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**O. O. Chepok**

# ASYMPTOTIC REPRESENTATIONS OF A CLASS OF REGULARLY VARYING SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH RAPIDLY AND REGULARLY VARYING NONLINEARITIES

**Abstract.** The asymptotic representations of solutions of a class of differential equations of the second order with rapidly and regularly varying nonlinearities are established.

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**რეზიუმე.** მიღებულია გარკვეული კლასის სწრაფად და რეგულარულად ცვლადი არაწრფივობის მქონე მეორე რიგის დიფერენციალური განტოლებების ამონახსნთა ასიმპტოტური წარმოდგენები.

# 1 Introduction

We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'), \tag{1.1}$$

where $\alpha_0 \in \{-1; 1\}$, the functions $p : [a;\omega[ \to ]0;+\infty[$ $(-\infty < a < \omega \le +\infty)$, and $\varphi_i : \Delta_{Y_i} \to ]0;+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, $\Delta_{Y_i}$ is either an interval $[y_i^0, Y_i[$ [1] or an interval $]Y_i; y_i^0]$. We suppose that $\varphi_1$ is a regularly varying function of index $\sigma_1$ as $y \to Y_1$ $(y \in \Delta_{Y_1})$ [7, pp. 10–15], and the function $\varphi_0$ is strongly monotonous on $\Delta_{Y_0}$, twice continuously differentiable on $\Delta_{Y_0}$ and satisfies the following conditions:

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{1.2}$$

The second order differential equations with both power-type and exponential-type nonlinearities in the right-hand side play an important role in the qualitative theory of differential equations. Such equations have a lot of applications in practice. The fact takes place, for example, during investigations of distribution of electrostatic potential in a cylindrical plasma volume of combustion products. The corresponding equation can be reduced to the following one:

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

This equation is of type (1.1), in which $\varphi_1(z) = |z|^{\lambda}$, $\varphi_0(z) = e^{\sigma z}$. Under some restrictions on the function $p(t)$, certain results for the asymptotic behavior of all regular solutions of that equation have been obtained in the papers by V. M. Evtukhov and N. G. Dric (see, for example, [2]).

The differential equation

$$y'' = \alpha_0 p(t)\varphi(y)$$

with a rapidly varying function $\varphi$ has been considered in the paper by V. M. Evtukhov and V. M. Kharkov [3]. But in that paper the introduced class of solutions of the equation depends on the function $\varphi$ that in most cases not useful for practical applications.

Equation (1.1) is a natural generalization of two previous ones.

The solution $y$ of equation (1.1) defined on the interval $[t_0,\omega[ \subset [a,\omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the conditions

$$y^{(i)} : [t_0,\omega[ \to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0 \tag{1.3}$$

are satisfied.

The goal of the present paper is to find for $\lambda_0 \in R \setminus \{0; 1\}$ the necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1.1) together with asymptotic representations of those solutions and their first order derivatives as $t \uparrow \omega$. According to the definition, such solutions are the regularly varying functions as $t \uparrow \omega$ of index $\frac{1}{\lambda_0 - 1}$.

# 2 Main results

First of all, we introduce some notations that will be necessary in the sequel. We consider

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty,\\ t-\omega, & \text{if } \omega < +\infty,\end{cases} \qquad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1},$$

[1] If $Y_i = +\infty$ (resp. $Y_i = -\infty$), we will take $y_i^0 > 0$ (resp. $y_i^0 < 0$).

$$\Phi_0(y)=\int\limits_{A_\omega}^{y}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz,\quad A_\omega=\begin{cases} y_0^0, & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=\pm\infty,\\ Y_0, & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=const,\end{cases}$$

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_0(y)}{y},\quad \Phi_1(y)=\int\limits_{A_\omega}^{y}\frac{\Phi_0(\tau)}{\tau}\,d\tau,\quad Z_1=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y),$$

$$F(t)=\frac{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}{\pi_\omega(t)I_1'(t)}\,.$$

If $y_1^0\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}=Y_1$, we put

$$I(t)=|\lambda_0-1|^{\frac{1}{1-\sigma_1}}\cdot y_1^0\cdot\int\limits_{B_\omega^0}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^0=\begin{cases} b, & \text{if } \int\limits_{b}^{\omega}\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega, & \text{if } \int\limits_{b}^{\omega}\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=<+\infty,\end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases} b, & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=\pm\infty,\\ \omega, & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0 |I(\tau)|}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=const.\end{cases}$$

Here, the number $b\in[a,\omega[$ is chosen in such a way that $y_1^0|\pi_\omega(t))|^{\frac{1}{\lambda_0-1}}\in\Delta_{Y_1}$ as $t\in[b;\omega]$.

