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THE ROBEN BOUNDARY VALUE PROBLEM OF THERMO-ELECTRO-MAGNETO ELASTICITY FOR HALF SPACE

Abstract. We prove the uniqueness theorem for the Roben boundary value problem of statics of the thermo-electro-magneto-elasticity theory in the case of a half-space. The corresponding unique solution is represented explicitly by means of the inverse Fourier transform under some natural restrictions imposed on the boundary vector function.

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რეზიუმე. ნახევარსივრცის შემთხვევაში დამტკიცებულია თერმო-ელექტრო-მაგნიტო დრეკადობის თეორიის ერთადერთობის თეორემა რობენის სასაზღვრო ამოცანისთვის. გარკვეულ ბუნებრივ შეზღუდვებში, რომელსაც ვადებთ სასაზღვრო ვექტორ-ფუნქციას, შესაბამისი რობენის სასაზღვრო ამოცანის ერთადერთი ამონახსნი წარმოდგენილია ცხადი სახით, შებრუნებული ფურიეს გარდაქმნის მეშვეობით.

1 Introduction

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields.

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. In [17], it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [1–7, 9–11, 14, 18].

The mathematical model of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

In the present paper, we prove the uniqueness theorem of a solution to the Roben boundary value problem of statics for a half-space. Under some natural restriction on the boundary vector functions, the corresponding unique solution is constructed explicitly by the transform technique.

2 Basic equations and formulation of boundary value problems

2.1 Field equations

Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, $E = (E_1, E_2, E_3)^\top = -\operatorname{grad}\varphi$ and $H = (H_1, H_2, H_3) = -\operatorname{grad}\psi$ are electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials, ϑ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and S is the entropy density. We employ the notation $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial_j$, $\partial_t = \partial/\partial_t$; the superscript $(\,\cdot\,)^\top$ denotes transposition operation; the summation over the repeated indices is meant from 1 to 3, unless otherwise stated.

In this subsection, we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators [12].

Constitutive relations:

$$\begin{aligned}
\sigma_{rj} = \sigma_{jr} &= c_{rjkl}\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \quad r, j = 1, 2, 3,\\
D_j &= e_{jkl}\varepsilon_{kl} + \varkappa_{jl}E_l + a_{jl}H_l + p_j\vartheta, \quad j = 1, 2, 3,\\
B_j &= q_{jkl}\varepsilon_{kl} + a_{jl}E_l + \mu_{jl}H_l + m_j\vartheta, \quad j = 1, 2, 3,\\
S &= \lambda_{kl}\varepsilon_{kl} + p_k E_k + m_k H_k + \gamma\vartheta.
\end{aligned}$$

Fourier Law:

$$q_j = -\eta_{jl}\partial_l\vartheta, \quad j = 1, 2, 3.$$

Equations of motion:

$$\partial_j\sigma_{rj} + X_r = \varrho\partial_t^2 u_r, \quad r = 1, 2, 3.$$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free):

$$\partial_j D_j = \varrho_e, \quad \partial_j B_j = 0.$$

Linearised equation of the entropy balance:

$$T_0\partial_t S - Q = -\partial_j q_j.$$

Here, ϱ is the mass density, ϱ_e is the electric density, c_{rjki} are the elastic constants, e_{jki} are the piezoelectric constants, q_{jki} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electro-magnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho c T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity. The constants involved in these equations satisfy the symmetry conditions

$$\begin{gathered} c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \quad \varkappa_{kj} = \varkappa_{jk}, \\ \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk}, \ \ r, j, k, l = 1, 2, 3. \end{gathered} \tag{2.1}$$

From physical considerations it follows (see, e.g., [8, 13]) that

$$c_{rjkl}\xi_{rj}\xi_{kl} \geq c_0\xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq c_1|\xi|^2, \quad \mu_{kj}\xi_k\xi_j \geq c_2|\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq c_3|\xi|^2, \tag{2.2}$$

for all $\xi_{kj} = \xi_{jk} \in \mathbb{R}$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c_0, c_1, c_2 and c_3 are positive constants. More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure positive definiteness of the matrix

$$\Xi = \begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} & [p_j]_{3\times 1} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} & [m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & \gamma \end{bmatrix}_{7\times 7}. \tag{2.3}$$

