Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 27–38

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EXISTENCE RESULTS FOR A NEW CLASS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAY

Abstract. In this paper we investigate the existence and uniqueness of solutions on a compact interval for non-linear fractional integro-differential equations with state-dependent delay and non-instantaneous impulses. Our results are based on the Banach contraction principle and the Krasnoselkii fixed point theorem. For the illustration of the results, an example is also discussed.

2010 Mathematics Subject Classification. 26A33, 34A37, 34G20.

Key words and phrases. Fractional integro-differential equations, sectorial operators, state-dependent delay, resolvent of operators, non-instantaneous impulse conditions.

რეზიუმე. სტატიაში ჩვენ შევისწავლით კომპაქტურ ინტერვალზე განსაზღვრულ ამონახსნთა არსებობასა და ერთადერთობას არაწრფივი ფრაქციული ინტეგროდიფერენციალური განტოლებებისათვის შინაგან მდგომარეობაზე დამოკიდებული დაგვიანებითა და არამყისიერი იმპულსებით. ჩვენი შედეგები ეფუძნება ბანახის კუმშვის პრინციპს და კრასნოსელსკის უძრავი წერტილის თეორემას. შედეგების ილუსტრაციისთვის განხილულია შესაბამისი მაგალითი.

1 Introduction

This paper is concerned with the existence of solutions defined on a compact real interval for semilinear integro-differential equations of fractional order for which impulses are not instantaneous of the form

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)\,ds = f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in (s_i, t_{i+1}] \subset J, \quad i = 0, \dots, N, \tag{1.1}$$

$$y(t) = g_i(t, y_{\rho(t,y_t)}), \quad t \in (t_i, s_i], \quad i = 1, \dots, N, \tag{1.2}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.3}$$

where $1 < \alpha < 2$, $J = [0, b]$, $b > 0$, $A : D(A) \subset E \to E$ is a closed linear operator of sectorial type on a complex Banach space $(E, \|\cdot\|_E)$, the convolution integral in the equation is known as the Riemann–Liouville fractional integral, $f : J \times \mathcal{B} \to E$ and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are suitable functions. For any function y defined on $(-\infty, b]$ and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here, $y_t(\cdot)$ represents the history of the state from each time $\theta \in (-\infty, 0]$ up to the present time t. We assume that the histories y_t belong to some abstract phase space $\mathcal{B}$, to be specified later, let $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_{N-1} \leq s_N \leq t_N \leq t_{N+1} = b$ be pre-fixed numbers, and $g_i \in C((t_i, s_i] \times \mathcal{B}, E)$, for all $i = 1, 2, \dots, N$, stand for the impulsive conditions.

Fractional differential equations have been of great interest recently, in both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, economy and so on (see [20, 21, 23]). In particular, the question on the existence of solutions of the Cauchy problem for fractional integro-differential equations was studied in numerous works; we refer the reader to the books by Abbas *et al.* [1, 2], Kilbas *et al.* [16], Lakshmikantham *et al.* [18], and the papers by Anguraj *et al.* [3], Balachandran *et al.* [5]. Cuevas *et al.* [6, 8, 9], studying S-asymptotically w-periodic solutions of some classes of semilinear differential and integro-differential equations. Recently, Wang and Chen [24] considered a class of retarded integro-differential equations with nonlocal initial conditions where the existence results of solutions are given over the half-line $[0, \infty)$. In [11], Gautam and Dabas studied the existence of local and global mild solution for an impulsive fractional integro-differential equation with state dependent delay.

Recently, Hernández and O'Regan [13] initiated the study on the Cauchy problems for a new type of first order evolution equations with non instantaneous impulses. In the model analyzed in [13], the impulses start abruptly at the points t_i and their action continue on a finite time interval $[t_i, s_i]$. This type of problem motivates to study certain dynamical changes of evolution processes in pharmacotherapy. For example, as in [13], we note the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorbtion for the body are gradual and continuous processes, we can interpret this situation as an impulsive action which starts abruptly and stays active on a finite time interval.

In this paper, we provide sufficient conditions for the existence of solutions for problem (1.1)–(1.3). Our approach is based on the Banach contraction principle and on the Krasnoselskii fixed point theorem.