**Note 2.1.** From conditions (1.2) it follows that $Z_0,Z_1\in\{0,+\infty\}$ and

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_0''(y)\cdot\Phi_0(y)}{(\Phi_0'(y))^2}=1,\quad \lim_{\substack{\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_1''(y)\cdot\Phi_1(y)}{(\Phi_1'(y))^2}=1. \tag{2.1}$$

**Note 2.2.** The following statements are valid:

1)

$$\Phi_0(y)=(\sigma_1-1)\,\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(y)}{\varphi_0'(y)}\,[1+o(1)]\ \text{ when }\ y\to Y_0\quad (y\in\Delta_{Y_0})$$

and therefore

$$\operatorname{sign}(\varphi_0'(y)\Phi_0(y))=\operatorname{sign}(\sigma_1-1),\ \text{ when }\ y\in\Delta_{Y_0}.$$

2)

$$\Phi_1(y)=\frac{\Phi_0^2(y)}{y\Phi_0'(y)}\,[1+o(1)],\ \text{ when }\ y\to Y_0\quad (y\in\Delta_{Y_0})$$

and therefore

$$\operatorname{sign}(\Phi_1(y))=y_0^0\ \text{ when }\ y\in\Delta_{Y_0}.$$

Note that, by (2.1), the relation

$$\lim_{z\to Z_0}\frac{\Phi''(\Phi_1^{-1}(z))z}{(\Phi'(\Phi_1^{-1}(z)))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(\Phi_1^{-1}(\Phi_1(y)))\Phi_1(y)}{(\Phi_1'(\Phi_1^{-1}(\Phi_1(y))))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(y)\Phi_1(y)}{(\Phi_1'(y))^2}=1$$

is valid, and from the latter it follows that

$$\lim_{z\to Z_0}\frac{z\cdot\Big(\frac{\Phi_1'(\Phi_1^{-1}(z))}{\Phi_1(\Phi_1^{-1}(z))}\Big)'}{\frac{\Phi_1'(\Phi_1^{-1}(z))}{\Phi_1(\Phi_1^{-1}(z))}}=\lim_{y\to Z_0}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}-1=0.$$

Thus the function $\frac{\Phi_1'(\Phi_1^{-1}(z))}{\Phi_1(\Phi_1^{-1}(z))}$ is slowly varying as $z\to Z_0$. The function $\Phi_1^{-1}(z)$ is also slowly varying as an inverse to the rapidly varying function. So, we have the following

**Note 2.3.** The function $\Phi^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ is slowly varying as $z\to Z_1$.

Let $Y\in\{0,\infty\}$, $\Delta_Y$ be some one-sided neighborhood of $Y$. The continuously differentiable function $L:\Delta_Y\to]0;+\infty[$ is called [6, p. 2–3] normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$, if

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0. \tag{2.2}$$

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition $S$ as $z\to Y$, if for any normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0;+\infty[$ the following equality takes place: $z\to Y$ $(z\in\Delta_Y)$

$$\theta(zL(z))=\theta(z)(1+o(1)).$$

We will consider that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L_0:\Delta_Y\to]0;+\infty[$ satisfies the condition $S_1$ as $z\to Y$, if for any finite segment $[a;b]\subset]0;+\infty[$ the inequality

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}}\Big|\ln|z|\cdot\Big(\frac{L(\lambda z)}{L(z)}-1\Big)\Big|<+\infty\ \text{ for all }\ \lambda\in[a;b]$$

is true.

Conditions $S$ and $S_1$ are satisfied by the functions $\ln|y|$, $|\ln|y||^\mu$ $(\mu\in R)$, $\ln|\ln|y||$ and by many others.

The following theorem has been obtained.

**Theorem 2.1.** *Let for equation* (1.1) $\sigma_1\neq1$*, the function* $\theta_1(z)$ *satisfy the condition* $S$ *as* $z\to Y_1$ $(z\in\Delta_{Y_1})$*, and the function* $\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ *satisfy the condition* $S_1$ *as* $z\to Z_1$*. Then for the existence of* $P_\omega(Y_0,Y_1,\lambda_0)$*-solutions of equation* (1.1)*, where* $\lambda_0\in\mathbb{R}\setminus\{0,1\}$*, it is necessary and, if*

$$I(t)I_1(t)\lambda_0(\sigma_1-1)>0\quad\text{as}\ \ t\in]b,\omega[\,, \tag{2.3}$$

*and the finite or infinite limits*

$$\lim_{t\uparrow\omega}\pi_\omega(t)F'(t)\ \ \text{and}\ \ \lim_{t\uparrow\omega}\frac{\sqrt{\big|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\big|}}{\ln|I_1(t)|}\quad\text{exist,} \tag{2.4}$$

*sufficient the fulfilment of the following conditions:*

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0;\quad \pi_\omega(t)y_1^0\alpha_0(\lambda_0-1)>0\quad\text{as}\ \ t\in[a;\omega[\,, \tag{2.5}$$

$$y_1^0\cdot\lim_{t\uparrow\omega}|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}=Y_1,\quad \lim_{t\uparrow\omega}I_1(t)=Z_1, \tag{2.6}$$

$$\lim_{t\uparrow\omega}\frac{I_1''(t)I_1(t)}{(I_1'(t))^2}=1,\quad \lim_{t\uparrow\omega}F(t)=\frac{\lambda_0-1}{\lambda_0}. \tag{2.7}$$

*Moreover, for each such solution there take place the following asymptotic representations as* $t\uparrow\omega$*:*

$$\Phi_1(y(t))=I_1(t)[1+o(1)],\qquad \frac{\pi_\omega(t)y'(t)}{y(t)}=\frac{\lambda_0}{\lambda_0-1}\,[1+o(1)]. \tag{2.8}$$

*Proof. Necessity.* Let the function $y : [t_0, \omega[ \to \Delta_{Y_0}$ be a $P_\omega(Y_0, Y_1, \lambda_0)$-solution of equation (1.1), for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$. Then, according to the properties of such solutions established by V. M. Evtukhov (see, e.g., [4]), we have

$$\frac{y(t)}{y'(t)} = \frac{\lambda_0}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)], \quad \frac{y''(t)}{y'(t)} = \frac{1}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)] \quad \text{as } \ t \in [a;\omega[\,. \tag{2.9}$$

Thus we obtain (2.5).

