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Short Communication

MALKHAZ ASHORDIA, MALKHAZ KUTSIA, MZIA TALAKHADZE

ON THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The modified criterion of the Opial type condition is given for the well-posedness of the Cauchy problem for the systems of linear generalized ordinary differential equations. Moreover, there are established the sufficient conditions guaranteeing the nearness of the left and right limits of the solutions of the perturbed problems to the left and right limits of the solution of the limit problem, respectively.

რეზიუმე. განზოგადებულ ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა სისტემებისთვის მოცემულია კოშის ამოცანის კორექტულობის ოპიალის ტიპის პირობის მოდიფიცირებული კრიტერიუმი. გარდა ამისა, დადგენილია საკმარისი პირობები, რომლებიც უზრუნველყოფს შეშფოთებული ამოცანების ამონახსნების მარჯვენა და მარცხენა ზღვრების სიახლოვეს ზღვრული ამოცანის ამონახსნის მარჯვენა და მარცხენა ზღვრებთან, შესაბამისად.

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Let $A_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV_{loc}(I;\mathbb{R}^n)$ and $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval, non-degenerated at the point. Consider the system

$$dx = dA_0(t)\cdot x + df_0(t) \ \text{ for } \ t \in I \tag{1}$$

under the Cauchy condition

$$x(t_0) = c_0, \tag{2}$$

where $c_0 \in \mathbb{R}^n$ is an arbitrary constant vector.

Let x_0 be a unique solution of problem $(1), (2)$.

Along with the Cauchy problem $(1), (2)$, consider the sequence of the Cauchy problems

$$dx = dA_k(t)\cdot x + df_k(t), \tag{1_k}$$

$$x(t_k) = c_k \tag{2_k}$$

$(k = 1, 2, \dots)$, where $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $f_k \in BV_{loc}(I;\mathbb{R}^n)$, $t_k \in I$ and $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$.

Without loss of generality, we assume that either **(a)** $t_k < t_0$ $(k = 1, 2, \dots)$, or **(b)** $t_k > t_0$ $(k = 1, 2, \dots)$, or **(c)** $t_k = t_0$ $(k = 1, 2, \dots)$.

In the paper we establish:

1. the sufficient conditions for the Cauchy problem $(1_k), (2_k)$ to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|x_k(t) - x_0(t)\| = 0 \tag{3}$$

in the case, where

$$\lim_{k\to+\infty} c_{kj} = c_{0j} \ \text{ if } \ j\in\{1,2\} \ \text{ is such that } \ (-1)^j(t_k-t_0)\geq 0 \ \ (k=0,1,\dots), \tag{3_j}$$

where

$$\begin{aligned} c_{k1} &= x_k(t_k-) = c_k - \big(d_1A_k(t_k)\cdot c_k + d_1f_k(t_k)\big),\\ c_{k2} &= x_k(t_k+) = c_k + \big(d_2A_k(t_k)\cdot c_k + d_2f_k(t_k)\big)\end{aligned} \quad (j=1,2;\ \ k=0,1,\dots); \tag{4}$$

2. the sufficient conditions for the Cauchy problem $(1_k),(2_k)$ to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty}\ \sup_{t\in I,\, t\neq t_k} \big\|x_k(t)-x_0(t)-x_{0j}(t)\big\| = 0 \tag{5}$$

in the case, where

$$\lim_{k\to+\infty} c_{kj} = c_{*j} \ \text{ if } \ j\in\{1,2\} \ \text{ is such that } \ (-1)^j(t_k-t_0)\geq 0 \ \ (k=0,1,\dots), \tag{5_j}$$

where c_{kj} $(j=1,2;\ k=1,2,\dots)$ are defined by (4), $c_{*j}\in\mathbb{R}^n$ $(j=1,2)$ are arbitrary vectors, differing from c_{0j} $(j=1,2)$, in general; the function x_{01} is a solution of the homogeneous system

