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Short Communication

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ON THE SOLVABILITY AND THE WELL-POSEDNESS OF THE MODIFIED CAUCHY PROBLEM FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Abstract. Effective sufficient conditions are given for the unique solvability and for the so-called H-well-posedness of the modified Cauchy problem for linear systems of generalized ordinary differential equations with singularities.

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1 Statement of the problem and basic notation

Let $I\subset\mathbb{R}$ be an interval non-degenerate at the point, $t_0\in I$, and

$$I_{t_0}=I\setminus\{t_0\},\quad I_{t_0}^{-}=]-\infty,t_0[\cap I,\quad I_{t_0}^{+}=]t_0,+\infty[\cap I.$$

Consider the linear system of generalized ordinary differential equations

$$dx=dA(t)\cdot x+df(t)\ \text{ for }\ t\in I_{t_0}, \tag{1.1}$$

where

$$A=(a_{ik})_{i,k=1}^{n}\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n}),\quad f=(f_k)_{k=1}^{n}\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n}).$$

Let $H=\operatorname{diag}(h_1,\dots,h_n):I_{t_0}\to\mathbb{R}^{n\times n}$ be arbitrary diagonal matrix-functions with continuous diagonal elements

$$h_k:I_{t_0}\to]0,+\infty[\ \ (k=1,\dots,n).$$

We consider the problem of finding a solution $x\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^n)$ of system (1.1) satisfying the modified Cauchy condition

$$\lim_{t\to t_0-}(H^{-1}(t)\,x(t))=0\ \text{ and }\ \lim_{t\to t_0+}(H^{-1}(t)\,x(t))=0. \tag{1.2}$$

Along with system (1.1), consider the perturbed singular system

$$dy=d\widetilde{A}(t)\cdot y+d\widetilde{f}(t)\ \text{ for }\ t\in I_{t_0}, \tag{1.3}$$

where

$$\widetilde{A}=(\widetilde{a}_{ik})_{i,k=1}^{n}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n}),\quad \widetilde{f}=(\widetilde{f}_k)_{k=1}^{n}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n})$$

are, as above, the matrix- and vector-functions, respectively.

In the present paper, we give sufficient conditions for the unique solvability of problem (1.1), (1.2). Moreover, we investigate the question when the unique solvability of problem (1.1), (1.2) guarantees unique solvability of problem (1.3), (1.2) and, as well, the nearness of their solutions in the definite sense if the matrix-functions A and $\widetilde{A}$ and the vector-functions f and $\widetilde{f}$ are near, respectively.

The analogous problems for system of ordinary differential equations with singularities

$$\frac{dx}{dt}=P(t)\,x+q(t)\ \text{ for }\ t\in I, \tag{1.4}$$

where

$$P\in L_{loc}(I_{t_0},\mathbb{R}^{n\times n}),\quad q\in L_{loc}(I_{t_0},\mathbb{R}^{n}),$$

have been investigated in the papers [6–8].

The singularity of system (1.4) is considered in the sense that the matrix-function P and the vector-function q are, in general, not integrable at the point t_0. In general, a solution of problem (1.4), (1.2) is not continuous at the point t_0 and, therefore, it cannot be a solution in the classical sense. But its restriction on every interval from I_{t_0} is a solution of system (1.4). In this connection we give the example from [8].

Let $\alpha>0$ and $\varepsilon\in\,]0,\alpha[\,$. Then the problem

$$\frac{dx}{dt}=-\frac{\alpha\,x}{t}+\varepsilon|t|^{\varepsilon-1-\alpha},\quad \lim_{t\to 0}(t^{\alpha}x(t))=0$$

has the unique solution $x(t)=|t|^{\varepsilon-\alpha}\operatorname{sgn} t$. This function is not a solution of the equation in the set $I=\mathbb{R}$, but its restrictions on $]-\infty,0[$ and $]0,+\infty[$ are the solutions of these equation.

The singularity of system (1.1) is considered in the sense that the matrix-function A and the vector-function f may have non-bounded total variation at the point t_0, i.e., on some closed interval $[a,b]$ from I such that $t_0\in[a,b]$.

As is known, such a problem for generalized differential system (1.1) has not been studied. So, the problem remains actual.

Some singular two-point boundary problems for generalized differential system (1.1) are investigated in [3–5].

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to study ordinary differential, impulsive and difference equations from a unified point of view (see [2–5, 10, 11] and the references therein).

In the paper the use will be made of the following notation and definitions.

$\mathbb{R}=\,]-\infty,+\infty[\,$, $\mathbb{R}_+=[0,+\infty[\,$, $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are, respectively, the closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ik})_{i,k=1}^{n,m}$ with the norm $\|X\|=\max\limits_{k=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ik}|$.