From (2.9), it also follows that $y'(t)$ as $t \in [a;\omega[$ is a regularly varying function of index $\frac{1}{\lambda_0 - 1}$. It can be represented in the form

$$y'(t) = |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} L_1(t) \quad \text{as } \ t \uparrow \omega, \tag{2.10}$$

where $L_1(t)$ is a regularly varying function as $t \uparrow \omega$ (see [7, p. 10]).

Hence, taking into account the properties of regularly varying functions [7, p. 10–15], we obtain the first of conditions (2.6).

From (1.1) and (2.9), it follows that as $t \uparrow \omega$

$$\frac{|y'(t)|^{1-\sigma_1}\operatorname{sign} y_1^0}{\varphi_0(y(t))} = \alpha_0(\lambda_0 - 1)\pi_\omega(t)\varphi_1(y'(t))|y'(t)|^{-\sigma_1}p(t)[1 + o(1)]. \tag{2.11}$$

Substituting (2.10) into (2.11), we get as $t \uparrow \omega$ the equality

$$\frac{y'(t)}{|\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}} = y_1^0|\lambda_0 - 1|^{\frac{1}{1-\sigma_1}}\left|\pi_\omega(t)\theta_1\Big(|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}L_1(t)y_1^0\Big)p(t)\right|^{\frac{1}{1-\sigma_1}}[1 + o(1)]. \tag{2.12}$$

In (2.10), the function $L_1$ is a slowly varying when its argument tends to $Y_1$. The function $\theta_1$ satisfies the condition $S$. So, from (2.12), we have as $t \uparrow \omega$

$$\frac{y'(t)}{|\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}} = y_1^0|\lambda_0 - 1|^{\frac{1}{1-\sigma_1}}\left|\pi_\omega(t)\theta_1\Big(|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)p(t)\right|^{\frac{1}{1-\sigma_1}}[1 + o(1)]. \tag{2.13}$$

Integrating the relation from $t_0$ to $t$, we get as $t \uparrow \omega$

$$\int\limits_{y(t_0)}^{y(t)} \frac{dz}{|\varphi_0(z)|^{\frac{1}{1-\sigma_1}}} = y_1^0|\lambda_0 - 1|^{\frac{1}{1-\sigma_1}}\int\limits_{t_0}^{t}\left|\pi_\omega(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)p(\tau)\right|^{\frac{1}{1-\sigma_1}}[1 + o(1)]\,d\tau.$$

Taking into account the choice of $A_\omega$, and that $y \to Y_0$ $(Y_0 \in \Delta_{Y_0})$, we have

$$\Phi_0(y(t)) = I(t)[1 + o(1)] \quad \text{as } \ t \uparrow \omega. \tag{2.14}$$

From (2.13) and (2.14), according to (2.9), we get

$$\frac{\pi_\omega(t)y'(t)}{y(t)} \cdot \frac{y(t)\Phi_0'(y(t))}{\Phi_0(y(t))} = \frac{\pi_\omega(t)I'(t)}{I(t)}\,[1 + o(1)] \quad \text{as } \ t \uparrow \omega. \tag{2.15}$$

By conditions (1.2), the function $\Phi_0(y)$ is rapidly varying as $y \to Y_0$ $(Y_0 \in \Delta_{Y_0})$. Thus from (2.15) it follows that

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \infty. \tag{2.16}$$

Taking into account equalities (2.14) and (2.9), we get

$$\frac{y'(t)\Phi_0(y(t))}{y(t)} = \frac{\lambda_0 I(t)}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)] \quad \text{as } \ t \uparrow \omega. \tag{2.17}$$

From here in the same way as equality (2.14) was obtained, we get the equality

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)] \quad \text{as } \ t \uparrow \omega. \tag{2.18}$$

Thus, the correctness of the first representation of (2.8) and the first equality of (2.6) are justified. We get the correctness of the second representation of (2.8) as a result of division (2.17) by (2.18).

The second representation of (2.8) can be rewritten in the form

$$\frac{\pi_\omega(t)y'(t)}{y(t)}\cdot\frac{y(t)\Phi_1'(y(t))}{\Phi_1(y(t))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\,[1+o(1)] \quad \text{as } \ t\uparrow\omega.$$

With the help of (2.9), from the above representation we get

$$\frac{\lambda_0}{\lambda_0-1}\cdot\frac{y(t)\Phi_1'(y(t))}{\Phi_1(y(t))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\,[1+o(1)] \quad \text{as } \ t\uparrow\omega. \tag{2.19}$$

From conditions (1.2) imposed on the function $\varphi_0(y(t))$ and Note 2.2, we find that $\Phi_1(y)$ is a rapidly varying function as $y\to Y_0$ $(Y_0\in\Delta_{Y_0})$. Then, taking into account (2.19), we get