$$dx = dA_0(t)\cdot x \tag{1_0}$$

on the set $\{t\in I:\ t<t_0\}$ under the condition

$$x_{01}(t_0-) = c_{*1} - x_0(t_0-),$$

and the function x_{02} is a solution of the homogeneous system (1_0) on the set $\{t\in I:\ t>t_0\}$ under the condition

$$x_{02}(t_0+) = c_{*2} - x_0(t_0+).$$

We note that the condition

$$\det\big(I_n + (-1)^j d_jA_0(t)\big)\neq 0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_0)<0 \ \ (j=1,2)$$

guarantees the unique solvability of the Cauchy problem (1), (2) for every $f_0\in BV_{loc}(I;\mathbb{R}^n)$ and $c_0\in\mathbb{R}^n$. Therefore, the vector functions x_{01} and x_{02} defined above exist and are uniquely defined.

In earlier works (see [3–5]) there are investigated the analogous question for the convergence in a general case, i.e., without any restrictions on the sequence t_k $(k=1,2,\dots)$, when

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on I}, \tag{6}$$

under the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{7}$$

and some condition guaranteeing the equalities

$$\lim_{k\to+\infty} d_jA_k(t_k) = d_jA_0(t_0), \quad \lim_{k\to+\infty} d_jf_k(t_k) = d_jf_0(t_0) \ \ (j=1,2). \tag{7_j}$$

Note that if $j\in\{1,2\}$ is such that (7_j) holds, then condition (3_j) follows from (4) and (7)

In the present paper we assume that (7) holds, but the fulfilment of conditions (7_j) $(j=1,2)$ is not required.

Analogous and some related questions for the initial and general linear boundary value problems are investigated e.g. in [1, 2, 9, 10, 12, 14] (see also the references therein) for systems of ordinary differential equations, in [3, 4, 8, 11, 13] for systems of generalized ordinary differential equations, and in [6] for systems of linear impulsive differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate linear ordinary differential, impulsive and difference equations from a unified point of view; in particular, these different type equations (linear) can be rewritten in form (1). Moreover, the convergence conditions for difference schemes corresponding to systems of ordinary differential and impulsive equations can be obtained from the results on the well-posedness of the corresponding problems for systems of generalized ordinary differential equations (see [5, 14, 15] and the references therein).

In the paper the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary finite or infinite interval from $\mathbb{R}$. We say that some property is valid in I if it is valid on every closed interval from I.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix. We designate the zero n vector by 0, as well.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}^n_+ = \mathbb{R}^{n\times 1}_+$.

If $X \in \mathbb{R}^{n\times n}$, then $\det(X)$ is the determinant of X.

I_n is the identity $n \times n$-matrix.

$\bigvee\limits_a^b(x)$ is the total variation of the function $x : [a, b] \to \mathbb{R}$; $\bigvee\limits_b^a(x) = -\bigvee\limits_a^b(x)$.

If $x : I \to \mathbb{R}$, then $\bigvee\limits_I(x)$ is the total variation of x on I, i.e. $\bigvee\limits_I(x) = \lim\limits_{a\to\alpha+,\, b\to\beta-} \bigvee\limits_a^b(x)$, where $\alpha = \inf I$ and $\beta = \sup I$.

$\bigvee\limits_a^b(X)$ is the sum of the total variations of the components x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$.

$\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$, $\bigvee\limits_I(X) = \lim\limits_{a\to\alpha+,\, b\to\beta-} \bigvee\limits_a^b(X)$, where $\alpha = \inf I$ and $\beta = \sup I$, $\bigvee\limits_{(b,a)}(X) = -\bigvee\limits_{(b,a)}(X)$.

If $X : I \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_I(X)$ is the sum of total variations on I of its components x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, m)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(\alpha-) = X(\alpha)$ if $\alpha \in I$ and $X(\beta+) = X(\beta)$ if $\beta \in I$; if α or β do not belong to I, then $X(t)$ is defined by the continuity outside of I).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}(I; \mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ (i.e. such that $\bigvee\limits_I(X) < \infty$).