2 Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\big\{\|y(t)\|_E : \; t \in J\big\}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|_E$ is Lebesgue integrable.

Let $L^1(J,E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b \|y(t)\|_E \, dt.$$

We define

$$\begin{aligned}\mathrm{PC}(J,E) = \big\{y :\ J \to E;\ y \in C((t_k, t_{k+1}], E),\ k = 0,1,\ldots,N \\ \text{and } y(t_k^+), y(t_k^-) \text{ exist with, } y(t_k^-) = y(t_k),\ k = 1,\ldots,N\big\}.\end{aligned}$$

Obviously, $\mathrm{PC}(J,E)$ is a Banach space with the norm

$$\|y\|_{\mathrm{PC}} = \sup_{t\in J} \|y(t)\|_E.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [12] and follow the terminology used in [15]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms:

(A_1) If $y : (-\infty, b) \to E$, $b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ we have the following conditions:

- (i) $y_t \in \mathcal{B}$;
- (ii) there exists a positive constant H such that $\|y(t)\|_E \le H\|y_t\|_{\mathcal{B}}$;
- (iii) there exist two functions $K(\,\cdot\,), M(\,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \le K(t) \sup\big\{\|y(s)\|_E :\ 0 \le s \le t\big\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$K_b = \sup\big\{K(t) :\ t \in J\big\} \ \text{ and } \ M_b = \sup\big\{M(t) :\ t \in J\big\}.$$

Remark 2.1.

1. (A_1)(ii) is equivalent to $\|\phi(0)\| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi-\psi\|_{\mathcal{B}} = 0$, we necessarily have $\phi(0) = \psi(0)$.

Definition 2.2. A function $f : J \times \mathcal{B} \to E$ is said to be a Carathéodory function if

- (i) for each $t \in J$, the function $f(t,\,\cdot\,) : \mathcal{B} \to E$ is continuous;
- (ii) for each $y \in \mathcal{B}$, the function $f(\,\cdot\,, y) : J \to E$ is measurable.

Definition 2.3. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space E. We recall that A is the generator of a solution operator if there exists $\mu \in \mathbb{R}$ and a strongly continuous function $S : \mathbb{R}^+ \to B(E)$ such that

$$\big\{\lambda^\alpha :\ \mathrm{Re}(\lambda) > \mu\big\} \subset \rho(A)$$

and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S(t)x \, dt, \ \ \mathrm{Re}\,\lambda > \mu, \ \ x \in E.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Remark 2.4. The concept of a solution operator, as defined above, is closely related to the concept of a resolvent family (see Prüss [22]). Because of the uniqueness of the Laplace transform, in the border case $\alpha = 1$, the family $S(t)$ corresponds to a C_0 semigroup (see [10]), whereas in the case $\alpha = 2$, a solution operator corresponds to the concept of a cosine family (see [4]). We note that solution operators, as well as resolvent families, are a particular case of (a, k)-regularized families introduced in [19]. According to [19], a solution operator $S_\alpha(t)$ corresponds to a $(1, \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-regularized family. The following result is a direct consequence of [19, Proposition 3.1 and Lemma 2.2].

Proposition 2.5. *Let $S_\alpha(t)$ be a solution operator on E with generator A. Then the following conditions are satisfied:*

(a) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$.*

(b) *Let $x \in D(A)$ and $t \geq 0$,*

$$S_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(s)\, ds.$$

(c) *Let $x \in E$. Then $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)x\, ds \in D(A)$ and*

$$S_\alpha(t)x = x + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)\, ds.$$

Definition 2.6. Let $A : D(A) \subset E \to E$ be a closed linear operator. A is said to be sectorial of the type (M, θ, μ) if there exist $\mu \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$ and $M > 0$ such that the resolvent of A exists outside the sector and following two conditions are satisfied:

(1) $\mu + S_\theta = \{\mu + s :\ \lambda \in \mathbb{C},\ |\arg(-\lambda)| < \theta\}$;

(2) $\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\mu|}$, $\lambda \notin \mu + S_\theta$.