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\infty. \tag{2.20}$$

By (2.1), (2.15), (2.16) and (2.19), we have

$$\lim_{t\uparrow\omega}\frac{I_1''(t)I_1(t)}{(I_1'(t))^2}=\lim_{t\uparrow\omega}\frac{\frac{\pi_\omega(t)I'(t)}{I(t)}}{\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}}=\lim_{t\uparrow\omega}\frac{\frac{y(t)\Phi_0'(y(t))}{\Phi_0(y(t))}}{\frac{y(t)\Phi_1'(y(t))}{\Phi_1(y(t))}}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_1''(y)\cdot\Phi_1(y)}{(\Phi_1'(y))^2}=1. \tag{2.21}$$

It means that the first of conditions (2.7) holds.

Note that the function $\Phi_1^{-1}(y)$ is slowly varying as $y\to Z_0$, since it is inverse to a rapidly varying as $y\to Y_0$ $(Y_0\in\Delta_{Y_0})$ function $\Phi_1$. Taking into account this fact and (2.18), we get as $t\uparrow\omega$

$$y(t)=\Phi_1^{-1}(I_1(t))[1+o(1)].$$

The correctness of the second of conditions (2.6) follows from this fact.

Note that (2.19) can be written in the form

$$\frac{\lambda_0}{\lambda_0-1}\cdot\Phi_1^{-1}(I_1(t))\cdot\frac{\Phi_1'(\Phi_1^{-1}(I_1(t)))}{\Phi_1(\Phi_1^{-1}(I_1(t)))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\,[1+o(1)] \quad \text{as } \ t\uparrow\omega.$$

The validity the second of conditions (2.7) is justified, and hence the necessity is proved.

*Sufficiency.* Let us suppose that conditions (2.3)–(2.7) of the theorem take place.

We apply to equation (1.1) the transformation

$$\begin{cases}\Phi_1(y(t))=I_1(t)[1+v_1(x)],\\ \dfrac{y'(t)}{y(t)}=\dfrac{\lambda_0}{\lambda_0-1}\cdot\dfrac{1}{\pi_\omega(t)}\,[1+v_2(x)]\end{cases} \tag{2.22}$$

and reduce system (2.22) to the following system of differential equations:

$$\begin{cases}v_1'=\dfrac{I_1'(t)}{I_1(t)}\,[1+v_1]\cdot\Big(\dfrac{\lambda_0}{\lambda_0-1}\cdot F(t)\cdot M(t,v_1)[1+v_2]-1\Big),\\[2ex] v_2'=\dfrac{1}{\pi_\omega(t)}\,[1+v_2]\cdot\Big[Q(t,v_1,v_2)(1+v_1)^{\sigma_1-1}(1+v_2)^{\sigma_1-1}-\dfrac{1}{\lambda_0}-v_2\Big].\end{cases} \tag{2.23}$$

Here,

$$M(t,v_1)=\frac{Y(t,v_1)\,\frac{\Phi_1'}{\Phi_1}\,(\Phi_1^{-1}(Y(t,v_1)))}{\Phi_1^{-1}(I_1(t))\,\frac{\Phi_1'}{\Phi_1}\,(\Phi_1^{-1}(I_1(t)))}\,,\quad Y(t,v_1)=\Phi_1^{-1}\big(I_1(t)[1+v_1]\big),$$

$$Q(t,v_1,v_2)=\frac{N(t,v_1,v_2)}{\lambda_0}\,\bigg(F(t)\Big(\frac{\lambda_0}{\lambda_0-1}\Big)^2\cdot M(t,v_1)$$

$$\times\Big(\frac{L(t)}{1+L(t)}+F(t)M(t,v_1)\cdot\frac{\Phi_1''(Y(t,v_1))\Phi_1(Y(t,v_1))}{(\Phi_1(Y(t,v_1)))^2}\cdot\frac{1}{\frac{I_1(t)I_1''(t)}{(I_1'(t))^2}+G(t)}\Big)\Big)^{\sigma_1-1},$$

$$N(t,v_1,v_2)=\frac{\theta_1\big(\frac{\lambda_0 Y(t,v_1)}{(\lambda_0-1)\pi_\omega(t)}\cdot[1+v_2]\big)}{\theta_1(|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}\operatorname{sign}y_1^0)},\quad G(t)=\frac{I_1(t)}{\pi_\omega(t)I_1'(t)},\quad L(t)=\frac{I_1'(t)}{\pi_\omega(t)I_1''(t)}.$$

From the first of conditions (2.7) we have

$$\lim_{t\uparrow\omega}G(t)=0. \tag{2.24}$$

We have already proved that the function $\Phi_1^{-1}(z)$ is slowly varying as $z\to Z_1$. So, taking into account the second of conditions (2.6), we have

$$\lim_{t\uparrow\omega}Y(t,v_1)=Y_0 \ \text{ uniformly by } \ v_1:|v_1|<\frac12\,. \tag{2.25}$$

By Note 2.3, we have

$$\lim_{t\uparrow\omega}M(t,v_1)=1 \ \text{ uniformly by } \ v_1:|v_1|<\frac12\,. \tag{2.26}$$