$\mathrm{BV}(I; D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all bounded variation matrix-functions $X : I \to D$.

$\mathrm{BV}_{loc}(I; D)$ is the set of all $X : I \to D$ for which the restriction on $[a, b]$ belong to $\mathrm{BV}([a, b]; D)$ for every closed interval $[a, b]$ from I.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

We introduce the operators. Let $a \in I$ be a fixed point, and $X \in \mathrm{BV}_{loc}(I, \mathbb{R}^{l\times n})$ and $Y \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times m})$. Then we put

$$\mathcal{B}(X, Y)(t) = X(t)Y(t) - X(a)Y(a) - \int_a^t dX(\tau) \cdot Y(\tau) \ \text{ for } \ t \in I,$$

$$\mathcal{I}(X, Y)(t) = \int_a^t d\big(X(\tau) + \mathcal{B}(X, Y)(\tau)\big) \cdot X^{-1}(\tau) \ \text{ for } \ t \in I,$$

$$\mathcal{D}_{\mathcal{B}}(Y_1,X_1;Y_2,X_2)(t)=\mathcal{B}(X_1,Y_1)(t)-\mathcal{B}(X_2,Y_2)(t) \ \text{ for } \ t\in I,$$
$$\mathcal{D}_{\mathcal{I}}(Y_1,X_1;Y_2,X_2)(t)=\mathcal{I}(X_1,Y_1)(t)-\mathcal{I}(X_2,Y_2)(t) \ \text{ for } \ t\in I.$$

Definition 1. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A_0,f_0;t_0)$ if for every $c_0\in\mathbb{R}^n$ and a sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (7), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (6) holds.

In [4,7], the necessary and sufficient (effectively sufficient) conditions are established that guarantee the inclusion

$$\big((A_k,f_k;t_k)\big)_{k=1}^{+\infty}\in S(A_0,f_0;t_0). \tag{8}$$

Analogous results are established for the general linear boundary value problems in [3,4].

Definition 2. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_{loc}(A_0,f_0;t_0-)$ if $t_k<t_0$ $(k=1,2,\dots)$ and for every $c_0\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (3_1), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Definition 3. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_{loc}(A_0,f_0;t_0+)$ if $t_k>t_0$ $(k=1,2,\dots)$ and for every $c_0\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (3_2), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Definition 4. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_{loc}(A_0,f_0;t_0\pm)$ if $t_k=t_0$ $(k=1,2,\dots)$ and for every $c_0\in\mathbb{R}^n$, the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ and $j\in\{1,2\}$ satisfying condition (3_j), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Definition 5. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}^*_{loc}(A_0,f_0;t_0-)$ if $t_k<t_0$ $(k=1,2,\dots)$ and for every $c_{*1}\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (5_1), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3_1) holds.

Definition 6. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}^*_{loc}(A_0,f_0;t_0+)$ if $t_k>t_0$ $(k=1,2,\dots)$ and for every $c_{*2}\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (5_2), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3_2) holds.

Definition 7. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}^*_{loc}(A_0,f_0;t_0\pm)$ if $t_k=t_0$ $(k=1,2,\dots)$ and for every $c_{*j}\in\mathbb{R}^n$ $(j=1,2)$ and the sequences $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying conditions (5_j) $(j=1,2)$, the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and conditions (3_j) $(j=1,2)$ hold.