Further, we introduce the following generalised stress operator:

$$\mathcal{T}(\partial, n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} & [-\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_jn_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}.$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$, we have

$$\mathcal{T}(\partial, n)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -q_jn_j)^\top. \tag{2.4}$$

The components of the vector $\mathcal{T}U$ given by (2.4) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are, respectively, the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

From the above equations of dynamics, in the case of statics, we get the following system of partial differential equations in matrix form:

$$A(\partial)U(x) = \Phi(x),$$

where $U = (u_1, \dots, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought vector function and $\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top$ is a given vector function; $A(\partial) = [A_{pq}(\partial)]_{6\times 6}$ is the matrix differential operator

$$A(\partial) = \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 3} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}.$$

From the symmetry conditions (2.1), inequalities (2.2) and positive definiteness of the matrix (2.3), it follows that $A(\partial)$ is a formally non-self adjoint strongly elliptic operator.

2.2 Formulation of boundary value problems

Let $\mathbb{R}^3$ be divided by some plane into two half-spaces. Without loss of generality, we can assume that these half-spaces are

$$\mathbb{R}^3_1 := \{x \mid \ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \text{ and } \ x_3 > 0\},$$
$$\mathbb{R}^3_2 := \{x \mid \ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \text{ and } \ x_3 < 0\};$$

$n = (n_1, n_2, n_3) = (0, 0, -1)$ is the outward unit normal vector with respect to $\mathbb{R}^3_1$; $S := \partial\mathbb{R}_1 = \partial\mathbb{R}_2$.

Now we formulate the **Roben type boundary-value problems** $(\mathbf{R})^{\pm}$ of the thermo-electro-magneto-elasticity theory for a half-space:

Find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\mathbb{R}^3_j})]^6 \cap [C^2(\mathbb{R}^3_j)]^6$, $j = 1, 2$, to the system of equations

$$A(\partial)U = 0 \ \text{ in } \ \mathbb{R}^3_j, \quad j = 1, 2, \tag{2.5}$$

satisfying the Roben type boundary condition

$$\{\mathcal{T}(\partial, n)U + aU\}^{\pm} = F \ \text{ on } \ S, \tag{2.6}$$

where a is a positive constant.

The symbols $\{\,\cdot\,\}^{\pm}$ denote the one-sided limits on S from $\mathbb{R}^3_1$ (sign "+") and $\mathbb{R}^3_2$ (sign "−").

We require that the boundary data involved in the above setting possess the following smoothness property: $F \in \overset{\circ}{C}{}^{\infty}(\mathbb{R}^2)$, where $\overset{\circ}{C}{}^{\infty}(\mathbb{R}^2)$ is the space of infinitely differentiable functions with compact support.

Let $\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}$ and $\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}$ denote the direct and inverse generalized Fourier transforms in the space of tempered distributions (the Schwartz space $\mathcal{S}'(\mathbb{R}^2)$) which for regular summable functions f and g read as follows:

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f] = \int_{\mathbb{R}^2} f(\widetilde{x})\, e^{i\widetilde{x}\cdot\widetilde{\xi}}\, d\widetilde{x}, \quad \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[g] = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} g(\widetilde{\xi})\, e^{-i\widetilde{x}\cdot\widetilde{\xi}}\, d\widetilde{\xi}, \tag{2.7}$$

where

$$\widetilde{x} = (x_1, x_2), \ \ \widetilde{\xi} = (\xi_1, \xi_2), \ \ d\widetilde{x} = dx_1\, dx_2, \ \ \widetilde{x}\cdot\widetilde{\xi} = x_1\xi_1 + x_2\xi_2.$$

Note that if $f(x) = f(x_1, x_2, x_3) = f(\widetilde{x}, x_3)$, then

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\partial_{x_j} f(x)] = -i\xi_j \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f] = -i\xi_j \widehat{f}(\widetilde{\xi}, x_3), \ \ j = 1, 2,$$

and hence

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\nabla_x f(x)] = \begin{bmatrix} -i\xi_1 \\ -i\xi_2 \\ \partial_{x_3} \end{bmatrix} \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f(x)] = P(-i\widetilde{\xi}, \partial x_3)\widehat{f}(\widetilde{\xi}, x_3) \tag{2.8}$$

with $\widehat{f}(\widetilde{\xi}, x_3) = \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f]$ and

$$P = P(-i\widetilde{\xi}, \partial_{x_3}) = (-i\xi_1, -i\xi_2, \partial_{x_3})^\top. \tag{2.9}$$