From the second of conditions (2.7), we get

$$\lim_{t\uparrow\omega}F(t)=\frac{\lambda_0}{\lambda_0-1}\,. \tag{2.27}$$

Now, we can prove that

$$\lim_{t\uparrow\omega}N(t,v_1,v_2)=1 \ \text{ uniformly by } \ v_1:|v_1|<\frac12 \ \text{ and uniformly by } \ v_2:|v_2|<\frac12\,. \tag{2.28}$$

From (2.26) and (2.27), it follows that

$$\lim_{t\uparrow\omega}\frac{\Big(\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}}\Big)'\cdot\pi_\omega(t)}{\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}}}=\lim_{t\uparrow\omega}\frac{1}{F(t)M(t,v_1)}-\frac{\lambda_0}{(1-\gamma_0)(\lambda_0-1)}=0.$$

Hence

$$\Big(\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}}\Big)$$

is a normalized slowly varying function as $t\uparrow\omega$. Statement (2.28) follows from the above according to the fact that the function $\Phi_1^{-1}$ is slowly variable as its argument tends to $Z_1$, and the function $\theta_1$ satisfies condition $S$.

Taking into account the first of conditions (2.7), we have

$$\lim_{t\uparrow\omega}L(t)=0. \tag{2.29}$$

From (2.24)–(2.29), it follows that

$$\lim_{t\uparrow\omega}Q(t,v_1,v_2)=\frac{1}{\lambda_0} \ \text{ uniformly by } \ v_1:|v_1|<\frac12 \ \text{ and uniformly by } \ v_2:|v_2|<\frac12\,. \tag{2.30}$$

By (2.6), from the fact that the function $\Phi_1^{-1}$ is slowly varying as the argument tends to $Z_1$, it follows that there exists a number $t_0\in[a,\omega[$ such that

$$\Phi_1^{-1}\big(I_1(t)(1+v_1)\big)\in\Delta_{Y_0}\quad\text{as}\ \ t\in[t_0,\omega[\,,\ \ |v_1|\le\frac12\,.$$

Further, we consider the system of differential equations (2.23) on the set

$$\Omega = [t_0, \omega[\, \times D, \quad D = \Big\{ (v_1, v_2): \ |v_i| \le \frac{1}{2}\,, \ i = 1, 2 \Big\}$$

and rewrite the system in the form

$$\begin{cases} v_1' = \dfrac{I_1'(t)}{I_1(t)} \Big[ A_{11}(t) v_1 + A_{12}(x) v_2 + R_1(x, v_1, v_2) + R_2(x, v_1, v_2) \Big], \\ v_2' = \dfrac{1}{\pi_\omega(t)} \Big[ A_{21} v_1 + A_{22} v_2 + R_3(x, v_1, z_2) + R_4(x, v_1, v_2) \Big], \end{cases} \tag{2.31}$$

where

$$A_{11}(t) = \frac{\lambda_0}{\lambda_0 - 1}\, F(t) - 1, \quad A_{12}(t) = \frac{\lambda_0}{\lambda_0 - 1}\, F(t),$$

$$R_1(t, v_1, v_2) = \frac{\lambda_0}{\lambda_0 - 1}\, F(t) - 1 + \frac{\lambda_0}{\lambda_0 - 1}\, F(t)(M(t, v_1) - 1)(1 + v_1 + v_2),$$

$$R_2(t, v_1, v_2) = \frac{\lambda_0}{\lambda_0 - 1}\, F(t) M(t, v_1) v_1 v_2,$$

$$A_{21} = \frac{\sigma_1 - 1}{\lambda_0}, \quad A_{22} = \frac{\sigma_1 - 1 - \lambda_0}{\lambda_0}\,,$$

$$R_3(t, v_1, z_2) = \frac{1}{\lambda_0} \big(1 + (\sigma_1 - 1) v_1 + \sigma_1 v_2\big) \cdot \big(\lambda_0 Q(t, v_1, v_2) - 1\big),$$

$$R_4(t, v_1, v_2) = Q(t, v_1, v_2) \Big[ (1 + \sigma_1 v_2) \big( (1 + v_1)^{\sigma_1 - 1} - 1 - (\sigma_1 - 1) v_1 \big) + \sigma_1 (\sigma_1 - 1) v_1 v_2 + \big( (1 + v_2)_1^{\sigma} - 1 - \sigma_1 v_2 \big) (1 + v_1)_1^{\sigma} \Big] - v_2^2.$$

By virtue of equalities (2.24)–(2.29), for $k \in \{2, 4\}$, we get

$$\lim_{|v_1| + |v_2| \to 0} \frac{R_k(t, v_1, v_2)}{|v_1| + |v_2|} = 0 \ \text{ uniformly by } t \text{ as } \ t \in [t_0, \omega[\,, \tag{2.32}$$

and for $k \in \{1, 3\}$,

$$\lim_{t \uparrow \omega} R_k(t, z_1, z_2) = 0 \ \text{ uniformly by } v_1, \ v_2 \text{ as } \ (v_1, v_2) \in D. \tag{2.33}$$

At the next stage of the proof we apply to system (2.31) the following transformation:

$$\begin{cases} v_1 = r_1, \\ v_2 = r_2 - H(t). \end{cases} \tag{2.34}$$

Here,

$$H(t) = \frac{\frac{\lambda_0}{\lambda_0 - 1}\, F(t) - 1}{\frac{\lambda_0}{\lambda_0 - 1}\, F(t)}\,.$$

