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ON BEHAVIOR OF OSCILLATING SOLUTIONS TO SECOND-ORDER EMDEN–FOWLER TYPE DIFFERENTIAL EQUATIONS

Abstract. The second-order Emden–Fowler type differential equation with positive bounded potential is considered. Asymptotic behavior of maximally extended oscillating solutions to the equation is described.[1]

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1 Introduction

Consider the second-order Emden–Fowler type differential equation

$$y'' + p(x, y, y')|y|^k \operatorname{sgn} y = 0, \quad k > 0, \quad k \neq 1, \tag{1.1}$$

with continuous in x and Lipschitz continuous in u, v positive function $p(x, u, v)$ defined on $\mathbb{R} \times \mathbb{R}^2$. The asymptotic behavior of all solutions to equation (1.1) in the case $p = p(x)$ was described by I. T. Kiguradze and T. A. Chanturia (see [11]). The results on asymptotic classification of maximally extended solutions to third- and fourth-order similar differential equations for $k > 0$, $k \neq 1$, were given by I. V. Astashova (see [1–5]). The asymptotic classification of solutions to equation (1.1) with negative function $p(x, u, v)$ for regular ($k > 1$) and singular ($0 < k < 1$) nonlinearities is contained in [6, 7].

Using the methods described in [2], we investigate the behavior of solutions to equation (1.1) in the case $p(x, u, v) > 0$ (see [8]). Further, suppose that the function $p(x, u, v)$ additionally satisfies the inequalities

$$0 < m \leq p(x, u, v) \leq M < +\infty. \tag{1.2}$$

2 Oscillation of solutions and their first derivatives

Consider the trajectories $\{(y(x), y'(x))\} \subset \mathbb{R}^2$ generated by nontrivial solutions to equation (1.1). Divide $\mathbb{R}^2$ by four closed sets crossing over the boundaries only

$$\begin{bmatrix} + \\ + \end{bmatrix}, \quad \begin{bmatrix} + \\ - \end{bmatrix}, \quad \begin{bmatrix} - \\ - \end{bmatrix}, \quad \begin{bmatrix} - \\ + \end{bmatrix}. \tag{2.1}$$

For the sets boundaries we use the following notation:

$$\begin{bmatrix} + \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ - \end{bmatrix}, \quad \begin{bmatrix} - \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ + \end{bmatrix}.$$

For example,

$$\begin{bmatrix} + \\ - \end{bmatrix} = \big\{(y_0, y_1) \in \mathbb{R}^2 : \ y_0 \geq 0, \ y_1 \leq 0\big\},$$
$$\begin{bmatrix} 0 \\ + \end{bmatrix} = \big\{(y_0, y_1) \in \mathbb{R}^2 : \ y_0 = 0, \ y_1 \geq 0\big\}.$$

Lemma 2.1. *Suppose $k \in (0, 1) \cup (1, +\infty)$, the function $p(x, u, v)$ is continuous in x, Lipschitz continuous in u, v, satisfies inequalities* (1.2) *and $y(x)$ is a nontrivial maximally extended solution to equation* (1.1)*. Then neither $y(x)$ nor its first derivative $y'(x)$ can be constant-sign functions in the neighborhood of domain boundaries.*

Proof. Using the substitutions $x \mapsto -x$, $y(x) \mapsto -y(x)$, we obtain an equation of the same type as (1.1). That is why we further investigate behavior of nontrivial solutions to equation (1.1) and their first derivatives near the right-side boundary of the domain only.

Prove the statement for solution $y(x)$, the proof for its first derivative $y'(x)$ is similar. Assume that a solution $y(x)$ to equation (1.1) is defined on a finite or on an infinite interval (a, b) and is positive in some neighborhood of b. According to the type of equation (1.1), the second derivative is negative in this neighborhood, therefore the first derivative decreases monotonously and has a finite or an infinite limit as $x \to b - 0$. It means that the first derivative is a constant-sign function in the neighborhood of b. That is why $y(x)$ is monotonous in the neighborhood of b and tends to a finite or an infinite value as $x \to b - 0$.