Applying Fourier transform (2.7) in (2.5)–(2.6) and taking into account (2.9), we arrive at the problems

$$A(P)\widehat{U}(\widetilde{\xi}, x_3) = 0, \quad x_3 \in (0; +\infty) \ \text{ or } \ x_3 \in (-\infty; 0), \tag{2.10}$$
$$\big\{\mathcal{T}(\partial, n)\widehat{U}(\widetilde{\xi}, x_3) + a\widehat{U}(\widetilde{\xi}, x_3)\big\}^{\pm}_{(x_3\to 0\pm)} = \widehat{F}(\widetilde{\xi}), \ \ \widetilde{\xi} \in \mathbb{R}^3 \setminus \{0\}. \tag{2.11}$$

We see that (2.10) is a system of ordinary differential equations of second order for each $\widetilde{\xi} \in \mathbb{R}^2$. We denote these problems by $\widehat{\mathbf{R}}^{\pm}$.

3 Uniqueness theorems

We start with constructing a system of linear independent solutions to system (2.10).

Let us denote by $k_j = k_j(\widetilde{\xi})$, $j = \overline{1,12}$, the roots of the equation

$$\det A(-i\xi) = 0 \tag{3.1}$$

with respect to ξ_3, where $A(-i\xi)$ is the symbol matrix of the operator $A(\partial)$.

Note that $\det A(-i\xi)$ is a homogeneous polynomial of order 12 and equation (3.1) has no real roots, i.e., $\operatorname{Im} k_j \neq 0$, $j = \overline{1,12}$. These roots are continuously dependent on the coefficients of (3.1) and the number of roots with positive and negative imaginary parts are equal. Denote by $k_1, k_2, \dots, k_6$ the roots with positive imaginary parts and by $k_7, \dots, k_{12}$ with negative ones.

Let us construct the following matrices:

$$\Phi^{(+)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^+} A^{-1}(-i\xi)\, e^{-i\xi_3 x_3}\, d\xi_3, \tag{3.2}$$

$$\Phi^{(-)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^-} A^{-1}(-i\xi)\, e^{-i\xi_3 x_3}\, d\xi_3, \tag{3.3}$$

where ℓ^+ (respectively, ℓ^-) is a closed simple curve of positive counter-clockwise orientation (respectively, negative clockwise orientation) in the upper (respectively, lower) complex half-plane $\operatorname{Re}\xi_3 > 0$ (respectively, $\operatorname{Re}\xi_3 < 0$) enclosing all the roots with respect to ξ_3 of the equation $\det A(-i\xi) = 0$ with positive (respectively, negative) imaginary parts (see Figure 1). Clearly, (3.2) and (3.3) do not depend on the shape of ℓ^+ (respectively, ℓ^-).

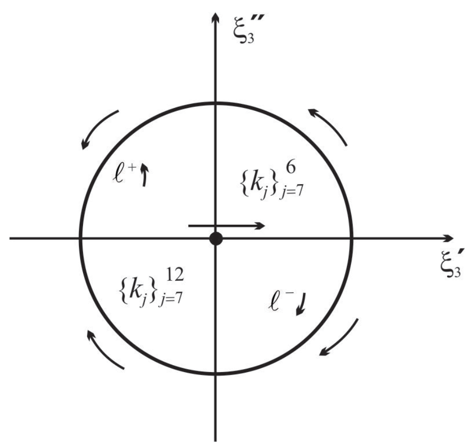

Figure 1. Complex $\xi_3 = \xi +'_3 +i\xi''_3$ plane.

With the help of the Cauchy integral theorem for analytic functions, we conclude that the entries of the matrix $\Phi^{(+)}(\widetilde{\xi}, x_3) = [\Phi^{(+)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to +\infty$ and are decreasing exponentially as $x_3 \to -\infty$ (since $-i\xi_3 x_3 = -i(\xi'_3 + i\xi''_3)x_3 = -i\xi'_3 x_3 + \xi''_3 x_3$ for $\xi_3 = \xi +'_3 +i\xi''_3$).

Analogously, the entries of the matrix $\Phi^{(-)}(\widetilde{\xi}, x_3) = [\Phi^{(-)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to -\infty$ and vanish exponentially as $x_3 \to +\infty$.

Due to Lemma 3.1 in [15], the columns of $\Phi^{(\pm)}(\widetilde{\xi}, x_3)$ are linearly independent solutions to system (2.10).