If $X=(x_{ik})_{i,k=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ik}|)_{i,k=1}^{n,m}$, $[X]_+=\frac{|X|+X}{2}$, $[X]_-=\frac{|X|-X}{2}$.

$\mathbb{R}_+^{n\times m}=\{(x_{ik})_{i,k=1}^{n,m}:\ x_{ik}\geq 0\ (i=1,\dots,n;\ k=1,\dots,m)\}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^{n}$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_{a}^{b}(X)$ is the sum of total variations on $[a,b]$ of its components x_{ik} $(i=1,\dots,n;\ k=1,\dots,m)$; if $a>b$, then we assume $\bigvee\limits_{a}^{b}(X)=-\bigvee\limits_{b}^{a}(X)$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t ($X(a-) = X(a)$, $X(b+) = X(b)$).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X) < \infty$).

$\mathrm{BV}_{loc}(J;D)$, where $J \subset \mathbb{R}$ is an interval and $D \subset \mathbb{R}^{n\times m}$, is the set of all $X : J \to D$ whose restriction on $[a,b]$ belongs to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from J.

$\mathrm{BV}_{loc}(I_{t_0};D)$ is the set of all $X : I \to D$ whose restriction on $[a,b]$ belongs to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from I_{t_0}.

Everywhere we assume that $a_1 \in I_{t_0}^-$ and $a_2 \in I_{t_0}^+$ are some fixed points.

If $X \in \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$, then $V(X)(t) = (v(x_{ik})(t))_{i,k=1}^{n,m}$ for $t \in I_{t_0}$, where $v(x_{ik})(a_j) = 0$, $v(x_{ik})(t) \equiv \bigvee\limits_{a_j}^t(x_{ik})$ for $(t-t_0)(a_j-t_0) > 0$ $(j=1,2)$.

$[X(t)]_+^v \equiv \frac{V(X)(t)+X(t)}{2}$, $[X(t)]_-^v \equiv \frac{V(X)(t)-X(t)}{2}$.

s_1, s_2, s_c and $\mathcal{J} : \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}) \to \mathrm{BV}_{loc}(I_{t_0};\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a_j) = s_2(x)(a_j) = 0, \quad s_c(x)(a_j) = x(a_j);$$

$$s_1(x)(t) = s_1(x)(s) + \sum_{s<\tau\le t} d_1x(\tau), \quad s_2(x)(t) = s_2(x)(s) + \sum_{s\le\tau<t} d_2x(\tau)$$

$$s_c(x)(t) = s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^{2}(s_j(x)(t) - s_j(x)(s))$$

$$\text{for } s<t<t_0 \text{ if } a_j<t_0 \text{ and for } t_0<s<t \text{ if } a_j>t_0 \ (j=1,2)$$

and

$$\mathcal{J}(x)(a_j) = x(a_j),$$

$$\mathcal{J}(x)(t) = \mathcal{J}(x)(s) + s_c(x)(t) - s_c(x)(s) - \sum_{s<\tau\le t}\ln|1-d_1x(\tau)| + \sum_{s\le\tau<t}\ln|1+d_2x(\tau)|$$

$$\text{for } s<t<t_0 \text{ if } a_j<t_0 \text{ and for } t_0<s<t<t_0 \text{ if } a_j>t_0 \ (j=1,2).$$

If $X \in \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^jd_jX(t)) \ne 0$ for $t \in I_{t_0}$ $(j=1,2)$, and $Y \in \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(a_j) = O_{n\times m},$$

$$\mathcal{A}(X,Y)(t) - \mathcal{A}(X,Y)(s) = Y(t) - Y(s) + \sum_{s<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\,d_1Y(\tau)$$

$$- \sum_{s\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\,d_2Y(\tau)$$

$$\text{for } s<t<t_0 \text{ if } a_j<t_0 \text{ and for } t_0<s<t<t_0 \text{ if } a_j>t_0 \ (j=1,2).$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)\,d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)\,d_2g(\tau),$$

where $\int\limits_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$. If $a=b$, then we assume $\int\limits_a^b x(t)\,dg(t) = 0$, and if $a>b$, then $\int\limits_a^b x(t)\,dg(t) = -\int\limits_b^a x(t)\,dg(t)$. So, $\int\limits_s^t x(\tau)\,dg(\tau)$ is the Kurzweil integral [9–11].