In this paper, we assume that in problem (1.1)–(1.3) the operator A is sectorial of type μ with $0 \leq \theta < \pi(\frac{1-\alpha}{2})$. Then A is the generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma \exp^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\, d\lambda,$$

where γ is a suitable path lying outside the sector $\mu + S_\theta$.

Cuesta [7] has proved that if A is a sectorial operator of type μ, for some $M > 0$ and $0 < \theta < \pi(1 - \frac{\alpha}{2})$, there is $C > 0$ such that

$$\|S_\alpha(t)\|_{B(E)} \leq \frac{CM}{1 + |\mu|t^\alpha} \ \text{ if }\ \mu < 0$$

and

$$\|S_\alpha(t)\|_{B(E)} \leq CM(1 + \mu t^\alpha)e^{\mu^{\frac{1}{\alpha}} t} \ \text{ if }\ \mu \geq 0.$$

Note that $S_\alpha(t)$ is, in fact, integrable on $[0, b]$.

Theorem 2.7 (Krasnoselkii's fixed point theorem [17])**.** *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that*

(i) *$Px + Qy \in B$, whenever $x, y \in B$;*

(ii) *P is compact and continuous;*

(iii) *Q is a contraction mapping.*

Then there exists $z \in B$ such that $z = Pz + Qz$.

3 Main results

Motivated by [9], we give the following definition of a mild solution of (1.1)–(1.3).

Definition 3.1. We say that the function $y : (-\infty, b] \to E$ is a mild solution of (1.1)–(1.3) if $y_0 = \phi \in \mathcal{B}$ on $(-\infty, b]$, $y|_{[0,b]} \in \mathrm{PC}([0, b], E)$ and

$$y(t) = \begin{cases} S_\alpha(t)\phi(0) + \int\limits_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)})\, ds, & t \in [0, t_1], \\ g_i(t, y_{\rho(t,y_t)}), & t \in (t_i, s_i], \quad i = 1, 2, \dots, N, \\ S_\alpha(t - s_i) g_i(s_i, y_{\rho(s_i, y_{s_i})}) + \int\limits_{s_i}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)})\, ds, & t \in (s_i, t_{i+1}]. \end{cases}$$

Set

$$\mathcal{R}(\rho^-) = \big\{\rho(s, \varphi) : \ (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \le 0\big\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t) \|\phi\|_{\mathcal{B}} \ \text{ for every } \ t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_φ) is frequently verified by the functions, continuous and bounded. For more details, see, e.g., [15].

Lemma 3.3 ([14, Lemma 2.4])**.** *If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \le (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\big\{|y(\theta)| : \ \theta \in [0, \max\{0, s\}]\big\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Proposition 3.4. *From (H_φ), $(A1)$ and Lemma* 3.3*, for all $t \in [0, b]$ we have*

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \le (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b\|y(t)\|.$$

Our first result is based on the Banach contraction principle.

Theorem 3.5. *Assume:*

$(H1)$ *The solution operator $S_\alpha(t)$ is compact for $t > 0$, and there exists $M > 0$ such that $\|S_\alpha(t)\| \le M$ for every $t \in J$.*

$(H2)$ *There exists $l > 0$ such that*

$$\|f(t, u) - f(t, v)\|_E \le l_f \|u - v\|_{\mathcal{B}} \ \textit{ for all } \ u, v \in \mathcal{B}.$$

$(H3)$ *The functions $g_i : (t_i, s_i] \times \mathcal{B} \to E$, $i = 1, \dots, N$, are continuous and there exist the constants $h_i > 0$, $i = 1, \dots, N$ such that*

$$\|g_i(t, u) - g_i(t, v)\|_E \le l'_g \|u - v\|_{\mathcal{B}} \ \textit{ for all } \ u, v \in \mathcal{B}.$$