Theorem 3.1. *The boundary value problems* $\widehat{\mathbf{R}}^{\pm}$ (2.10)–(2.11) *have only one solution in the space of functions vanishing at infinity.*

Proof. Let $x_3 \in (0; +\infty)$. We look for a solution of problem (2.10)–(2.11) in the following form:

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(-)}(\widetilde{\xi}, x_3) C, \quad x_3 > 0,$$

where $C = (C_1, \dots, C_6)$ is an unknown vector depending only on $\widetilde{\xi}$, $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$.

From (2.11), we have

$$\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0)C + a\Phi^{(-)}(\widetilde{\xi}, 0)C = \widehat{F}(\widetilde{\xi}), \quad \widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\},$$

and since

$$\det\big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0) + a\Phi^{(-)}(\widetilde{\xi}, 0)\big] \neq 0, \quad |\widetilde{\xi}| \neq 0,$$

due to Lemma 3.1 in [15], we obtain

$$C(\widetilde{\xi}) = \big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0) + a\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1}\widehat{F}(\widetilde{\xi}).$$

Therefore, the unique solution of the problem $\widehat{\mathbf{R}}^+$ has the form

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(-)}(\widetilde{\xi}, x_3)\big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0) + a\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1}\widehat{F}(\widetilde{\xi}), \quad x_3 > 0. \tag{3.4}$$

Similarly, if $x_3 \in (-\infty; 0)$, then the unique solution of $\widehat{\mathbf{R}}^-$ has the form

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(+)}(\widetilde{\xi}, x_3)\big[\mathcal{T}(-i\xi, n)\Phi^{(+)}(\widetilde{\xi}, 0) + a\Phi^{(+)}(\widetilde{\xi}, 0)\big]^{-1}\widehat{F}(\widetilde{\xi}), \quad x_3 < 0. \tag{3.5}$$

The theorem is proved. □

Using Lemma 2 from [16], we can write

$$\big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0) + a\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1} = \begin{bmatrix} [\mathcal{O}(|\widetilde{\xi}|)]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(|\widetilde{\xi}|) \end{bmatrix}_{6\times 6}, \tag{3.6}$$

where

$$\mathcal{T}(-i\xi, n) := \begin{bmatrix} [c_{rjkl}n_j(-i\xi_l)]_{3\times 3} & [e_{lrj}n_j(-i\xi_l)]_{3\times 3} & [q_{lrj}n_j(-i\xi_l)]_{3\times 1} & [-\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j(-i\xi_l)]_{1\times 3} & \varkappa_{jl}n_j(-i\xi_l) & a_{jl}n_j(-i\xi_l) & -p_jn_j \\ [-q_{jkl}n_j(-i\xi_l)]_{1\times 3} & a_{jl}n_j(-i\xi_l) & \mu_{jl}n_j(-i\xi_l) & -m_jn_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j(-i\xi_l) \end{bmatrix}_{6\times 6}.$$

Remark 3.2. For arbitrary $x_3 > 0$ (see [15]),

$$\Phi^{(-)}(\widetilde{\xi}, x_3) = \begin{bmatrix} [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times 5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(|\widetilde{\xi}|^{-1}) \end{bmatrix}_{6\times 6}$$

and, due to (3.6),

$$\Phi^{(-)}(\widetilde{\xi}, x_3)\big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0) + a\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1} = \begin{bmatrix} [\mathcal{O}(1)]_{5\times 5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(1) \end{bmatrix}_{6\times 6}. \tag{3.7}$$

Similarly, for arbitrary $x_3 < 0$,

$$\Phi^{(+)}(\widetilde{\xi}, x_3)\big[\mathcal{T}(-i\xi, n)\Phi^{(+)}(\widetilde{\xi}, 0) + a\Phi^{(+)}(\widetilde{\xi}, 0)\big]^{-1} = \begin{bmatrix} [\mathcal{O}(1)]_{5\times 5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(1) \end{bmatrix}_{6\times 6}. \tag{3.8}$$