Moreover, we put

$$\int\limits_s^{t+} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int\limits_s^{t+\delta} x(\tau)\,dg(\tau),\quad \int\limits_s^{t-} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int\limits_s^{t-\delta} x(\tau)\,dg(\tau).$$

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_s^t x(\tau)\,dg_1(\tau)-\int\limits_s^t x(\tau)\,dg_2(\tau)\ \text{ for }\ s,t\in\mathbb{R}.$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int\limits_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}\quad\text{for }\ a\le s\le t\le b,$$
$$S_c(G)(t)\equiv(s_c(g_{ik})(t))_{i,k=1}^{l,n},\ \ S_j(G)(t)\equiv\big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\ \ (j=1,2).$$

If $G_j:[a,b]\to\mathbb{R}^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G=G_1-G_2$ and $X:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\int\limits_s^t dG_1(\tau)\cdot X(\tau)-\int\limits_s^t dG_2(\tau)\cdot X(\tau)\ \text{ for }\ s,\ t\in\mathbb{R},$$
$$S_c(G)=S_c(G_1)-S_c(G_2),\ \ S_j(G)=S_j(G_1)-S_j(G_2)\ \ (j=1,2).$$

A vector-function $x:I_{t_0}\to\mathbb{R}^n$ is said to be a solution of system (1.1) if $x\in\mathrm{BV}([a,b],\mathbb{R}^n)$ for every closed interval $[a,b]$ from I_{t_0} and

$$x(t)=x(s)+\int\limits_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\quad\text{for}\quad a\le s<t\le b.$$

We assume that

$$\det(I_n+(-1)^jd_jA(t))\neq 0\quad\text{for}\quad t\in I_{t_0}\quad (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems (see [9–11]), i.e., for the case when $A\in\mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and $f\in\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Let the matrix-function $A_0\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ be such that

$$\det\big(I_n+(-1)^jd_jA_0(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2).\tag{1.5}$$

Then a matrix-function $C_0:I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the generalized differential system

$$dx=dA_0(t)\cdot x,\tag{1.6}$$

if for every interval and $J\subset I$ and $\tau\in J$, the restriction of the matrix-function $C_0(.,\tau):I_{t_0}\to\mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (1.6) satisfying the condition

$$C_0(\tau,\tau)=I_n.$$

Therefore, C_0 is the Cauchy matrix of system (1.6) if and only if the restriction of C_0 on every interval $J\times J$ is the Cauchy matrix of the system in the sense of definition given in [11].

We assume

$$I_{t_0}^-(\delta)=[t_0-\delta,t_0[\cap I_{t_0},\ \ I_{t_0}^+(\delta)=]t_0,t_0+\delta]\cap I_{t_0},\ \ I_{t_0}(\delta)=I_{t_0}^-(\delta)\cup I_{t_0}^+(\delta)$$

for every $\delta>0$.

2 Existence and uniqueness of solutions of the Cauchy problem

In this section we give sufficient conditions for the unique solvability of problem (1.1), (1.2).

Theorem 2.1. *Let there exist a matrix-function* $A_0 \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ *and constant matrices* B_0 *and* B *from* $\mathbb{R}_+^{n\times n}$ *such that conditions* (1.5) *and*

$$r(B) < 1 \tag{2.1}$$

hold, and the estimates

$$|C_0(t,\tau)| \leq H(t)\, B_0\, H^{-1}(\tau) \ \text{ for } \ t \in I_{t_0}(\delta), \quad (t-t_0)(\tau-t_0) > 0, \ \ |\tau - t_0| \leq |t - t_0| \tag{2.2}$$

and

$$\bigg| \int\limits_{t_0\mp}^{t} |C_0(t,\tau)|\, dV(\mathcal{A}(A_0, A - A_0)(\tau)) \cdot H(\tau) \bigg| \leq H(t)\, B$$
$$\text{for } \ t \in I_{t_0}^-(\delta) \ \text{ and } \ t \in I_{t_0}^+(\delta), \ \text{ respectively,} \tag{2.3}$$

are valid for some $\delta > 0$, *where* C_0 *is the Cauchy matrix of system* (1.4). *Let, moreover, respectively,*

$$\lim_{t \to t_0\mp} \bigg\| \int\limits_{t_0\mp}^{t} H^{-1}(\tau)\, |C_0(t,\tau)|\, dV(\mathcal{A}(A_0, f))(\tau) \bigg\| = 0. \tag{2.4}$$