By (2.27), we have

$$\lim_{t \uparrow \omega} H(t) = 0. \tag{2.35}$$

Thus get a system

$$\begin{cases} r_1' = \dfrac{I_1'(t)}{I_1(t)}\, \dfrac{\lambda_0}{\lambda_0 - 1}\, F(t) \big[ r_2 + r_1 r_2 + R(t; r_1; r_2) \big], \\ r_2' = \dfrac{1}{\pi_\omega(t)} \big[ A_{21} r_1 + A_{22} r_2 + V_3(t, r_1, r_2) + V_4(t, r_1, r_2) \big], \end{cases} \tag{2.36}$$

where

$$R(t, r_1, r_2) = (M(t, r_1) - 1)(1 + r_1)(1 + r_2 - H(t)),$$
$$V_3(t, r_1, r_2) = R_4(t, r_1, r_2 - H(t)) - R_4(t, r_1, r_2) + \pi_\omega(t)H'(t) - A_{22}H(t) + R_3(t, r_1, r_2 - H(t)),$$
$$V_4(t, r_1, r_2) = R_4(t, r_1, r_2).$$

Let us show that

$$\lim_{t\uparrow\omega} \pi_\omega(t)H'(t) = 0. \tag{2.37}$$

According to condition (2.4) of the theorem, there exists the following finite or infinite limit

$$\lim_{t\uparrow\omega} \pi_\omega(t)H'(t).$$

Let

$$\pi_\omega(t)H'(t) = q(t) \ \text{ and } \ \lim_{t\uparrow\omega} q(t) \neq 0. \tag{2.38}$$

Then

$$H'(t) = \frac{q(t)}{\pi_\omega(t)}\,.$$

As a result of integration of the above equality from $t_0$ to $t$, we have

$$H(t) - H(t_0) = \int\limits_{t_0}^{t} \frac{q(\tau)}{\pi_\omega(\tau)}\, d\tau. \tag{2.39}$$

From (2.35) and (2.39), it follows that the integral $\int\limits_{t_0}^{\omega} \frac{q(\tau)}{\pi_\omega(\tau)}\, d\tau$ must be finite. But this is possible only if

$$\lim_{t\uparrow\omega} q(t) = 0.$$

Thus, taking into account (2.38), we have proved the correctness of statement (2.37).

Owing to the properties of the function $R_4$, by (2.28) and (2.35), it follows that

$$\lim_{t\uparrow\omega} \big[R_4(t, r_1, r_2 - H(t)) - R_4(t, r_1, r_2)\big] = 0 \ \text{ uniformly by } r_1 \text{ and } r_2 \text{ as } \ |r_i| < \frac{1}{2}\,, \ \ i = 1, 2. \tag{2.40}$$

Applying the transformation

$$\begin{cases} r_1 = w_1, \\ r_2 = \sqrt{|G(t(x))|}\, w_2, \end{cases} \tag{2.41}$$

where

$$x = \beta \ln |I_1(t)|, \quad \beta = \begin{cases} 1, & \text{if } \lim\limits_{t\uparrow\omega} I_1(t) = \infty, \\ -1, & \text{if } \lim\limits_{t\uparrow\omega} I_1(t) = 0, \end{cases} \tag{2.42}$$

to system (2.31) and taking into account (2.3), we obtain the system

$$\begin{cases} w_1' = \beta\sqrt{|G(t(x))|}\,\Big[\dfrac{\lambda_0}{\lambda_0 - 1}\, F(t(x))w_2 + \dfrac{\lambda_0}{\lambda_0 - 1}\, F(t(x))w_1 w_2 + W(x; w_1; w_2)\Big], \\ w_2' = \beta\sqrt{|G(t(x))|}\,\Big[\operatorname{sign} G(t(x))A_{21}w_1 \\ \qquad + \Big(\sqrt{|G(t(x))|}\operatorname{sign} G(t(x))A_{22}(x) - \widetilde{N}(x)\Big)w_2 + W_3(x, w_1, w_2) + W_4(x, w_1, w_2)\Big], \end{cases} \tag{2.43}$$

where

$$W(x; w_1; w_2) = \frac{\lambda_0}{\lambda_0 - 1}\, F(t(x)) \cdot \frac{(M(t(x), w_1) - 1)}{\sqrt{|G(t(x))|}}\, (1 + w_1)\Big(1 + \sqrt{|G(t(x))|}\, w_2 - H(t(x))\Big),$$

$$W_3(x,w_1,w_2) = V_3\Big(t(x), w_1, \sqrt{|G(t(x))|}\,w_2\Big),$$
$$W_4(x,w_1,w_2) = V_4\Big(t(x), w_1, \sqrt{|G(t(x))|}\,w_2\Big),$$
$$\widetilde{N}(x) = \frac{\operatorname{sign}(G(t(x)))G'(t(x))I(t(x))}{2G(t(x))\sqrt{|G(t(x))|}\,I'(t(x))}\,.$$