If

$$C = M K_b (l'_g + l_f b) < 1,$$

then there exists a unique solution of problem (1.1)–(1.3)*.*

Proof. Let $Y = \{u \in \mathrm{PC}(E):\ u(0) = \phi(0) = 0\}$ be endowed with the uniform convergence topology and $P : Y \to Y$ be the operator defined by

$$(Py)(t) = \begin{cases} S_\alpha(t)\phi(0) + \displaystyle\int_0^t S_\alpha(t-s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\, ds, & t \in [0, t_1],\ \ i = 0, \\ g_i(t, \overline{y}_{\rho(t,\overline{y}_t)}), & t \in (t_i, s_i],\ \ i \ge 1, \\ S_\alpha(t-s_i) g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})}) + \displaystyle\int_{s_i}^t S_\alpha(t-s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\, ds, & t \in (s_i, t_{i+1}],\ \ i \ge 1, \end{cases}$$

where $\overline{y} : (-\infty, b] \to E$ is such that $\overline{y}_0 = \phi$ and $\overline{y} = y$ on J. Let $\overline{\phi} : (-\infty, b] \to E$ be the extension of ϕ to $(-\infty, b]$ such that $\overline{\phi}(\theta) = \phi(0) = 0$ on J. It is clear that P is a well-defined operator from Y into Y. We show that P has a fixed point which is, in turn, a mild solution of problem (1.1)–(1.3).

For any $t \in [0, t_1]$ and $y, y^* \in Y$, from $(H1)$–$(H2)$ we have

$$\begin{aligned} \|(Py)(t) - (Py^*)(t)\|_E &\le \int_0^t \|S_\alpha(t-s)\|_{B(E)} \big\| f(s, \overline{y}_{\rho(s,\overline{y}_s)}) - f(s, \overline{y}^*_{\rho(s,\overline{y}^*_s)}) \big\|_E\, ds \\ &\le \int_0^t Ml_f \big\| \overline{y}_{\rho(s,\overline{y}_s)} - \overline{y}^*_{\rho(s,\overline{y}^*_s)} \big\|_{\mathcal{B}}\, ds. \end{aligned}$$

Using Proposition 3.4, we get

$$\begin{aligned} \|(Py)(t) - (Py^*)(t)\|_E &\le \int_0^t Ml_f K_b \big\| \overline{y}(s) - \overline{y}^*(s) \big\|_E\, ds \le Ml_f K_b \int_0^t \big\| \overline{y}(s) - \overline{y}^*(s) \big\|_E\, ds \\ &= Ml_f K_b \int_0^t \|y(s) - y^*(s)\|_E\, ds \qquad (\text{since } \overline{y} = y \text{ on } [0, b]) \\ &\le Ml_f K_b b \|y - y^*\|_{\mathrm{PC}}. \end{aligned}$$

For any $t \in (t_i, s_i]$, $i = 1, \dots, N$, we have

$$\|(Py)(t) - (Py^*)(t)\|_E = \big\| g_i(t, \overline{y}_{\rho(t,\overline{y}_t)}) - g_i(t, \overline{y}^*_{\rho(t,\overline{y}^*_t)}) \big\|_E \le l'_g K_b \|y - y^*\|_{\mathrm{PC}}.$$

Similarly, for any $t \in (s_i, t_{i+1}]$, $i = 1, \dots, N$, we have

$$\begin{aligned} \|(Py)(t) - (Py^*)(t)\|_E &\le \Big\| S_\alpha(t-s_i) \big[g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})}) - g_i(s_i, \overline{y}^*_{\rho(s_i,\overline{y}^*_{s_i})}) \big] \Big\|_E \\ &\quad + \int_{s_i}^t \|S_\alpha(t-s)\|_{B(E)} \big\| f(s, \overline{y}_{\rho(s,\overline{y}_s)}) - f(s, \overline{y}^*_{\rho(s,\overline{y}^*_s)}) \big\|_E\, ds \\ &\le Ml'_g K_b \|y - y^*\|_{\mathrm{PC}} + Ml_f K_b b \|y - y^*\|_{\mathrm{PC}} \le (Ml'_g K_b + Ml_f K_b b) \|y - y^*\|_{\mathrm{PC}}. \end{aligned}$$

Thus, for all $t \in [0, b]$, we obtain $\|(Py) - (Py^*)\|_{\mathrm{PC}} \le C \|y - y^*\|_{\mathrm{PC}}$. Hence, P is a contraction on Y and has a unique fixed point $y \in P$, which is, obviously, a unique mild solution of problem (1.1)–(1.3) on $[0, b]$. □

To obtain an existence result via Krasnoselskii's fixed point theorem, we need the following assumptions.

