \mathfrak{B}_2\overline{\mathfrak{u}}|,\\ \left|\kappa_1(\tau, s, \mathfrak{u}) - \kappa_1(\tau, s, \overline{\mathfrak{u}})\right| &\leqslant \rho_1|\mathfrak{u} - \overline{\mathfrak{u}}|, \quad (\tau, s) \in \mathfrak{d},\\ \left|\kappa_2(\tau, s, \mathfrak{u}) - \kappa_2(\tau, s, \overline{\mathfrak{u}})\right| &\leqslant \rho_2|\mathfrak{u} - \overline{\mathfrak{u}}|, \quad (\tau, s) \in \mathfrak{d}_0,\end{aligned}$$
and
$$\left|\mathfrak{h}(\tau, \mathfrak{u}, \mathfrak{B}_1(\mathfrak{u}), \mathfrak{B}_2(\mathfrak{u})) - \mathfrak{h}(\tau, \overline{\mathfrak{u}}, \mathfrak{B}_1(\overline{\mathfrak{u}}), \mathfrak{B}_2(\overline{\mathfrak{u}}))\right| \geqslant \overline{\gamma}|\mathfrak{u} - \overline{\mathfrak{u}}| - \overline{\eta}_1|\mathfrak{B}_1\mathfrak{u} - \mathfrak{B}_1\overline{\mathfrak{u}}| - \overline{\eta}_2|\mathfrak{B}_2\mathfrak{u} - \mathfrak{B}_2\overline{\mathfrak{u}}|,$$
with $\mathfrak{d} = \{(\tau, s) :\ 1 \leqslant s \leqslant \tau \leqslant e\}$ and $\mathfrak{d}_0 = [1, e] \times [1, e]$, which imply that (A1), (A2), (A3) and (A4) are satisfied with
$$\gamma = \frac{5}{6e^2}\,, \quad \overline{\gamma} = \frac{1}{6e^2}\,, \quad \eta_1 = \overline{\eta}_1 = \frac{1}{7\sqrt{\pi}}\,, \quad \eta_2 = \overline{\eta}_2 = \frac{1}{23}\,, \quad \rho_1 = \frac{1}{e^7}\,, \text{ and } \rho_2 = \frac{1}{37e^{12}}\,.$$

By simple calculations, we see that

$$\frac{(\overline{\eta}_1\rho_1+\overline{\eta}_2\rho_2)(\mathfrak{b}-\mathfrak{a})}{\overline{\gamma}}+\frac{2[\gamma+(\eta_1\rho_1+\eta_2\rho_2)(\mathfrak{b}-\mathfrak{a})]}{\Gamma(\alpha+1)}\,(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{\alpha}\approx 0.2604<1.$$

So, by Theorem 3.2, our problem has a unique solution.

5 Conclusions

The main contribution of this work is to provide some sufficient conditions ensuring the existence and uniqueness of periodic solutions to a large class of nonlinear Volterra–Fredholm integro-differential fractional equations involving the ψ-Caputo fractional derivative, by using the coincidence degree theory of Mawhin [15]. To illustrate the applicability of our research, we have discussed two examples.

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