Note that

$$\widetilde{N}(x) = \frac{\operatorname{sign}(G(t(x)))G'(t(x))I(t(x))}{2G(t(x))\sqrt{|G(t(x))|}\,I'(t(x))} = \frac{\operatorname{sign}(G(t(x)))G'(t(x))\pi_\omega(t(x))}{2\sqrt{|G(t(x))|}}\,.$$

At the same time, the equality

$$\frac{(M(t,w_1)-1)}{\sqrt{|G(t(x))|}} = \ln|I_1(t)|\cdot\Big(\frac{\Phi_1^{-1}(I_1(t)[1+v_1])\psi(\Phi_1^{-1}(I_1(t)[1+v_1]))}{\Phi^{-1}(I_1(t))\psi(\Phi^{-1}(I_1(t)))} - 1\Big)\cdot\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|}$$

is true. Next, let us prove that

$$\lim_{t\uparrow\omega}\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|} = 0. \tag{2.44}$$

By de L'Hospital rule we have

$$\lim_{t\uparrow\omega}\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|} = -\frac{1}{2}\lim_{t\uparrow\omega}\frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}}\,.$$

The last limit has a finite or infinite boundary, since the second limit in (2.4) exists.

Now let us prove that

$$\lim_{t\uparrow\omega}\frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}} = 0. \tag{2.45}$$

According to condition (2.4), there exists the following finite or infinite limit

$$\lim_{t\uparrow\omega}\frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}}\,.$$

Suppose that

$$\frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}} = q_1(t) \quad\text{and}\quad \lim_{t\uparrow\omega} q_1(t) \neq 0. \tag{2.46}$$

Then

$$\frac{G'(t)}{\sqrt{|G(t)|}} = \frac{q_1(t)}{\pi_\omega(t)}\,.$$

As a result of integration of this equality from $t_0$ to $t$, we have

$$2\sqrt{|G(t)|} - 2\sqrt{|G(t_0)|} = \int\limits_{t_0}^{t}\frac{q_1(\tau)}{\pi_\omega(\tau)}\,d\tau. \tag{2.47}$$

From (2.24) and (2.47), it follows that the integral $\int\limits_{t_0}^{\omega}\frac{q_1(\tau)}{\pi_\omega(\tau)}\,d\tau$ must be finite. But this is possible only if

$$\lim_{t\uparrow\omega} q_1(t) = 0. \tag{2.48}$$

The last one is in contradiction with assumption (2.46). So, statement (2.44) is true.

Let us now prove that

$$\lim_{x\to+\infty} \widetilde{N}(x) = 0. \tag{2.49}$$

The function $\Phi^{-1}(z)\cdot\frac{\Phi_1'(\Phi^{-1}(z))}{z}$ satisfies condition $B$, hence

$$\left|\ln|I_1(t(x))|\cdot\Big(\frac{\Phi_1^{-1}(I_1(t)[1+v_1])\psi(\Phi_1^{-1}(I_1(t)[1+v_1]))}{\Phi^{-1}(I_1(t))\psi(\Phi^{-1}(I_1(t)))}-1\Big)\right|<\infty.$$

From the above equality and statement (2.49), it follows that

$$\lim_{x\to+\infty} W(x;w_1;w_2)=0 \text{ uniformly towards } w_1 \text{ and } w_2 \text{ if } |w_i|<\frac{1}{2}\,,\ \ i=1,2. \tag{2.50}$$

Note that the characteristic equation of a matrix

$$\begin{pmatrix} 0 & \beta \\ \beta\,\mathrm{sign}(\lambda_0(\sigma_1-1))A_{21} & 0 \end{pmatrix}$$

has the form

$$\mu^2-\frac{|\sigma_1-1|}{|\lambda_0|}=0.$$

This equation has no roots with real part equal to zero. Let us consider $\int\limits_{x_0}^{\infty} G(t(x))\,dx$. Taking into account the presentation $G(t(x))=\frac{I(t(x))}{\pi_\omega(t(x))I'(t(x))}$, we have

$$\int\limits_{x_0}^{\infty} G(t(x))\,dx=\int\limits_{x_0}^{\infty}\frac{I_1(t(x))}{\pi_\omega(t(x))I_1'(t(x))}\,dx=\int\limits_{t(x_0)}^{\omega}\frac{I_1(t)}{\pi_\omega(t)I_1'(t)}\,\frac{I_1'(t)}{I_1(t)}\,dt=\ln|\pi_\omega(t)|_{d_1}^{\omega}\longrightarrow\infty \quad \text{as} \ \ t\to\omega.$$

Since in some neighborhood of zero the inequality

$$\int\limits_{x_0}^{\infty}\sqrt{|G(t(x))|}\,dx\ge \mathrm{sign}(G(t(x)))\int\limits_{x_0}^{\infty} G(t(x))\,dx$$

takes place, it is true that

$$\int\limits_{x_0}^{\infty}\sqrt{|G(t(x))|}\,dx\longrightarrow+\infty.$$

We have got that for the system of differential equations (2.43) all conditions of Theorem 2.2 from [5] are fulfilled. According to this theorem, system (2.43) has a one-parameter family of solutions $\{w_i\}_{i=1}^2:[x_1,+\infty[\,\to\mathbb{R}^2$ ($x_1\ge x_0$, $x_0=\beta\ln|I_1(t_0)|$) that tend to zero as $x\to+\infty$. By (2.42), (2.22) these solutions correspond to those solutions $y$ of equation (1.1) that admit asymptotic representations (2.8) as $t\uparrow\omega$.