$(H4)$ The function $f : J \times \mathcal{B} \to E$ is Carathéodory one.

$(H5)$ There exist a function $p \in L^1(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+ \to (0,\infty)$ such that
$$\|f(t,u)\|_E \le p(t)\psi(\|u\|_{\mathcal{B}}) \ \text{ for a.e. } \ t\in J \ \text{ and each } \ u\in\mathcal{B}.$$

$(H6)$ The functions $t \to g_i(t,0)$ are bounded with
$$G^* = \max_{i=1,\dots,N} \|g_i(t,0)\|_E.$$

Theorem 3.6. *Assume that* $(H1)$, $(H3)$–$(H6)$ *and* (H_φ) *hold. Then problem* (1.1)–(1.3) *has a mild solution.*

Proof. Let P be the operator introduced in the proof of Theorem 3.5. We introduce the decomposition $Py(t) = P^1y(t) + P^2y(t)$, where
$$(P^1y)(t) = \begin{cases} S_\alpha(t-s_i)g_i(s_i,\overline{y}_{\rho(s_i,\overline{y}_{s_i})}) + \int\limits_{s_i}^{t} S_\alpha(t-s)f(s,\overline{y}_{\rho(s,\overline{y}_s)})\,ds, & \text{if } \ t\in(s_i,t_{i+1}], \ \ i\ge 1,\\ S_\alpha(t)\phi(0) + \int\limits_0^t S_\alpha(t-s)f(s,\overline{y}_{\rho(s,\overline{y}_s)})\,ds, & \text{if } \ t\in[0,t_1],\\ 0, & \text{if } \ t\in(t_i,s_i], \ \ i\ge 1,\end{cases}$$
and
$$(P^2y)(t) = \begin{cases} g_i(t,\overline{y}_{\rho(t,\overline{y}_t)}), & \text{if } \ t\in(t_i,s_i], \ \ i\ge1,\\ 0, & \text{if } \ t\in(s_i,t_{i+1}], \ \ i\ge 1,\\ 0, & \text{if } \ t\in[0,t_1].\end{cases}$$
Let $B_r = \{y\in Y: \ \|y\|_{\mathrm{PC}} \le r\}$. The proof of the theorem will be given in a couple of steps.

Step 1: For any $y\in B_r$, we have $P^1y + P^2y \in B_r$.

Case 1. For each $t\in[0,t_1]$, we obtain
$$\begin{aligned}\|(P^1y+P^2y)(t)\|_E &\le \|S_\alpha(t)\|_{B(E)}\|\phi(0)\|_{\mathcal{B}} + \int\limits_0^t \|S_\alpha(t-s)f(s,\overline{y}_{\rho(s,\overline{y}_s)})\|_E\,ds\\ &\le M\|\phi\|_{\mathcal{B}} + M\int\limits_0^t \|f(s,\overline{y}_{\rho(s,\overline{y}_s)}\|_E\,ds \le M\|\phi\|_{\mathcal{B}} + M\int\limits_0^t p(s)\psi\big(\|\overline{y}_{\rho(s,\overline{y}_s)}\|_{\mathcal{B}}\big)\,ds.\end{aligned}$$
Set
$$d = (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b r.$$
Then we have
$$\|(P^1y+P^2y)(t)\|_E \le M\|\phi\|_{\mathcal{B}} + M\psi(d)\int\limits_0^t p(s)\,ds.$$
Thus,
$$\|P^1(y) + P^2(y)\| \le M\|\phi\|_{\mathcal{B}} + M\psi(d)\|p\|_{L^1[0,t_1]} \le r.$$

Case 2. For each $t\in[t_i,s_i)$, $i=1,2,\dots,N$, we have
$$\begin{aligned}\|(P^1y+P^2y)(t)\|_E &\le \|g_i(t,\overline{y}_{\rho(t,\overline{y}_t)})\|_E\\ &\le \big\|g_i(t,\overline{y}_{\rho(t,\overline{y}_t)}) - g_i(t,0)\big\|_E + \|g_i(t,0)\|_E \le l_g'\|\overline{y}_{\rho(t,\overline{y}_t)}\|_{\mathcal{B}} + G^* \le l_g' d + G^*.\end{aligned}$$
Then
$$\|P^1y + P^2y\| \le l_g' d + G^* \le r.$$