Then problem (1.1), (1.2) *has the unique solution.*

Theorem 2.2. *Let there exist a constant matrix* $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *such that conditions* (2.1) *and*

$$\begin{gathered} \big[(-1)^j d_j a_{ii}(t)\big]_+ > -1 \ \text{ for } \ t < t_0 \ \ (j = 1,2;\ i = 1,\dots,n), \\ \big[(-1)^j d_j a_{ii}(t)\big]_- < 1 \ \text{ for } \ t > t_0 \ \ (j = 1,2;\ i = 1,\dots,n) \end{gathered} \tag{2.5}$$

hold, and the estimates

$$|c_i(t,\tau)| \leq b_0 \frac{h_i(t)}{h_i(\tau)} \ \text{ for } \ t \in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0) > 0, \ \ |\tau - t_0| \leq |t - t_0| \ \ (i = 1,\dots,n), \tag{2.6}$$

$$\bigg| \int\limits_{t_0\mp}^{t} c_i(t,\tau) h_i(\tau)\, d\big[a_{ii}(\tau)\operatorname{sgn}(\tau - t_0)\big]_+^v \bigg|$$
$$\leq b_{ii}(t)\, h_i(t) \ \text{ for } \ t \in I_{t_0}^-(\delta) \ \text{ and } \ t \in I_{t_0}^+(\delta), \ \text{ respectively } \ (i = 1,\dots,n) \tag{2.7}$$

and

$$\bigg| \int\limits_{t_0\mp}^{t} c_i(t,\tau) h_k(\tau)\, dV(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \bigg| \leq b_{ik}(t)\, h_i(t)$$
$$\text{for } \ t \in I_{t_0}^-(\delta) \ \text{ and } \ t \in I_{t_0}^+(\delta), \ \text{ respectively } \ (i \neq k;\ i,k = 1,\dots,n) \tag{2.8}$$

are valid for some $b_0 > 0$ *and* $\delta > 0$. *Let, moreover, respectively,*

$$\lim_{t \to t_0\mp} \int\limits_{t_0\mp}^{t} \frac{c_i(t,\tau)}{h_i(t)}\, dV(\mathcal{A}(a_{0ii}, f_i))(\tau) = 0 \ \ (i = 1,\dots,n), \tag{2.9}$$

where $a_{0ii}(t) \equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v \operatorname{sgn}(t-t_0)$ $(i = 1,\dots,n)$ *and* c_i *is the Cauchy function of the equation* $dx = x\, da_{0ii}(t)$ *for* $i \in \{1,\dots,n\}$. *Then problem* (1.1), (1.2) *has the unique solution.*

Remark 2.1. The Cauchy functions $c_i(t,\tau)$ $(i=1,\dots,n)$, mentioned in the theorem, for $t,\tau\in I_{t_0}^-$ and $t,\tau\in I_{t_0}^+$, have the form

$$c_i(t,\tau)=\begin{cases}\exp\big(s_0(a_{0ii})(t)-s_0(a_{0ii})(\tau)\big)\prod\limits_{\tau<s\le t}(1-d_1a_{0ii}(s))^{-1}\prod\limits_{\tau\le s<t}(1+d_2a_{0ii}(s)) & \text{for } t>\tau,\\ \exp\big(s_0(a_{0ii})(t)-s_0(a_{0ii})(\tau)\big)\prod\limits_{t<s\le \tau}(1-d_1a_{0ii}(s))\prod\limits_{t\le s<\tau}(1+d_2a_{0ii}(s))^{-1} & \text{for } t<\tau,\\ \qquad 1 & \text{for } t=\tau.\end{cases}$$

Corollary 2.1. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that conditions* (2.1) *and* (2.5) *hold, and the estimates*

$$\bigg|\int\limits_{t_0\mp}^{t}|\tau-t_0|\,d\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\bigg|$$
$$\le b_{ii}\,|t-t_0| \ \textit{ for } \ t\in I_{t_0}^-(\delta) \ \textit{ and } \ t\in I_{t_0}^+(\delta), \ \textit{ respectively } \ (i=1,\dots,n) \tag{2.10}$$

and

$$\bigg|\int\limits_{t_0\mp}^{t}|\tau-t_0|\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg|$$
$$\le b_{ik}\,|t-t_0| \ \textit{ for } \ t\in I_{t_0}^-(\delta) \ \textit{ and } \ t\in I_{t_0}^+(\delta), \ \textit{ respectively } \ (i\ne k;\ i,k=1,\dots,n) \tag{2.11}$$

are valid for some $\delta>0$. Let, moreover, respectively,

$$\lim_{t\to t_0\mp}\frac{1}{|t-t_0|}\,\Big|\bigvee_{t_0}^{t}(\mathcal{A}(a_{0ii},f_i))(\tau)\Big|=0 \ \ (i=1,\dots,n), \tag{2.12}$$