(A) The results concerning the sets $\mathcal{S}(A_0,f_0;t_0)$, $\mathcal{S}_{loc}(A_0,f_0;t_0-)$, $\mathcal{S}_{loc}(A_0,f_0;t_0+)$ and $\mathcal{S}_{loc}(A_0,f_0;t_0\pm)$

Theorem 1. *Let $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in BV(I;\mathbb{R}^n)$, $t_0\in I$, and the sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that*

$$\det\big(I_n+(-1)^jd_jA_0(t)\big)\neq\ 0 \ \textit{ for } \ t\in I, \ \ (-1)^j(t-t_0)<0 \ \textit{ and for } \ t=t_0$$
$$\textit{if } \ j\in\{1,2\} \quad \textit{is such that } \ (-1)^j(t_k-t_0)>0 \ \textit{ for every } \ k\in\{1,2,\dots\} \tag{9}$$

and

$$\lim_{k\to+\infty}t_k=t_0. \tag{10}$$

Then inclusion (8) holds if and only if there exists a sequence of matrix-functions $H_k\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that

$$\inf\big\{|\det(H_0(t))|:\ t\in I\big\}>0, \tag{11}$$

and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t), \tag{12}$$

$$\lim_{k\to+\infty}\left\{\left\|\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)(\tau)\Big|_{t_k}^{t}\right\|\left(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\big)\Big|\right)\right\}=0$$

and

$$\lim_{k\to+\infty}\left\{\left\|\mathcal{D}_{\mathcal{B}}(f_k,H_k;f_0,H_0)(\tau)\Big|_{t_k}^{t}\right\|\left(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\big)\Big|\right)\right\}=0$$

hold uniformly on I.

The last two conditions are analogy to the Opial conditions concerning to the well-posed question for the ordinary differential case (see [14]). Note that, the Opial condition has only the sufficient character for the last case.

We offer another form of criterion for inclusion (8), differing from Theorem 1.

Theorem 1′. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1,2,\dots)$ be such that conditions* (9) *and* (10) *hold. Then inclusion* (8) *holds if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$ such that conditions* (11) *and*

$$\limsup_{k\to+\infty}\bigvee_{I}(H_k+\mathcal{B}(H_k,A_k))<+\infty$$

hold, and conditions (12),

$$\lim_{k\to+\infty}(\mathcal{B}(H_k,A_k)(t)-\mathcal{B}(H_k,A_k)(t_k))=\mathcal{B}(H_0,A_0)(t)-\mathcal{B}(H_0,A_0)(t_0)$$

and

$$\lim_{k\to+\infty}(\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k))=\mathcal{B}(H_0,f_0)(t)-\mathcal{B}(H_0,f_0)(t_0)$$

hold uniformly on I.

Remark 1. Without loss of generality, we can assume that $H_0(t)\equiv I_n$ in Theorems 1 and 1′. So

$$\mathcal{B}(I_n,Y)(t)-\mathcal{B}(I_n,Y)(s)=Y(t)-Y(s)\ \text{ and }$$
$$\mathcal{I}(I_n,Y)(t)-\mathcal{I}(I_n,Y)(s)=Y(t)-Y(s)\ \text{ for }\ Y\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m}).$$

Theorem 2. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1,2,\dots)$ be such that conditions* (9) *and* (10) *hold. Let, moreover, the sequences of matrix– and vector-functions $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $f_k \in BV_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ be such that the conditions*

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\left\{\|A_{kj}(t)-A_{0j}(t)\|\left(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\right)\right\}=0 \tag{13}$$

and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\left\{\|f_{kj}(t)-f_{0j}(t)\|\left(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\right)\right\}=0 \tag{14}$$

hold if $j\in\{1,2\}$ is such that $(-1)^j(t_k-t_0)\geq 0$ for every $k\in\{1,2,\dots\}$, where

$$A_{kj}(t)\equiv(-1)^j(A_k(t)-A_k(t_k))-d_jA_k(t_k)\ \ (j=1,2;\ \ k=0,1,\dots), \tag{15}$$
$$f_{kj}(t)\equiv(-1)^j(f_k(t)-f_k(t_k))-d_jf_k(t_k)\ \ (j=1,2;\ \ k=0,1,\dots). \tag{16}$$