Case 3. For each $t \in (s_i, t_{i+1})$, $i = 1, 2, \ldots, N$, we obtain

$$\|(P^1y + P^2y)(t)\|_E \le \left\|S_\alpha(t - s_i)g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})})\right\|_E + \int_{s_i}^{t} \left\|S_\alpha(t - s)f(s, \overline{y}_{\rho(s,\overline{y}_s)})\right\|_E ds \le M(l'_g d + G^*) + M\psi(d)\int_{s_i}^{t} p(s)\, ds.$$

Then

$$\|P^1y + P^2y\| \le M\left(l'_g d + G^* + \psi(d)\int_{s_i}^{t} p(s)\, ds\right) \le r.$$

Thus, we obtain $P^1y + P^2y \in B_r$ for any $y \in B_r$.

Step 2: We show that P^2 is a contraction on B_r.

Case 1. For $y_1, y_2 \in B_r$ and for $t \in [0, t_1]$, we have

$$\left\|(P^2y_1)(t) - (P^2y_2)(t)\right\|_E = 0.$$

Case 2. For $y_1, y_2 \in B_r$ and for $t \in [t_i, s_i)$, $i = 1, 2, \ldots, N$, we have

$$\left\|(P^2y_1)(t) - (P^2y_2)(t)\right\|_E \le l'_g d.$$

Case 3. For $y_1, y_2 \in B_r$ and for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, N$, we obtain

$$\left\|(P^2y_1)(t) - (P^2y_2)(t)\right\|_E = 0.$$

Thus, we obtain

$$\|P^2y_1 - P^2y_2\|_{\mathrm{PC}} \le l'_g d = L,$$

which implies that P^2 is a contraction due to $L < 1$.

Step 3: P^1 is continuous.

Let y^n be a sequence such that $y^n \to y$ in B_r.

Case 1. For each $t \in [0, t_1]$, we have

$$\left\|P^1(y^n)(t) - P^1(y)(t)\right\|_E = \left\|S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t - s)\Big[f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)})\Big]\, ds\right\|_E \le M\int_0^t \left\|f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)})\right\|_E ds.$$

Case 2. For each $t \in [t_i, s_i)$, $i = 1, 2, \ldots, N$, we have

$$\left\|P^1(y^n)(t) - P^1(y)(t)\right\|_E = 0.$$

Case 3. For each $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, N$, we obtain

$$\left\|P^1(y^n)(t) - P^1(y)(t)\right\|_E = \left\|S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t - s)\Big[f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)})\Big]\, ds\right\|_E \le M\int_0^t \left\|f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)})\right\|_E ds.$$

Then by $(H4)$, by the Lebesgue dominated convergence theorem, we have

$$\|P^1y^n - P^1y\|_{\mathrm{PC}} \to 0 \ \text{ as } \ n \to +\infty.$$

Step 4: P^1 is compact.

I. $P^1(B_r) \subset B_r$ because $\|P^1 y\| \le r$.

II. We show that P^1 maps a bounded set into a equicontinuous set of B_r.

Case 1. For the interval $t \in [0, t_1]$, $0 \le \tau_1 \le \tau_2 \le t_1$, any $y \in B_r$, one has

$$\begin{aligned}\big\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\big\|_E &\le \|S_\alpha(\tau_2) - S_\alpha(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\ &+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|f(s, \overline{y}_{\rho(s,\overline{y}_s)})\|_E\, ds + \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\|\|_E\, ds \\ &\le \|S_\alpha(\tau_2) - S_\alpha(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \psi(d)\int_0^{\tau_1} \big\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\big\|_{B(E)} p(s)\, ds + M\psi(d)\int_{\tau_1}^{\tau_2} p(s)\, ds.\end{aligned}$$