where $a_{0ii}(t)\equiv-[a_{ii}(t)\operatorname{sgn}(t-t_0))]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$. Then system (1.1) *has the unique solution satisfying the initial condition*

$$\lim_{t\to t_0\mp}\frac{\|x(t)\|}{t-t_0}=0. \tag{2.13}$$

Remark 2.2. In Corollary 2.2, if the estimates

$$\bigg|\int\limits_{s}^{t}|\tau-t_0|\,d\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\bigg|\le b_{ii}|t-s|$$
$$\text{for } \ t,s\in I_{t_0}(\delta), \ \ (t-t_0)(s-t_0)>0, \ \ |s-t_0|\le|t-t_0| \ \ (i=1,\dots,n)$$

and

$$\bigg|\int\limits_{s}^{t}|\tau-t_0|\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg|\le b_{ik}\,|t-s|$$
$$\text{for } \ t,s\in I_{t_0}(\delta), \ \ (t-t_0)(s-t_0)>0, \ \ |s-t_0|\le|t-t_0| \ \ (i\ne k;\ i,k=1,\dots,n)$$

hold instead of (2.10) and (2.11), respectively, then the solution of problem (1.1), (2.13) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Corollary 2.2. *Let conditions* (2.5) *and*

$$\mathcal{J}(a_{0ii})(t)-\mathcal{J}(a_{0ii})(\tau)\le-\lambda_i\ln\frac{t-t_0}{\tau-t_0}+a_{ii}^*(t)-a_{ii}^*(\tau)$$
$$\textit{for } \ t,\tau\in I_{t_0}, \ \ (t-t_0)(\tau-t_0)>0, \ \ |\tau-t_0|\le|t-t_0| \ \ (i=1,\dots,n) \tag{2.14}$$

hold, where $a_{0ii}(t) \equiv -[a_{ii}(t)\,\mathrm{sgn}(t-t_0)]_-^v\,\mathrm{sgn}(t-t_0)$ $(i=1,\dots,n)$, $\lambda_i \geq 0$ $(i=1,\dots,n)$, a_{ii}^* $(i=1,\dots,n)$ *are nondecreasing functions on the intervals* $I_{t_0}^-$ *and* $I_{t_0}^+$. *Let, moreover,*

$$\bigg|\int\limits_{t_0\mp}^{t} |\tau-t_0|^{\lambda_i-\lambda_k}\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg| < +\infty$$
$$\textit{for } t\in I_{t_0}^- \textit{ and } t\in I_{t_0}^+, \textit{ respectively } (i\neq k;\ i,k=1,\dots,n), \tag{2.15}$$

and

$$\bigg|\int\limits_{t_0\mp}^{t} |\tau-t_0|^{\lambda_i}\,dV(\mathcal{A}(a_{0ii},f_i))(\tau)\bigg| < +\infty$$
$$\textit{for } t\in I_{t_0}^- \textit{ and } t\in I_{t_0}^+, \textit{ respectively } (i=1,\dots,n). \tag{2.16}$$

Then system (1.1) *has the unique solution satisfying the initial condition*

$$\lim_{t\to t_0\mp}\big(|t-t_0|^{\lambda_i}\,x_i(t)\big)=0 \ \ (i=1,\dots,n). \tag{2.17}$$

3 Well-posedness of the Cauchy problem

Let $I_{t_0t} =]\min\{t_0,t\},\max\{t_0,t\}[$ for $t\in I$.

Definition 3.1. Problem (1.1), (1.2) is said to be H-well-posed if it has the unique solution x and for every $\varepsilon>0$ there exists $\eta>0$ such that problem (1.3), (1.2) has the unique solution y and the estimate

$$\|H(t)\,(x(t)-y(t))\| < \varepsilon \ \text{ for } \ t\in I$$

holds for every $\widetilde{A}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and $\widetilde{f}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^n)$ such that

$$\det\big(I_n+(-1)^j d_j\widetilde{A}(t)\big)\neq 0 \ \text{ for } \ t\in I_{t_0} \ \ (j=1,2);$$

$$\bigg\|\int\limits_{t_0\mp}^{t} H^{-1}(s)\,dV(\widetilde{A}-A)(s)\cdot H(s)\bigg\| + \sum_{j=1}^{2}\bigg\|\sum_{\tau\in I_{t_0t}} H^{-1}(\tau)|d_j(\widetilde{A}-A)(\tau)|H(\tau)\bigg\| < \eta$$
$$\text{for } \ t\in I_{t_0}^- \ \text{ and } \ t\in I_{t_0}^+, \ \text{ respectively } \ \ \text{(j=1,2)},$$

and

$$\bigg\|\int\limits_{t_0\mp}^{t} H^{-1}(s)\,dV(\widetilde{f}-f)(s)\cdot H(s)\bigg\| + \sum_{j=1}^{2}\bigg\|\sum_{\tau\in I_{t_0t}} H^{-1}(\tau)|d_j(\widetilde{f}-f)(\tau)|H(\tau)\bigg\| < \eta$$
$$\text{for } \ t\in I_{t_0}^- \ \text{ and } \ t\in I_{t_0}^+, \ \text{ respectively } \ \ \text{(j=1,2)}.$$