Theorem 3.3. *The Roben boundary value problems* (2.5)–(2.6) *have at most one solution* $U = (u, \varphi, \psi, \vartheta)^\top$ *in the space* $[C^1(\overline{\mathbb{R}^3_{1,2}})]^6 \cap [C^2(\mathbb{R}^3_{1,2})]^6$ *provided*

$$\vartheta(x) = \mathcal{O}(|x|^{-1}), \tag{3.9}$$

$$\partial^\alpha \widetilde{U}(x) = \mathcal{O}\big(|x|^{-1-|\alpha|}\ln|x|\big) \ \ as \ \ |x| \to \infty \tag{3.10}$$

for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$*. Here* $\widetilde{U} = (u, \varphi, \psi)^\top$*.*

Proof. Let $U^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})$ be two solutions of the problem under consideration with properties indicated in the theorem for $\mathbb{R}^3_1$. It is evident that the difference

$$V = (u', \varphi', \psi', \vartheta') = U^{(1)} - U^{(2)}$$

solves the corresponding homogeneous problem.

Therefore for the temperature function we get the separated homogeneous Roben problem

$$[A(\partial)V]_6 = \eta_{jl}\partial_j\partial_l\vartheta' = 0 \ \text{ in } \ \mathbb{R}^3_1, \tag{3.11}$$

$$\{\eta_{jl}n_j\partial_l\vartheta' + a\vartheta'\}^+ = 0 \ \text{ on } \ S. \tag{3.12}$$

Note that (3.9) implies $\partial\vartheta(x) = O(|x|^{-2})$.

By Green's formula (see (2.83) in [12]), for

$$B^+(0;R) := \big\{(x_1, x_2, x_3) \mid \ x_1^2 + x_2^2 + x_3^2 \le R^2 \text{ and } x_3 > 0\big\}$$

and (3.11)–(3.12), we have

$$\begin{aligned}\int\limits_{B^+(0;R)} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx &= \int\limits_{\partial B^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+\,dS\\ &= \int\limits_{\Sigma^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+\,d\Sigma = -a \int\limits_{\Sigma^+(0;R)} \{(\vartheta')^2\}\,d\Sigma.\end{aligned} \tag{3.13}$$

Here, $\Sigma^+(0;R)$ is the upper half-sphere centered at the origin and radius R. Relation (3.13) implies

$$\int\limits_{B^+(0,R)} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx = 0.$$

Taking the limit as $R \to \infty$, in view of (2.2) and (3.9), we get

$$\int\limits_{\mathbb{R}^3_1} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx = 0.$$

Due to (2.2), $\vartheta' = const$ and from (3.9) we conclude that $\vartheta' = 0$.

Therefore, the five-dimensional vector $\widetilde{V} = (u', \varphi', \psi')^\top$, constructed by the first five components of the solution vector V, solves the following homogeneous boundary value problem:

$$\begin{aligned}\widetilde{A}(\partial)\widetilde{V} &= 0 \ \text{ in } \ \mathbb{R}^3_1,\\ \{\widetilde{\mathcal{T}}(\partial,n)\widetilde{V} + a\widetilde{V}\}^+ &= 0 \ \text{ on } \ S,\end{aligned} \tag{3.14}$$

where $\widetilde{A}(\partial)$ is the 5×5 differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects (see [12]):

$$\widetilde{A}(\partial) = [\widetilde{A}_{pq}(\partial)]_{5\times5} := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1}\\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l\\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times5}$$

and $\widetilde{\mathcal{T}}(\partial,n)$ is the corresponding 5×5 generalized stress operator:

$$\widetilde{\mathcal{T}}(\partial,n) = [\widetilde{\mathcal{T}}_{pq}(\partial,n)]_{5\times5} := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times3} & [e_{lrj}n_j\partial_l]_{3\times1} & [q_{lrj}n_j\partial_l]_{3\times1}\\ [-e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l\\ [-q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{5\times5}.$$

Using the limiting procedure as above in the corresponding Green's identity for the vectors satisfying decay conditions (3.10), we obtain

$$\int\limits_{\mathbb{R}^3_1} \big[\widetilde{A}(\partial)\widetilde{V}\cdot\widetilde{V}+\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})\big]\,dx=\lim_{R\to\infty}\int\limits_{\Sigma^+(0;R)}[\widetilde{\mathcal{T}}\widetilde{V}]^+\cdot[\widetilde{V}]^+\,d\Sigma=-a\lim_{R\to\infty}\int\limits_{\Sigma^+(0;R)}[\widetilde{V}^2]^+\,d\Sigma, \tag{3.15}$$

where $\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})$ has the form

$$\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})=c_{rjkl}\partial_l u'_k\partial_j u'_r+\varkappa_{jl}\partial_l\varphi'\partial_j\varphi'+a_{jl}(\partial_l\varphi'\partial_j\psi'+\partial_j\psi'\partial_l vf')+\mu_{jl}\partial_l\psi'\partial_j\psi'. \tag{3.16}$$