Then

$$((A_k,f_k;t_k))_{k=1}^{+\infty}\in S_{loc}(A_0,f_0;t_0-)\ \ \textit{if}\ \ j=1,$$

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S_{loc}(A_0, f_0; t_0+) \ \text{ if } \ j = 2$$

and

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S_{loc}(A_0, f_0; t_0\pm) \ \text{ if } \ j \in \{1, 2\}.$$

Remark 2. In Theorem 2, the sequence $x_k(t)$ $(k = 1, 2, \dots)$ converges to x_0 uniformly on the set $\{t \in I,\ t \le t_0\}$ if $t_k > t_0$ $(k = 1, 2, \dots)$, and on the set $\{t \in I,\ t \ge t_0\}$ if $t_k < t_0$ $(k = 1, 2, \dots)$; as for the case, where $t_k = t_0$ $(k = 1, 2, \dots)$, the sequence $x_k(t)$ $(k = 1, 2, \dots)$ converges to x_0 uniformly in both intervals $\{t \in I:\ t < t_0\}$ and $\{t \in I:\ t > t_0\}$. Moreover, if conditions (13) and (14) hold uniformly on the set I, then these conditions are equivalent to the conditions

$$\lim_{k\to+\infty} \Big\{ \big\|(A_k(t) - A_k(t_k)) - (A_0(t) - A_0(t_0))\big\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k - A_0)\Big|\Big)\Big\} = 0 \tag{17}$$

and

$$\lim_{k\to+\infty} \Big\{ \big\|(f_k(t) - f_k(t_k)) - (f_0(t) - f_0(t_0))\big\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k - A_0)\Big|\Big)\Big\} = 0 \tag{18}$$

uniformly on I, respectively, since (17) and (18) imply that

$$\lim_{k\to+\infty} d_j A_k(t) = d_j A_0(t) \ \text{ and } \ \lim_{k\to+\infty} d_j f_k(t) = d_j f_0(t)$$

uniformly on I for every $j \in \{1, 2\}$. In addition, equalities (7_j) $(j = 1, 2)$ hold and,therefore, as above, conditions (3_j) $(j = 1, 2)$ hold, as well. Thus, in the case under consideration, condition (3) holds uniformly on I, i.e., condition (6) holds, as well.

Theorem 3. *Let $A_0^* \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_0^* \in BV(I; \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1, 2, \dots)$ be such that condition* (10) *holds,*

$$\begin{aligned} \det\big(I_n + (-1)^j d_j A_0^*(t)\big) \neq \ 0 \ \ \text{for } \ t \in I, \ \ (-1)^j (t - t_0) < 0 \ \text{ and for } \ t = t_0 \\ \text{if } \ j \in \{1, 2\} \ \text{ is such that } \ (-1)^j (t_k - t_0) > 0 \ \text{ for every } \ k \in \{1, 2, \dots\}, \end{aligned}$$

and the Cauchy problem

$$dx = dA_0^*(t) \cdot x + df_0^*(t),$$
$$x(t_0) = c_0^*$$

has a unique solution x_0^*. Let, moreover, the sequences of matrix– and vector-functions $A_k, H_k \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ and $f_k, h_k \in BV_{loc}(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$ and of constant vectors $c_k^* \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ be such that the conditions

$$\inf\big\{|\det(H_k(t)|:\ t \in I,\ t \neq t_k\big\} > 0 \ \text{ for every sufficiently large } k,$$
$$\lim_{k\to+\infty} c_k^* = c_0^*, \quad \lim_{k\to+\infty} c_{kj}^* = c_{0j}^*, \tag{19}$$
$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \Big\{ \|A_{kj}^*(t) - A_{0j}^*(t)\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k^* - A_0^*)\Big|\Big)\Big\} = 0$$

and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \Big\{ \|f_{kj}^*(t) - f_{0j}^*\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k^* - A_0^*)\Big|\Big)\Big\} = 0$$