Theorem 3.1. *Let I be a closed interval and there exist a matrix-function* $A_0\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ *and constant matrices* B_0 *and* B *from* $\mathbb{R}_+^{n\times n}$ *such that conditions* (1.5), (2.1) *hold and estimates* (2.2),

$$|C_0(t,\tau)|\,|d_jA_0(\tau)(I_n+(-1)^j d_jA_0(\tau))^{-1}| \leq H(t)\,B_0\,H^{-1}(\tau)$$
$$\textit{for } \ t\in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0)>0, \ \ |\tau-t_0|\leq|t-t_0| \ \ (j=1,2)$$

and

$$\bigg\|\int\limits_{t_0\mp}^{t}|C_0(t,\tau)|\,dV(A)(s)\cdot H(s)\bigg\| \\ +\sum_{j=1}^{2}\bigg\|\sum_{l\in I_{t_0t}}|C_0(t,\tau)||d_jA_0(\tau)\cdot(I_n+(-1)^jd_jA_0(\tau))^{-1}|\,|d_jA(\tau)|H(\tau)\bigg\|<\eta$$

$$\text{for } t\in I_{t_0}^{-} \text{ and } t\in I_{t_0}^{+}, \text{ respectively,}$$

are valid for some $\delta>0$, where C_0 is the Cauchy matrix of system (1.6)*. Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\Bigg(\bigg\|\int\limits_{t_0\mp}^{t}H^{-1}(t)\,|C_0(t,\tau)|\,dV(f)(\tau)\bigg\| \\ +\sum_{j=1}^{2}\bigg\|\sum_{l\in I_{t_0t}}H^{-1}(t)|C_0(t,\tau)|\,|d_jA_0(\tau)\cdot(I_n+(-1)^jd_jA_0(\tau))^{-1}|\,|d_jf(\tau)|\bigg\|\Bigg)=0.$$

Then problem (1.1), (1.2) *is H-well-posed.*

Theorem 3.2. *Let I be a closed interval and there exist a constant matrix $B=(b_{ik})_{i,k=1}^{n}\in\mathbb{R}_+^{n\times n}$ such that conditions* (2.1), (2.5) *hold and estimates* (2.6), (2.7),

$$|c_i(t,\tau)|\,|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\leq b_0\frac{h_i(t)}{h_i(\tau)}$$

$$\text{for } t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\leq|t-t_0|\ \ (i=1,\dots,n;\ j=1,2)$$

and

$$\bigg|\int\limits_{t_0\mp}^{t}|c_i(t,\tau)|h_k(\tau)\,dv(a_{ik})(\tau)\bigg| \\ +\sum_{j=1}^{2}\bigg|\sum_{\tau\in I_{t_0t}}|c_i(t,\tau)||d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_ja_{ik}(\tau)|h_i(\tau)\bigg|\leq b_{ik}\,h_i(t)$$

$$\text{for } t\in I_{t_0}^{-}(\delta) \text{ and } t\in I_{t_0}^{+}(\delta), \text{ respectively } (i\neq k;\ i,k=1,\dots,n)$$

are valid for some $b_0>0$ and $\delta>0$. Let, moreover, respectively,

$$\lim_{t\to t_0\mp}\Bigg(\bigg|\int\limits_{t_0\mp}^{t}\frac{|c_i(t,\tau)|}{h_i(t)}\,dv(f_i)(\tau)\bigg| \\ +\sum_{j=1}^{2}\sum_{\tau\in I_{t_0t}}\frac{|c_i(t,\tau)|}{h_i(t)}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_jf_i(\tau)|\Bigg)=0\ \ (i=1,\dots,n),$$

where $a_{0ii}(t)\equiv-[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$, and c_i is the Cauchy function of the equation $dx=x\,da_{0ii}(t)$ for $i\in\{1,\dots,n\}$. Then problem (1.1), (1.2) *is H-well-posed.*