If $\widetilde{V}$ is a solution of (3.14) satisfying (3.10), then from (3.15) and (2.2) we have

$$\int\limits_{\mathbb{R}^3_1}\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})\,dx=0. \tag{3.17}$$

From (3.14), (3.16) and (3.17), along with (2.2), we get

$$u'(x)=\widetilde{a}\times x+\widetilde{b},\quad \varphi'(x)=b_4,\quad \psi'=b_5,$$

where $\widetilde{a}=(a_1,a_2,a_3)$ and $\widetilde{b}=(b_1,b_2,b_3)$ are arbitrary constant vectors and b_4, b_5 are arbitrary constants. Now, in view of (3.10), we arrive at the equalities $u'(x)=0$, $\varphi'(x)=0$, $\psi'(x)=0$ for all $x\in\mathbb{R}^3_1$, consequently, $U^{(1)}=U^{(2)}$ in $\mathbb{R}^3_1$.

The proof is similar for the domain $\mathbb{R}^3_2$. □

Theorem 3.4. *Let $F\in\overset{\circ}{C}{}^\infty(\mathbb{R}^2)$ and for arbitrary multi-index $\beta=(\beta_1,\beta_2)$,*

$$\int\limits_{\mathbb{R}^2}F(\widetilde{x})\widetilde{x}^\beta\,d\widetilde{x}=0,\quad |\beta|=0,1.$$

Then the Roben boundary value problems (2.5)–(2.6) *possess unique solution which can be represented in the form*

$$U(x)=\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\Big[\Phi^{(-)}(\widetilde{\xi},x_3)\big[\mathcal{T}(-i\xi,n)\Phi^{(-)}(\widetilde{\xi},0)+a\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1}\widehat{F}(\widetilde{\xi})\Big],\quad x_3>0, \tag{3.18}$$

or

$$U(x)=\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\Big[\Phi^{(+)}(\widetilde{\xi},x_3)\big[\mathcal{T}(-i\xi,n)\Phi^{(+)}(\widetilde{\xi},0)+a\Phi^{(+)}(\widetilde{\xi},0)\big]^{-1}\widehat{F}(\widetilde{\xi})\Big],\quad x_3<0. \tag{3.19}$$

Proof. It suffices to show that the vector functions (3.18) and (3.19) satisfy conditions (3.9)–(3.10). This will be done if we prove that the relations

$$x_j\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\big[\widehat{U}(\widetilde{\xi},x_3)\big]=\mathcal{O}(1),\quad j=1,2,3, \tag{3.20}$$

and

$$x_j^2\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\big[\widehat{U}(\widetilde{\xi},x_3)\big]=\mathcal{O}(1),\quad j=1,2,3, \tag{3.21}$$

hold for all $x\in\mathbb{R}^3$, where $\widehat{U}(\widetilde{\xi},x_3)$ is defined by (3.4) or (3.5).

Under the restriction on F, we conclude that $\widehat{F}\in\mathcal{S}(\mathbb{R}^2)$ and $\widehat{F}(\widetilde{\xi})=\mathcal{O}(|\widetilde{\xi}|^2)$ as $|\widetilde{\xi}|\to 0$, where $\mathcal{S}(\mathbb{R}^2)$ is the space of rapidly decreasing functions. Therefore, in view of (3.7)–(3.8), we have

$$\begin{aligned}
&\frac{\partial\widehat{U}(\widetilde{\xi},x_3)}{\partial\xi_j}=\mathcal{O}(1),\quad |\widetilde{\xi}|\to 0,\\
&\frac{\partial\widehat{U}(\widetilde{\xi},x_3)}{\partial\xi_j}=\mathcal{O}(|\widetilde{\xi}|^{-k}),\quad |\widetilde{\xi}|\to\infty,\quad k\geq 2,
\end{aligned} \tag{3.22}$$

uniformly for all $x \in \mathbb{R}^3$.