Case 2. For the interval $t \in [t_i, s_i)$, $i = 1, 2, \ldots, N$, $t_i \le \tau_1 \le \tau_2 \le s_i$, any $y \in B_r$, one has

$$\big\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\big\|_E = 0.$$

Case 3. For the interval $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, N$, $s_i \le \tau_1 \le \tau_2 \le t_i + 1$, any $y \in B_r$, one has

$$\begin{aligned}\big\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\big\|_E &\le \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)}\|g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})})\|_E \\ &+ \int_0^{\tau_1} \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)}\|f(s, \overline{y}_{\rho(s,\overline{y}_s)})\|_E\, ds + \int_{\tau_1}^{\tau_2} \big\|S_\alpha(\tau_2 - s_i) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\big\|_E\, ds \\ &\le \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)}(l_g' d + G^*) \\ &\quad + \psi(d)\int_0^{\tau_1} \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)} p(s)\, ds + M\psi(d)\int_{\tau_1}^{\tau_2} p(s)\, ds.\end{aligned}$$

From the aforementioned equation, we find that $\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\| \to 0$ as $\tau_2 \to \tau_1$, since $S_\alpha(t)$ is continuous in the uniform operator topology. So, P^1 is equicontinuous. As a consequence of Steps 3–4, together with the Arzelà–Ascoli theorem, we can conclude that $P^1 : B_r \to B_r$ is continuous and completely continuous. By using Krasnoselskii's fixed point theorem, the operator $P = P^1 + P^2$ has a fixed point, which is a solution of problem (1.1)–(1.3). □

4 An example

We consider the fractional differential equation with a state-dependent delay of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{1}{\Gamma(\alpha - 1)} \displaystyle\int_{-\infty}^{t} (t - s)^{\alpha - 2} L_x u(s, x)\, ds \\ \qquad = \dfrac{e^{-\gamma t + t}|u(t - \sigma(u(t, 0)), x)|}{3(e^{-t} + e^t)(1 + |u(t - \sigma(u(t, 0), x))|)}, \quad (t, x) \in \displaystyle\bigcup_{i=1}^{N} [s_i, t_{i+1}] \times [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, \quad t \in [0, b], \\ u(\tau, x) = u_0(\tau, x), \quad \theta \in (-\infty, 0], \quad x \in [0, \pi], \\ u(t, x) = G_i\big(t, u\big(t - \sigma(u(t, 0)), x\big)\big), \quad (t, x) \in (t_i, s_i] \times [0, \pi], \quad i = 1, 2, \ldots, N, \end{cases} \tag{4.1}$$

where $1 < \alpha < 2$, $0 = t_0 = s_0 < t_1 < t_2 < \cdots < t_N - 1 \le s_N \le t_N \le t_N + 1 = b$ are prefixed real numbers, $\sigma \in C(\mathbb{R}, [0, \infty))$, $\gamma > 0$, L_x stands for the operator with respect to the spatial variable x which is given by $L_x = \frac{\partial^2}{\partial x^2} - r$, with $r > 0$. Take $E = L^2([0,\pi],\mathbb{R})$ and the operator $A := L_x : D(A) \subset E \to E$ with domain $D(A) := \{u \in E : \ u'' \in E, \ u(0) = u(\pi) = 0\}$. Clearly, A is densely defined in E and is sectorial. Hence A is a generator of a solution operator on E. For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma = \Big\{\phi \in C((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\Big\}$$

with the norm

$$\|\phi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|\phi(\theta)|.$$

Notice that the phase space $\mathcal{B}_\gamma$ satisfies axioms (A_1), (A_2) and (A_3) (see [15] for more details). Set

$$\begin{aligned} y(t)(x) &= u(t,x), \\ \phi(\theta)(x) &= u_0(\theta,x), \\ f(t,\phi)(x) &= \frac{e^{-\gamma t+t}|\phi(0,x)|}{3(e^{-t}+e^t)(1+|\phi(0,x)|)}, \\ g_i(t,\phi)(x) &= G_i\big(t, u\big(t-\sigma(u(t,0)),x\big)\big), \\ \rho(t,\phi) &= t - \sigma(\phi(0,0)). \end{aligned}$$

Let $\phi \in \mathcal{B}_\gamma$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then by Theorem 3.5, there exists at least one mild solution of (4.1).

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(Received 18.09.2017)

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