Let $b < +\infty$. If a solution $y(x)$ (and hence $y''(x)$) or its first derivative has a finite limit, then integrating the second derivative or the first derivative, respectively, on a finite interval, we obtain the finite limits in both cases. So, we get a contradiction with the right-maximally extension of a solution. If solution and first derivative limits are infinite, then they must have the same sign. So, we get a contradiction with equation (1.1).

Let $b = +\infty$. If a solution $y(x)$ (and hence $y''(x)$) or its first derivative has a nontrivial limit, then integrating the second derivative or the first derivative, respectively, on the whole domain, we obtain infinite limits in both cases. Thus, they must be of the same sign, and therefore we get a contradiction with equation (1.1). If solution and first derivative limits are equal to zero, then the solution is positive in a neighborhood of $+\infty$, monotonously decreases to zero and its first derivative is negative and monotonously increases to zero as $x \to +\infty$. It means that the second derivative (as the solution is positive) decreases to zero at infinity. So, we get a contradiction with equation (1.1). $\square$

Theorem 2.1. *Suppose $k \in (0,1) \cup (1,+\infty)$, the function $p(x,u,v)$ is continuous in x, Lipschitz continuous in u, v and satisfies inequalities* (1.2). *Then all nontrivial maximally extended solutions and their first derivatives to equation* (1.1) *are oscillating at the left– and right-hand sides, zeroes x_j of solutions and zeroes x'_j of their first derivatives alternate, i.e.,*

$$\cdots < x_{j-1} < x'_j < x_j < x'_{j+1} < \cdots, \quad j \in \mathbb{Z}.$$

Moreover, for any $j \in \mathbb{Z}$, the following inequalities hold:

$$-\sqrt{\frac{M}{m}} \le \frac{y'(x_{j+1})}{y'(x_j)} \le -\sqrt{\frac{m}{M}}, \quad -\Big(\frac{M}{m}\Big)^{\frac{1}{k+1}} \le \frac{y(x'_{j+1})}{y(x'_j)} \le -\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}}.$$

Proof. As mentioned above, it suffices to investigate the asymptotic behavior of nontrivial maximally extended solutions at the right-hand side.

Prove that a trajectory generated by any nontrivial maximally extended solution $y(x)$ to equation (1.1) moves between the introduced sets (2.1) at the right-hand side only by the following scheme:

$$\begin{array}{ccc} \begin{bmatrix} + \\ + \end{bmatrix} & \longrightarrow & \begin{bmatrix} + \\ - \end{bmatrix} \\ \uparrow & & \downarrow \\ \begin{bmatrix} - \\ + \end{bmatrix} & \longleftarrow & \begin{bmatrix} - \\ - \end{bmatrix} \end{array}. \tag{2.2}$$

Indeed, suppose that $(y(x), y'(x))$ is an internal point for the set $\begin{bmatrix} + \\ + \end{bmatrix}$ at some moment. It means that $y(x) > 0$, $y'(x) > 0$ and $y''(x) < 0$. Therefore, $y(x)$ is positive and increases, $y'(x)$ is positive and decreases, while the trajectory generated by the solution $y(x)$ is located in the interior of $\begin{bmatrix} + \\ + \end{bmatrix}$. Then either $y'(x)$ is equal to zero and the corresponding trajectory will get to the boundary $\begin{bmatrix} + \\ 0 \end{bmatrix}$ of $\begin{bmatrix} + \\ + \end{bmatrix}$ or $y'(x)$ is nontrivial and have a nonnegative limit at the right-hand side, i.e., the first derivative will be a constant-sign function. So, we get a contradiction with Lemma 2.1. Thus, the case is possible if and only if the trajectory generated by the solution $y(x)$ gets to the boundary $\begin{bmatrix} + \\ 0 \end{bmatrix}$, i.e., the solution $y(x)$ is positive and has a local extremum at some point x'_0, moreover, $y''(x'_0) < 0$. Then there exists a constant $\delta > 0$ such that $y(x) > 0$, $y'(x) < 0$ for $x \in (x'_0, x'_0 + \delta)$. So, the trajectory will get to the interior of the set $\begin{bmatrix} + \\ - \end{bmatrix}$.