For $j = 1$ or $j = 2$, we find

$$\begin{aligned}
x_j \int\limits_{\mathbb{R}^2} \widehat{U}(\widetilde{\xi}, x_3)\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi} &= i \int\limits_{\mathbb{R}^2} \widehat{U}(\widetilde{\xi}, x_3)\, \frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial \xi_j}\, d\widetilde{\xi} = i \lim_{R\to\infty} \int\limits_{K(0;R)} \widehat{U}(\widetilde{\xi}, x_3)\, \frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial \xi_j}\, d\widetilde{\xi} \\
&= -i \lim_{R\to\infty} \bigg(\int\limits_{K(0;R)} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j}\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi} - \int\limits_{\partial K(0;R)} \widehat{U}(\widetilde{\xi}, x_3)\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, \frac{\xi_j}{R}\, ds \bigg) \\
&= -i \lim_{R\to\infty} \int\limits_{K(0;R)} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j}\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi} = -i \int\limits_{\mathbb{R}^2} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j}\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi}, \qquad (3.23)
\end{aligned}$$

where $K(0, R)$ is the circle of radius R centered at the origin.

It is clear that relations (3.22) and (3.23) imply (3.20). Condition (3.21) can be proved similarly if we note that

$$\begin{aligned}
\frac{\partial^2 \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j^2} &= \mathcal{O}\big(|\widetilde{\xi}|^{-1}\big), \ \ |\widetilde{\xi}| \to 0, \\
\frac{\partial^2 \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j^2} &= \mathcal{O}(|\widetilde{\xi}|^{-k-1}), \ \ |\widetilde{\xi}| \to \infty, \ \ k \geq 2,
\end{aligned}$$

uniformly for all $x \in \mathbb{R}^3$.

For arbitrary $x_3 > 0$, we can write

$$\begin{aligned}
& x_3 \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] \\
&\quad = x_3 \int\limits_{\mathbb{R}^2} \bigg(\int\limits_{\ell^-} A^{-1}(-i\xi)\, e^{-i\xi_3 x_3}\, d\xi_3 \bigg) \big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0) + a\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1} \widehat{F}(\widetilde{\xi})\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi}. \qquad (3.24)
\end{aligned}$$

Due to Lemma 3.3 in [15], the entries of the matrix $A^{-1}(-i\xi)$ are homogeneous functions in ξ and

$$A^{-1}(-i\xi) = \begin{bmatrix} [\mathcal{O}(|\xi|^{-2})]_{5\times5} & [\mathcal{O}(|\xi|^{-3})]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(|\xi|^{-2}) \end{bmatrix}_{6\times6}. \qquad (3.25)$$

Using the Cauchy integral theorem for analytic functions and relations (3.6), (3.25), from (3.24) we get

$$\begin{aligned}
& x_3 \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\big[\widehat{U}(\widetilde{\xi}, x_3)\big] \\
&\quad = x_3 \int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3} \begin{bmatrix} [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1}) \end{bmatrix} \begin{bmatrix} [\mathcal{O}(|\widetilde{\xi}|)]_{5\times5} & [\mathcal{O}(1)]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|) \end{bmatrix} \widehat{F}(\widetilde{\xi})\, d\widetilde{\xi} \\
&\quad = x_3 \int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3} \begin{bmatrix} [\mathcal{O}(1)]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(1) \end{bmatrix} \widehat{F}(\widetilde{\xi})\, d\widetilde{\xi} = I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= x_3 \int\limits_{|\xi|\leq M} e^{-|\widetilde{\xi}|x_3} \begin{bmatrix} [\mathcal{O}(1)]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(1) \end{bmatrix} \widehat{F}(\widetilde{\xi})\, d\widetilde{\xi}, \\
I_2 &= x_3 \int\limits_{|\xi|> M} e^{-|\widetilde{\xi}|x_3} \begin{bmatrix} [\mathcal{O}(1)]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(1) \end{bmatrix} \widehat{F}(\widetilde{\xi})\, d\widetilde{\xi}
\end{aligned}$$

for some positive number M.

Since $\widehat{F}(\widetilde{\xi}) \in \mathcal{S}(\mathbb{R}^2)$, it is easy to check that $I_1 = \mathcal{O}(1)$ and $I_2 = \mathcal{O}(1)$ and hence (3.20) holds.

We can prove the boundedness of the vector function $x_3^2 \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)]$ quite similarly by taking into account that $\widehat{F}(\widetilde{\xi}) = \mathcal{O}(|\widetilde{\xi}|^2)$ as $|\widetilde{\xi}| \to 0$. □

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