Corollary 3.1. *Let I be a closed interval and there exist a constant matrix $B=(b_{ik})_{i,k=1}^{n}\in\mathbb{R}_+^{n\times n}$*

such that conditions (2.1) *and* (2.5) *hold, and the estimates*

$$\mathcal{J}(a_{0ii})(t)-\mathcal{J}(a_{0ii})(\tau)\leq \mu_i \ln\frac{t-t_0}{\tau-t_0}$$
$$\text{for } t,\tau\in I_{t_0},\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\leq|t-t_0|\ \ (i=1,\dots,n), \tag{3.1}$$
$$\lim_{\tau\to t_0\mp}\Big|\big[a_{ii}(t)\operatorname{sgn}(t-t_0)\big]_+^v-\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\Big|$$
$$\leq b_{ii}\ \text{ for } t\in I_{t_0}^-(\delta)\ \text{ and } t\in I_{t_0}^+(\delta),\ \text{ respectively } (i=1,\dots,n)$$

and

$$\lim_{\tau\to t_0\mp}\big|v(a_{ik})(t)-v(a_{ik})(\tau)+\sum_{j=1}^{2}\sum_{s\in I_{t_0\tau}}|d_j a_{0ii}(s)\cdot(1+(-1)^j d_j a_{0ii}(s))^{-1}|\,|d_j a_{ik}(s)|\leq b_{ik}$$
$$\text{for } t\in I_{t_0}^-(\delta)\ \text{ and } t\in I_{t_0}^+(\delta),\ \text{ respectively } (i\neq k;\ i,k=1,\dots,n)$$

are valid for some $\mu_i\geq 0$ $(i=1,\dots,n)$ *and* $\delta>0$, *where* $a_{0ii}(t)\equiv-[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$. *Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\bigg(\bigg|\int\limits_{t_0\mp}^{t}\frac{1}{|\tau-t_0|^{\mu_i}}\,dv(f_i)(\tau)\bigg|$$
$$+\sum_{j=1}^{2}\sum_{\tau\in I_{t_0\tau}}\frac{1}{|\tau-t_0|^{\mu_i}}|d_j a_{0ii}(\tau)\cdot(1+(-1)^j d_j a_{0ii}(\tau))^{-1}|\,|d_j f_i(\tau)|\bigg)=0\ \ (i=1,\dots,n).$$

Then system (1.1) *under the condition*

$$\lim_{t\to t_0\mp}\frac{x_i(t)}{|t-t_0|^{\mu_i}}=0\ \ (i=1,\dots,n) \tag{3.2}$$

is H-well-posed.

Remark 3.1. Let, in addition to the conditions of Corollary 3.1, the condition

$$\lim_{t\to t_0\mp}\sup\xi_{ji}(t)<+\infty\ \ (j=1,2;\ i=1,\dots,n) \tag{3.3}$$

hold, where

$$\xi_{ji}(t)=\sum_{\tau\in I_{tj}}\sum_{k=1}^{n}|\tau-t_0|^{\mu_k}|d_j a_{ik}(\tau)|+|d_j f_i(\tau)|\ \text{ for } t\in I_{t_0}\cap]a_1,a_2[\ \ (j=1,2;\ i=1,\dots,n), \tag{3.4}$$

$I_{t1}=]a_1,t]$ and $I_{t2}=[a_1,t[$ for $a_1<t<t_0$, $I_{t1}=]t,a_2]$ and $I_{t2}=[t,a_2[$ for $t_0<t<a_2$. Then the solution of problem (1.1), (3.2) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Corollary 3.2. *Let I be a closed interval and there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that conditions* (2.1) *and* (2.5) *hold, and estimates* (2.10), (3.1) *for* $\mu_i=0$ $(i=1,\dots,n)$ *and*

$$\bigg|\int\limits_{t_0\mp}^{t}|\tau-t_0|\,dv(a_{ik}))(\tau)\bigg|+\sum_{j=1}^{2}\sum_{\tau\in I_{t_0t}}|\tau-t_0||d_j a_{0ii}(\tau)\cdot(1+(-1)^j d_j a_{0ii}(\tau))^{-1}|\,|d_j a_{ik}(\tau)|\leq b_{ik}|t-t_0|$$
$$\text{for } t\in I_{t_0}^-(\delta)\ \text{ and } t\in I_{t_0}^+(\delta),\ \text{ respectively } (i\neq k;\ i,k=1,\dots,n)$$

are valid for some $\delta > 0$, *where* $a_{0ii}(t) \equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v \operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$. *Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\frac{1}{|t-t_0|}\bigg(|v(f_i)(t)-v(f_i)(t_0\mp)| + \sum_{j=1}^{2}\sum_{\tau\in I_{t_0\tau}}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_jf_i(\tau)|\bigg)=0\ \ (i=1,\dots,n).$$