Further, we have $y(x) > 0$, $y'(x) < 0$ and $y''(x) < 0$. Therefore, $y(x)$ is positive and decreases, $y'(x)$ is positive and increases, while the corresponding trajectory is located in the interior of $\begin{bmatrix} + \\ - \end{bmatrix}$. According to Lemma 2.1, the solution $y(x)$ cannot be positive at the right-hand side, that is why it will be equal to zero at some point $x_0 > x'_0$, and the trajectory generated by this solution will get to

$\begin{bmatrix} 0 \\ - \end{bmatrix}$. As $y'(x_0) < 0$, there exists a constant $\widetilde{\delta} > 0$ such that $y(x) < 0$, $y'(x) < 0$ for $x \in (x_0, x_0 + \widetilde{\delta})$. Thus, the trajectory will get to the interior of the set $\begin{bmatrix} - \\ - \end{bmatrix}$.

Now, we have $y(x) < 0$ and $y'(x) < 0$. Similarly, prove that the trajectory generated by $y(x)$ at the right-hand side gets to the boundary $\begin{bmatrix} - \\ 0 \end{bmatrix}$, i.e., $y(x)$ has a local minimum at some point $x'_1 > x_0 > x'_0$. It moves further towards the interior of the set $\begin{bmatrix} - \\ + \end{bmatrix}$, and according to Lemma 2.1, tends to the boundary $\begin{bmatrix} 0 \\ + \end{bmatrix}$ for $x_1 > x'_1 > x_0 > x'_0$. Thereafter the trajectory goes to the interior of the set $\begin{bmatrix} + \\ + \end{bmatrix}$.

So, we have proved that the trajectory generated by any nontrivial maximally extended solution $y(x)$ to equation (1.1) can move between the introduced sets (2.1) at the right-hand side only by the scheme (2.2).

Besides, according to Lemma 2.1, it cannot stay in any set (2.1) at the left- and right-hand sides. Therefore, the solution $y(x)$ to equation (1.1) and its first derivative $y'(x)$ are oscillating at the left- and right-hand sides, zeroes x_j of solutions and zeroes x'_j of their first derivatives alternate, i.e.,

$$\cdots < x_{j-1} < x'_j < x_j < x'_{j+1} < \cdots, \quad j \in \mathbb{Z}.$$

Further, without any restrictions, we assume $y'(x_j) < 0$. Note

$$0 = |y(x_j)|^{k+1} - |y(x_{j+1})|^{k+1} = -(k+1) \int\limits_{y(x_j)}^{y(x_{j+1})} |y|^{k-1} y \, dy,$$

and from equation (1.1) we have

$$\begin{aligned} 0 = -(k+1) \int\limits_{y(x_j)}^{y(x_{j+1})} |y|^{k-1} y \, dy &= (k+1) \int\limits_{x_j}^{x_{j+1}} \frac{y''\, y'}{p(x,y,y')} \, dx \\ &= (k+1) \int\limits_{x_j}^{x'_{j+1}} \frac{y''\, y'}{p(x,y,y')} \, dx + (k+1) \int\limits_{x'_{j+1}}^{x_{j+1}} \frac{y''\, y'}{p(x,y,y')} \, dx. \end{aligned} \tag{2.3}$$

As $y'(x_j) < 0$, we have $y'(x) < 0$ and $y''(x) > 0$ for $x \in (x_j, x'_{j+1})$. Also, for $x \in (x'_{j+1}, x_{j+1})$, we have $y'(x) > 0$ and $y''(x) > 0$. So, $\frac{y''\, y'}{p(x,y,y')} < 0$ for $x \in (x_j, x'_{j+1})$ and $\frac{y''\, y'}{p(x,y,y')} > 0$ for $x \in (x'_{j+1}, x_{j+1})$.