hold for some $j \in \{1, 2\}$, where

$$A_{kj}^*(t) = (-1)^j (A_k^*(t) - A_k^*(t_k)) - d_j A_k^*(t_k),$$

$$f^*_{kj}(t) = (-1)^j(f^*_k(t) - f^*_k(t_k)) - d_j f^*_k(t_k) \ \ for \ \ t \in I \ \ (j = 1,2; \ k = 0,1,\dots);$$
$$A^*_k(t) = \mathcal{I}(H_k, A_k)(t),$$
$$f^*_k(t) = h_k(t) - h_k(t_k) + \mathcal{B}(H_k, f_k)(t) - \mathcal{B}(H_k, f_k)(t_k) - \int_{t_k}^{t} dA^*_k(s) \cdot h_k(s) \ \ for \ \ t \in I \ \ (k = 1,2,\dots);$$
$$c^*_k = H_k(t_k)c_k + h_k(t_k) \ \ (k = 1,2,\dots),$$
$$c^*_{kj} = c^*_k + (-1)^j\big(d_j A^*_k(t_k)c^*_k + d_j f^*_k(t_k)\big) \ \ (j = 1,2; \ \ k = 0,1,\dots).$$

Then problem $(1_k), (2_k)$ *has a unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \big\|H_k(t)x_k(t) + h_k(t) - x^*_0(t)\big\| = 0.$$

Remark 3. In Theorem 3, the vector-function $x^*_k(t) \equiv H_k(t)\,x_k(t) + h_k(t)$ is a solution of the problem

$$dx = dA^*_k(t) \cdot x + df^*_k(t),$$
$$x(t_k) = c^*_k$$

for every sufficiently large k.

Corollary 1. *Let* $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$*,* $f_0 \in BV(I;\mathbb{R}^n)$*,* $c_0 \in \mathbb{R}^n$*,* $t_0 \in I$*, and the sequences* $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$*,* $f_k \in BV(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$*,* $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ *and* $t_k \in I$ $(k = 1,2,\dots)$ *be such that conditions* (9)*,* (10)*,* (11)*,*

$$\lim_{k\to+\infty} (c_{kj} - \varphi_k(t_k)) = c_{0j},$$
$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|H_k(t) - H_0(t)\| = 0,$$
$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \left\{ \left\| \mathcal{D}_{\mathcal{I}}(A_k, H_k; A_0, H_0)(\tau)\Big|_{t_k}^{t} \right\| \left(1 + \left| \bigvee_{t_k}^{t} (\mathcal{D}_{\mathcal{I}}(A_k, H_k; A_0, H_0)) \right| \right) \right\} = 0$$

and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \left\{ \left\| \mathcal{D}_{\mathcal{B}}(f_k - \varphi_k, H_k; f_0, H_0)(\tau)\Big|_{t_k}^{t} + \int_{t_k}^{t} d\mathcal{I}(H_k, A_k)(\tau) \cdot \varphi_k(\tau) \right\| \left(1 + \left| \bigvee_{t_k}^{t} (\mathcal{D}_{\mathcal{I}}(A_k, H_k; A_0, H_0)) \right| \right) \right\} = 0$$

hold for some $j \in \{1,2\}$*, where* $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$*,* $\varphi_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$ *and* c_{kj} $(k = 0,1,\dots)$ *are defined by* (4)*. Then problem* $(1_k), (2_k)$ *has a unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|x_k(t) - \varphi_k(t) - x_0(t)\| = 0.$$

(B) The results concerning the sets $\mathcal{S}^*_{loc}(A_0, f_0; t_0-)$, $\mathcal{S}^*_{loc}(A_0, f_0; t_0+)$ and $\mathcal{S}^*_{loc}(A_0, f_0; t_0\pm)$

For the goal, we will use the following easy lemma.