Then problem (1.1), (2.13) *is* H*-well-posed.*

Remark 3.2. Let, in addition to the conditions of Corollary 3.2, condition (3.3) hold, where the functions ξ_{ji} $(j=1,2;\ i=1,\dots,n)$ are defined by (3.4), $\mu_i=1$ $(i=1,\dots,n)$, and the intervals I_{tj} $(j=1,2)$ are defined as in Remark 3.1. Then the solution of problem (1.1), (2.13) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Corollary 3.3. *Let* I *be a closed interval and let conditions* (2.5) *and* (2.14) *hold, where* $a_{0ii}(t)\equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$, $\lambda_i\geq 0$ $(i=1,\dots,n)$, *and the functions* $a_{ii}^*(t)\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$ *are nondecreasing on the interval* I. *Let, moreover,*

$$\bigg|\int_{t_0\mp}^{t}|\tau-t_0|^{\lambda_i-\lambda_k}\,dv(a_{ik}))\tau)\bigg| + \sum_{j=1}^{2}\bigg|\sum_{\tau\in I_{t_0t}}|\tau-t_0|^{\lambda_i-\lambda_k}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_ja_{ik}(\tau)|\bigg|<+\infty$$

for $t\in I_{t_0}^+$ *and* $t\in I_{t_0}^-$, *respectively* $(i\neq k;\ i,k=1,\dots,n)$

and

$$\bigg|\int_{t_0\mp}^{t}|\tau-t_0|^{\lambda_i}\,dv(f_i))(\tau)\bigg|+\sum_{j=1}^{2}\sum_{\tau\in I_{t_0t}}|\tau-t_0|^{\lambda_i-\lambda_k}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_jf_i(\tau)|<+\infty$$

for $t\in I_{t_0}^-$ *and* $t\in I_{t_0}^+$, *respectively* $(i=1,\dots,n)$.

Then system (1.1) *under the condition*

$$\lim_{t\to t_0\mp}\big(|t-t_0|^{\lambda_i}\,x_i(t)\big)=0\ \ (i=1,\dots,n) \tag{3.5}$$

is H*-well-posed.*

Remark 3.3. Let the conditions of Corollary (3.3) hold, where $\lambda_i=0$ $(i=1,\dots,n)$. Let, in addition, condition (3.3) hold, where the functions ξ_{ji} $(j=1,2;\ i=1,\dots,n)$ are defined by (3.4), $\mu_i=0$ $(i=1,\dots,n)$, and the intervals I_{tj} $(j=1,2)$ are defined as in Remark 3.1. Then the solution of problem (1.1), (3.5) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Remark 3.4. In Remarks 3.1–3.3, condition (3.3) is essential, i.e., if the condition is violated, then the conclusion of our remarks are not true. Below, we reduce the corresponding example. Let $I=[0,1]$, $n=1$, $t_0=0$, $t_n=1/\sqrt{n}$ $(n=1,2,\dots)$, the function $a: I\to\mathbb{R}$ is defined by

$$a(0)=0,\ \ a(1)=-\ln 2,\ \ a(t)=\ln\Big(k_n(t-t_n)+\frac{1}{n}\Big)\ \text{ for }\ t_n\leq t<t_{n-1}\ \ (n=2,3,\dots),$$

where $k_n=(n-2)(2n(n-1)(t_n-t_{n-1}))^{-1}$ $(n=2,3,\dots)$. It is evident that the singular Cauchy problem

$$dx=x\,da(t),\ \ \lim_{t\to 0}t^{-1}|x(t)|=0$$

has the unique solution x defined by the equalities

$$x(t) = k_n(t - t_n) + \frac{1}{n} \ \text{ for } \ t_n \le t < t_{n-1} \ \ (n = 2, 3, \dots), \ \ x(1) = -\ln 2.$$

Moreover, we have $d_2x(t) \equiv 0$ and $d_1x(t_n) = 1/2$ $(n = 2, 3, \dots)$. Thus we conclude that $x \in \mathrm{BV}_{loc}(I_{t_0}; \mathbb{R})$, but $x \notin \mathrm{BV}_{loc}(I; \mathbb{R})$. Besides, taking into account that the function $a(t)$ is nonincreasing on the intervals $t_n \le t < t_{n-1}$ $(n = 2, 3, \dots)$, we conclude that $[a(t)]_+^v = 0$ on these intervals. Therefore, due to the equalities $d_2a(t) \equiv 0$ and $d_1a(t_n) = 1/2$ $(n = 2, 3, \dots)$, all the conditions of our remarks are fulfilled with the exclusion of (3.3).

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