Estimate expression (2.3):

$$\begin{aligned} &(k+1) \int\limits_{x_j}^{x'_{j+1}} \frac{y''\, y'}{p(x,y,y')} \, dx + (k+1) \int\limits_{x'_{j+1}}^{x_{j+1}} \frac{y''\, y'}{p(x,y,y')} \, dx \\ &\quad \le \frac{k+1}{M} \int\limits_{x_j}^{x'_{j+1}} y''\, y' \, dx + \frac{k+1}{m} \int\limits_{x'_{j+1}}^{x_{j+1}} y''\, y' \, dx = \frac{k+1}{M} \int\limits_{y'(x_j)}^{y'(x'_{j+1})} y' \, dy' + \frac{k+1}{m} \int\limits_{y'(x'_{j+1})}^{y'(x_{j+1})} y' \, dy' \\ &\qquad = \frac{k+1}{2M} \, (y')^2 \Big|_{x_j}^{x'_{j+1}} + \frac{k+1}{2m} \, (y')^2 \Big|_{x'_{j+1}}^{x_{j+1}} = -\frac{k+1}{2M} \, (y'(x_j))^2 + \frac{k+1}{2m} \, (y'(x_{j+1}))^2, \end{aligned}$$

whence

$$\frac{k+1}{2M} \, (y'(x_j))^2 \le \frac{k+1}{2m} \, (y'(x_{j+1}))^2.$$

Obtain another estimate for (2.3):

$$
\begin{aligned}
&(k+1)\int\limits_{x_j}^{x'_{j+1}} \frac{y''\,y'}{p(x,y,y')}\,dx + (k+1)\int\limits_{x'_{j+1}}^{x_{j+1}} \frac{y''\,y'}{p(x,y,y')}\,dx \\
&\geq \frac{k+1}{m}\int\limits_{x_j}^{x'_{j+1}} y''\,y'\,dx + \frac{k+1}{M}\int\limits_{x'_{j+1}}^{x_{j+1}} y''\,y'\,dx = \frac{k+1}{m}\int\limits_{y'(x_j)}^{y'(x'_{j+1})} y'\,dy' + \frac{k+1}{M}\int\limits_{y'(x'_{j+1})}^{y'(x_{j+1})} y'\,dy' \\
&= \frac{k+1}{2m}\,(y')^2\Big|_{x_j}^{x'_{j+1}} + \frac{k+1}{2M}\,(y')^2\Big|_{x'_{j+1}}^{x_{j+1}} = -\frac{k+1}{2m}\,(y'(x_j))^2 + \frac{k+1}{2M}\,(y'(x_{j+1}))^2,
\end{aligned}
$$

whence

$$\frac{k+1}{2m}\,(y'(x_j))^2 \geq \frac{k+1}{2M}\,(y'(x_{j+1}))^2.$$

Therefore,

$$\sqrt{\frac{m}{M}}\,|y'(x_j)| \leq |y'(x_{j+1})| \leq \sqrt{\frac{M}{m}}\,|y'(x_j)| \tag{2.4}$$

and

$$\sqrt{\frac{m}{M}} \leq \Big|\frac{y'(x_{j+1})}{y'(x_j)}\Big| \leq \sqrt{\frac{M}{m}}\,.$$

Since zeroes x_j and extremum points x'_j of a nontrivial maximally extended solution to equation (1.1) alternate, for any $j \in \mathbb{Z}$ we have $y'(x_{j+1})\,y'(x_j) < 0$ and

$$-\sqrt{\frac{M}{m}} \leq \frac{y'(x_{j+1})}{y'(x_j)} \leq -\sqrt{\frac{m}{M}}\,.$$

Obtain the second estimate. We have $y(x'_j) > 0$. Note

$$|y(x'_j)|^{k+1} = |y(x'_j)|^{k+1} - |y(x_j)|^{k+1} = -(k+1)\int\limits_{y(x'_j)}^{y(x_j)} |y|^{k-1}y\,dy = (k+1)\int\limits_{x'_j}^{x_j} \frac{y''\,y'}{p(x,y,y')}\,dx.$$