Lemma 1. *Let* $j \in \{1,2\}$ *be such that condition* (5_j) *hold, where* $c_{*j} \in \mathbb{R}^n$*, and the vectors* $c_{kj} (k = 1,2,\dots)$ *are defined by* (4)*. Then the vector-function* $x_{*1}(t) \equiv x_0(t) + x_{01}(t)$ *will be a solution of system* (1) *under the condition* $x(t_0-) = c_{*1}$*, and the vector-function* $x_{*2}(t) \equiv x_0(t) + x_{02}(t)$ *will be a solution of system* (1) *under the condition* $x(t_0+) = c_{*2}$*.*

Theorem 2*. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that conditions* (9) *and* (10) *hold. Let, moreover, the sequences of matrix– and vector-functions $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $f_k \in BV_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ be such that conditions* (13) *and* (14) *hold if $j \in \{1,2\}$ is such that $(-1)^j(t_k-t_0)\ge 0$ for every $k\in\{1,2,\dots\}$, where $A_{kj}(t)$ $(j=1,2;\ k=0,1,\dots)$ and $f_{kj}(t)$ $(j=1,2;\ k=0,1,\dots)$ are defined by* (15) *and* (16), *respectively. Then*

$$((A_k,f_k;t_k))_{k=1}^{+\infty} \in S^*_{loc}(A_0,f_0;t_0-) \ \text{ if } \ j=1,$$
$$((A_k,f_k;t_k))_{k=1}^{+\infty} \in S^*_{loc}(A_0,f_0;t_0+) \ \text{ if } \ j=2$$

and

$$((A_k,f_k;t_k))_{k=1}^{+\infty} \in S^*_{loc}(A_0,f_0;t_0\pm) \ \text{ if } \ j\in\{1,2\}.$$

Theorem 3*. *Let the conditions of Theorem* 3 *be fulfilled, with the exclusion of* (19), *instead of which the condition*

$$\lim_{k\to+\infty} c^*_{kj} = c^*_j, \tag{20}$$

holds, where the vectors $c^*_{kj}\in\mathbb{R}^n$ $(k=1,2,\dots)$ are defined as in Theorem 3, *and $c^*_j\in\mathbb{R}^n$ is a vector differing from c^*_{0j}, in general. Then problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and*

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\ne t_k} \big\|H_k(t)x_k(t)+h_k(t)-x^*_0(t)-x^*_j(t)\big\| = 0,$$

where the function x^*_1 is a solution of the homogeneous system

$$dx = dA^*_0(t)\cdot x$$

on the set $\{t\in I: t<t_0\}$ under the condition

$$x^*_1(t_0-) = c^*_1 - x^*_0(t_0-),$$

and the function x^*_2 is a solution of the homogeneous system on the set $\{t\in I:\ t>t_0\}$ under the condition

$$x^*_2(t_0+) = c^*_2 - x^*_0(t_0+).$$

Corollary 1*. *Let the conditions of Corollary* 1 *be fulfilled with the exclusion of* (20), *instead of which the condition*

$$\lim_{k\to+\infty}(c_{kj}-\varphi_k(t_k)) = c_{*j}$$

holds for some $j\in\{1,2\}$, where the vectors $c^*_{kj}\in\mathbb{R}^n$ $(k=1,2,\dots)$ are defined as in Theorem 3, *and $c^*_j\in\mathbb{R}^n$ is a vector differing from c^*_{0j}, in general. Then problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and*

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\ne t_k} \big\|x_k(t)-\varphi_k(t)-x_0(t)-x_{0j}\big\| = 0,$$

where the vector-function x_{0j} is defined as above.

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Author's addresses:

Malkhaz Ashordia

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. Sokhumi State University, 9 A. Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: malkhaz.ashordia@tsu.ge, ashord@rmi.ge

Malkhaz Kutsia, Mzia Talakhadze

Sokhumi State University, 9 A. Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: maxo51@mail.ru; mzia.talakhadze01@gmail.com