By representations (2.8) and inequality (2.3) it is clear that the obtained solutions are indeed the $P_\omega(Y_0,Y_1,\lambda_0)$-solutions. The theorem is proved completely. □

# 3 Illustration of the results

To illustrate the results obtained above, we consider the following differential equation for $t\in[2,+\infty[$

$$y''=\psi(t)\exp\big(\exp(|y|^a)-\exp(t^d)\big)|y|^{\sigma_0}|y'|^{\sigma_1}. \tag{3.1}$$

Here, $\sigma_0,\sigma_1\in R$, $\sigma_1>1$, $a,d\in\,]0,+\infty[$ , the function $\psi:[2,+\infty[\,\to\,]0,+\infty[$ is continuous, regularly varying at infinity of index $\gamma$, $\gamma\in R$.

This equation is of type (1.1) for which

$$\alpha_0 = 1, \quad p(t) = \psi(t)\exp\big(-\exp(t^d)\big), \quad \varphi_0(y) = |y|^{\sigma_0}\exp\big(\exp(|y|^a)\big), \quad \varphi_1(y') = |y'|^{\sigma_1}.$$

Using the above proven theorem, let us investigate the question of the existence and asymptotic behavior as $t \to +\infty$ of $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions of equation (3.1) for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

In our case,

$$\pi_\omega(t) = t, \quad \theta_1(y) = 1.$$

Thus the function $\theta_1$ satisfies condition $S$.

Taking into account the choice of $B^0_{+\infty}$, as $t \to +\infty$, we have

$$I(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \frac{\sigma_1 - 1}{d} \cdot t^{1-d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - t^d\Big)[1 + o(1)].$$

In the same way, as $t \to +\infty$, we have

$$I_1(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \Big(\frac{\sigma_1 - 1}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)].$$

In addition, in our case, since $Y_0 = \infty$, taking into account the choice of $A^0_\infty$, we get

$$\Phi_0(y) = \frac{\sigma_1 - 1}{a} \cdot y^{\frac{\sigma_0}{\sigma_1 - 1} + 1 - a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - |y|^a\Big)[1 + o(1)] \quad \text{as } y \to \infty.$$

Similarly, we have

$$\Phi_1(y) = \Big(\frac{\sigma_1 - 1}{a}\Big)^2 \cdot y^{\frac{\sigma_0}{\sigma_1 - 1} + 1 - 2a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - 2|y|^a\Big)[1 + o(1)] \quad \text{as } y \to \infty. \tag{3.2}$$

We have

$$\lim_{t \uparrow +\infty} F(t) = \frac{a}{d}\,. \tag{3.3}$$

From (3.3) and the second condition of (2.7), it follows that equation (3.1) may have only $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions with

$$\lambda_0 = \frac{d}{d - a}\,.$$

Taking into account asymptotic representations for functions $I$, $I_1$, $\Phi_0$, $\Phi_1$, $\Phi_1^{-1}$, we get

$$\lim_{t \to +\infty} tF'(t) = 0.$$

So, the first condition of (2.4) is valid.

Note that

$$\frac{\sqrt{|\frac{\pi_\omega(t) I_1'(t)}{I_1(t)}|}}{\ln |I_1(t)|} = \sqrt{d(\sigma_1 - 1)}\,\frac{t^{\frac{d}{2}}}{\exp(\frac{t^d}{2})}\,[1 + o(1)] \quad \text{as } t \to \infty,$$

from which the second condition of (2.4) takes place.

At the same time,

$$\Phi_1^{-1}(y) \cdot \frac{\Phi_1'(\Phi_1^{-1}(y))}{y} = \frac{(\sigma_1 - 1)^2}{a} \ln y \cdot \big(\ln((\sigma_1 - 1)\ln y)\big)^{\frac{\frac{\sigma_0}{\sigma_1 - 1} - 2a + 1}{a}}[1 + o(1)] \quad \text{as } y \to \infty.$$

This means that condition $S_1$ is satisfied.

Thus, all conditions of Theorem 2.1 are satisfied. By virtue of this theorem, equation (3.1) may have only $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions. From Theorem 2.1 it also follows that equation (3.1) has one-parameter family of $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions.

Also, using the known asymptotic behavior of the function $\Phi_1^{-1}$, it is easy to find that every $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solution of equation (3.1) and its derivative satisfy the following asymptotic representations

$$(y(t))^{\frac{\sigma_0}{\sigma_1 - 1} + 1 - 2a} \cdot \exp\Big(\frac{\exp(|y(t)|^a)}{\sigma_1 - 1} - 2|y(t)|^a\Big)$$
$$= \Big|\frac{a}{d-a}\Big|^{\frac{1}{1-\sigma_1}} \cdot \Big(\frac{a}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot \psi^{\frac{1}{1-\sigma_1}}(t) \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)] \quad \text{as } t \to +\infty,$$
$$y'(t) = \frac{y(t)}{t}\,[1 + o(1)] \quad \text{as } t \to +\infty.$$

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**Author's address:**

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.
*E-mail:* olachepok@ukr.net