As $y(x'_j) > 0$, we have $y'(x) < 0$ and $y''(x) < 0$ for $x \in (x'_j, x_j)$, i.e., $\frac{y''\,y'}{p(x,y,y')} > 0$ for $x \in (x'_j, x_j)$. So,

$$\frac{k+1}{M}\int\limits_{x'_j}^{x_j} y''\,y'\,dx \leq (k+1)\int\limits_{x'_j}^{x_j} \frac{y''\,y'}{p(x,y,y')}\,dx \leq \frac{k+1}{m}\int\limits_{x'_j}^{x_j} y''\,y'\,dx,$$

then

$$\frac{k+1}{M}\int\limits_{y'(x'_j)}^{y'(x_j)} y'\,dy' \leq |y(x'_j)|^{k+1} \leq \frac{k+1}{m}\int\limits_{y'(x'_j)}^{y'(x_j)} y'\,dy'$$

and

$$\frac{k+1}{2M}\,(y'(x_j))^2 \leq |y(x'_j)|^{k+1} \leq \frac{k+1}{2m}\,(y'(x_j))^2. \tag{2.5}$$

Analogously, on the interval (x_j, x'_{j+1}) we obtain the estimates similar to (2.5):

$$\frac{k+1}{2M}\,(y'(x_j))^2 \leq |y(x'_{j+1})|^{k+1} \leq \frac{k+1}{2m}\,(y'(x_j))^2$$

and, therefore,
$$\frac{2m}{k+1}\,|y(x'_{j+1})|^{k+1} \le (y'(x_j))^2 \le \frac{2M}{k+1}\,|y(x'_{j+1})|^{k+1}.$$
So,
$$\frac{m}{M}\,|y(x'_{j+1})|^{k+1} \le |y(x'_j)|^{k+1} \le \frac{M}{m}\,|y(x'_{j+1})|^{k+1},$$
$$\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}}|y(x'_{j+1})| \le |y(x'_j)| \le \Big(\frac{M}{m}\Big)^{\frac{1}{k+1}}|y(x'_{j+1})|$$
and
$$\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}} \le \Big|\frac{y(x'_{j+1})}{y(x'_j)}\Big| \le \Big(\frac{M}{m}\Big)^{\frac{1}{k+1}}.$$

Since zeroes x_j and extremum point x'_j of a nontrivial maximally extended solution to equation (1.1) alternate, for any $j \in \mathbb{Z}$ we have $y(x'_{j+1})\,y(x'_j) < 0$ and
$$-\Big(\frac{M}{m}\Big)^{\frac{1}{k+1}} \le \frac{y(x'_{j+1})}{y(x'_j)} \le -\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}}. \qquad \square$$

Repeating the steps described in the proof of Theorem 2.1, T. Korchemkina has obtained the following

Corollary ([9])**.** *Introduce the notation*
$$m_j = \min_{x\in[x_j,x_{j+1}]} p(x,y(x),y'(x)), \quad M_j = \max_{x\in[x_j,x_{j+1}]} p(x,y(x),y'(x)), \;\; j \in \mathbb{Z}.$$
Then, for any $j \in \mathbb{Z}$, the following inequalities hold:
$$-\sqrt{\frac{M_j}{m_j}} \le \frac{y'(x_{j+1})}{y'(x_j)} \le -\sqrt{\frac{m_j}{M_j}} - \Big(\frac{M_j^2}{m_j m_{j-1}}\Big)^{\frac{1}{k+1}} \le \frac{y(x'_j)}{y(x'_{j+1})} \le -\Big(\frac{m_j^2}{M_j\,M_{j-1}}\Big)^{\frac{1}{k+1}}.$$

3 Asymptotic behavior of maximally extended solutions

I. T. Kiguradze and T. A. Chanturia in [11] proved that if $p = p(x)$ is a positive locally integrable function of locally bounded variation, then for both regular ($k > 1$) and singular ($0 < k < 1$) nonlinearities, any nontrivial right-maximally extended solution to equation (1.1) is proper, i.e., is defined in the neighborhood of $+\infty$.

For $k > 1$, an example is given [10] of a continuous function $p = p(x)$ satisfying inequalities (1.2) such that there exists a solution to (1.1) with a resonance asymptote $x = x^*$ ($\overline{\lim\limits_{x\to x^*-0}}\, y(x) = +\infty$, $\underline{\lim\limits_{x\to x^*-0}}\, y(x) = -\infty$), i.e., a non-proper solution. Step by step we construct a continuous function $p(x)$ and an oscillating solution $y(x)$ to equation (1.1). On each step we define p, construct a solution to equation (1.1) and estimate the distance between consecutive zeros $x_{j+1} - x_j$.

Moreover, the sufficient conditions on the function $p = p(x)$ are obtained under which all nontrivial maximally extended solutions are defined on the whole axis.

Theorem 3.1. *Suppose $k \in (0,1)\cup(1,+\infty)$, $p = p(x)$ is a continuous function of a globally bounded variation satisfying inequalities* (1.2)*. Then for any nontrivial maximally extended solution $y(x)$ to* (1.1) *there exist the finite positive limits $\lim\limits_{j\to\pm\infty}|y'(x_j)|$, $\lim\limits_{j\to\pm\infty}|y(x'_j)|$ and $\lim\limits_{j\to\pm\infty}(x_{j+1} - x_j)$.*

Proof. Let $y(x)$ be a nontrivial maximally extended solution to equation (1.1). Now we investigate an asymptotic behavior of $y(x)$ at the right-side boundary of the domain ($j \to +\infty$), the case $j \to -\infty$ is similar.

Let us use the following notation:

$$m_j = \min_{x\in[x_j,x_{j+1}]} p(x), \quad M_j = \max_{x\in[x_j,x_{j+1}]} p(x), \quad j\in\mathbb{Z},$$
$$m'_j = \min_{x\in[x'_j,x'_{j+1}]} p(x), \quad M'_j = \max_{x\in[x'_j,x'_{j+1}]} p(x), \quad j\in\mathbb{Z}.$$

By repeating the steps described in the proof of Theorem 2.1, for any $j\in\mathbb{N}$, we obtain similar to (2.5) estimates:

$$\frac{k+1}{2M'_j}\,(y'(x_j))^2 \le |y(x'_j)|^{k+1} \le \frac{k+1}{2m'_j}\,(y'(x_j))^2,$$
$$\frac{k+1}{2M'_j}\,(y'(x_j))^2 \le |y(x'_{j+1})|^{k+1} \le \frac{k+1}{2m'_j}\,(y'(x_j))^2,$$

whence

$$\Big(\frac{m'_j}{M'_j}\Big)^{\frac{1}{k+1}} \le \Big|\frac{y(x'_j)}{y(x'_{j+1})}\Big| \le \Big(\frac{M'_j}{m'_j}\Big)^{\frac{1}{k+1}}.$$

Moreover, due to the above estimate and estimate (2.4), for any $j\in\mathbb{N}$, we have

$$\big|\ln|y'(x_{j+1})| - \ln|y'(x_j)|\big| \le \frac{1}{2}\,(\ln M_j - \ln m_j) \le \frac{1}{2}\,V_{[x_j,x_{j+1}]}\ln p(x),$$
$$\big|\ln|y(x'_{j+1})| - \ln|y(x'_j)|\big| \le \frac{1}{k+1}\,(\ln M'_j - \ln m'_j) \le \frac{1}{k+1}\,V_{[x'_j,x'_{j+1}]}\ln p(x),$$
$$\sum_{j=1}^{+\infty} V_{[x_j,x_{j+1}]}\ln p(x) = V_{[x_1,+\infty)}\ln p(x) < +\infty,$$
$$\sum_{j=1}^{+\infty} V_{[x'_j,x'_{j+1}]}\ln p(x) = V_{[x'_1,+\infty)}\ln p(x) < +\infty,$$

where $V_{[a,b]}\ln p(x)$, $V_{[c,+\infty)}\ln p(x)$ are variations of the function $\ln p(x)$ on $[a,b]$ and $[c,+\infty)$, respectively. Due to the Weierstrass test, the series $\sum\limits_{j=1}^{+\infty}(\ln|y'(x_{j+1})| - \ln|y'(x_j)|)$ converges.

Therefore, there exists a finite $\lim\limits_{j\to+\infty}\ln|y'(x_j)|$, hence there exists a finite $\lim\limits_{j\to+\infty}|y'(x_j)|$. Analogously, we obtain the existence of a finite positive $\lim\limits_{j\to+\infty}|y(x'_j)|$.

Further, let us show that the distance between consecutive zeros $(x_{j+1}-x_j)$ has a limit as $j\to+\infty$. Multiplying equation (1.1) by y', integrating it on $[x'_{j+1},x]$, $x\le x_{j+1}$, and assuming without any restrictions that $y(x)\ge 0$ on $[x'_{j+1},x_{j+1}]$, we obtain

$$(y'(x))^2 = -2\int_{x'_{j+1}}^{x} p(s)y'(s)y^k(s)\,ds = 2\int_{x'_{j+1}}^{x} p(s)|y'(s)|y^k(s)\,ds \le \frac{2M'_{j+1}}{k+1}\,\big(H_{j+1}^{k+1} - y^{k+1}(x)\big).$$

Analogously, we obtain the estimate

$$(y'(x))^2 \ge \frac{2m'_{j+1}}{k+1}\,\big(H_{j+1}^{k+1} - y^{k+1}(x)\big),$$

so,

$$\sqrt{\frac{2m'_{j+1}}{k+1}}\,\sqrt{H_{j+1}^{k+1} - y^{k+1}(x)} \le |y'(x)| \le \sqrt{\frac{2M'_{j+1}}{k+1}}\,\sqrt{H_{j+1}^{k+1} - y^{k+1}(x)}\,.$$

Note that

$$x_{j+1} - x'_{j+1} = \int_{x_{j+1}}^{x'_{j+1}} \frac{y'(x)}{|y'(x)|}\,dx = \int_0^{H_{j+1}} \frac{dy}{|y'|} \le \sqrt{\frac{k+1}{2m'_{j+1}}}\int_0^{H_{j+1}} \frac{dy}{\sqrt{H_{j+1}^{k+1} - y^{k+1}}}$$

and making the replacement $y = uH_{j+1}$ in the last integral, we obtain

$$x_{j+1} - x'_{j+1} \le \sqrt{\frac{k+1}{2m'_{j+1}}}\, H_{j+1}^{-\frac{k-1}{2}} \int\limits_0^1 \frac{du}{\sqrt{1-u^{k+1}}}\,.$$

Analogously, the inequality

$$x_{j+1} - x'_{j+1} \ge \sqrt{\frac{k+1}{2M'_{j+1}}}\, H_{j+1}^{-\frac{k-1}{2}} \int\limits_0^1 \frac{du}{\sqrt{1-u^{k+1}}}$$

holds. Due to the assumptions of the theorem, the function $p(x)$ has a finite positive limit p_+ as $x \to +\infty$ and we have proved that there exists a finite positive $\lim\limits_{j\to+\infty} H_j$. Thus, passing to the limit in last inequalities, we can conclude that the distance between the extremum point and zero $(x_{j+1} - x'_{j+1})$ has a finite positive limit as $j \to +\infty$. Analogously, both the distance $(x'_{j+1} - x_j)$ and hence their sum $(x_{j+1} - x_j)$ have finite positive limits as $j \to +\infty$. □

Remark 3.1. Note that the theorem assumption of a globally bounded variation for the function $p(x)$ is essential for the existence of finite positive limits $\lim\limits_{j\to\pm\infty} |y'(x_j)|$, $\lim\limits_{j\to\pm\infty} |y(x'_j)|$ and $\lim\limits_{j\to\pm\infty} (x_{j+1} - x_j)$. An example of a continuous function $p(x) > 0$ (satisfying inequalities (1.2) but not of a globally bounded variation) is given [10] such that there exists an unbounded proper solution $\lim\limits_{j\to+\infty} |y'(x_j)| = \lim\limits_{j\to+\infty} |y(x'_j)| = +\infty$. Also, an example of a continuous function $p(x) > 0$ (satisfying inequalities (1.2) but not of a globally bounded variation) is given [10] such that there exists a nontrivial proper oscillating solution tending at $+\infty$ to zero with its first derivative.

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