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ON THE EXISTENCE OF SOLUTIONS TO HIGHER-ORDER REGULAR NONLINEAR EMDEN–FOWLER TYPE EQUATIONS WITH GIVEN NUMBER OF ZEROS ON THE PRESCRIBED INTERVAL

Abstract. The existence of solutions with a given number of zeros to higher-order regular-nonlinear Emden–Fowler type equations is proven.[1]

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1 Introduction

Consider the equation

$$y^{(n)} + p(t, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y = 0, \tag{1.1}$$

where $n \geq 2$, $k \in (1, +\infty)$, the function $p(t, y_1, y_2, y_3, \dots, y_n) \in C(\mathbb{R}^{n+1})$ is Lipschitz continuous in $(y_1, y_2, y_3, \dots, y_n)$ and for some m, $M > 0$ satisfies the inequalities

$$0 < m \leq p(t, y_1, y_2, \dots, y_n) \leq M < +\infty.$$

The problem of the existence of solutions to (1.1) with the given number of zeros on the prescribed domain is investigated.

Asymptotic classification of solutions to (1.1) with $n = 3, 4$, $k \in (1, +\infty)$, $p(t, y, y', \dots, y^{(n-1)}) \equiv$ const and with $n = 3$, $k \in (0, 1)$, $p(t, y, y', \dots, y^{(n-1)}) \equiv const$ is provided in [1, 3] by I. Astashova. Later, the existence of quasiperiodic solutions to the regular ($k \in (1, +\infty)$) higher-order Emden–Fowler type equations has been proved in [2].

Using [1], the existence of solutions with the given number of zeros was proved for the case of third- and fourth-order equations with the constant coefficient p and with $k \in (0, 1) \cup (1, +\infty)$ (see [4]). Later, the case of the higher-order differential equation (1.1) with the constant potential and regular nonlinearity ($k > 1$) was considered in [5]. In [6], the existence of solutions with the given number of zeros was proved for (1.1) with $n = 3$, $k \in (1, +\infty)$. In [7], the existence of such solutions was proved for the equation with $k \in (0, 1)$.

Now we generalize these results to the case of equation (1.1).

2 Main result

Theorem 2.1. *For any real a and b satisfying $-\infty < a < b < +\infty$ and any integer $S \geq 2$, equation (1.1) has a solution defined on the segment $[a, b]$, vanishing at its end points a, b and having exactly S zeros on $[a, b]$.*

3 Preliminary results

The following statements are used to prove the main theorem.

Lemma 3.1 (Generalization of 7.1 from [1])**.** *Let $y(t)$ be a solution to (1.1). If for some t_0 the inequalities*

$$y(t_0) \geq 0,\ y'(t_0) > 0,\ y''(t_0) \geq 0,\ \dots,\ y^{(n-1)}(t_0) \geq 0$$

hold, then there is a local supremum of y at some point $t_0' > t_0$ satisfying the inequalities

$$t_0' - t_0 \leq (\mu y'(t_0))^{-\frac{k-1}{k+n-1}},$$
$$y(t_0') > (\mu y'(t_0))^{\frac{n}{k+n-1}},$$

where $\mu > 0$ is a constant depending only on n, k, m, M.

Lemma 3.2 (Generalization of 7.2 from [1])**.** *Let $y(t)$ be a solution to (1.1). If for some t_0' the inequalities*

$$y(t_0') > 0,\ y'(t_0') \leq 0,\ \dots,\ y^{(n-1)}(t_0') \leq 0$$

hold, then y is equal to zero at some point $t_0 > t_0'$ satisfying the inequalities

$$t_0 - t_0' \leq (\mu y(t_0'))^{-\frac{k-1}{n}},$$
$$y'(t_0) < -(\mu y(t_0'))^{\frac{k+n-1}{n}},$$

where $\mu > 0$ is a constant depending only on n, k, m, M.

Lemma 3.3 (Generalization of 7.3 from [1])**.** *Under the conditions of Lemmas* 3.1*,* 3.2*, for any* $t_1 > t_0$ *such that* $y(t_0) = 0$*,* $y(t_1) = 0$*, the inequality*

$$|y'(t_1)| > Q|y'(t_0)|$$

holds, where $Q > 1$ *is a constant depending only on* k*,* m*,* M*.*

Lemma 3.4. *Let* D *be a subset of* $\mathbb{R}^n$ *and* $\widetilde{D}$ *be a subset of* $\mathbb{R}^{n+1}$*. Suppose that for any* $c \in D$ *there exists* $x_c > 0$ *such that* $\{c\} \times [0, x_c] \subset \widetilde{D}$*. Consider a continuous function* $f(c, x) : \widetilde{D} \to \mathbb{R}$ *and introduce the following conditions:*

- $f(c, 0) = 0$ *for any* $c \in D$*,*
- *for every* $c \in D$*, there exists a point* $x_1(c) \in (0, x_c)$ *such that* $f(c, x_1(c)) = 0$ *and* $f(c, x) \neq 0$ *whenever* $x \in (0, x_1(c))$*,*
- $f(c, x)$ *is differentiable in* x*, and* $\frac{df}{dx}(c, x_1(c)) \neq 0$ *for all* $c \in D$*.*

If these conditions hold, then $x_1(c) : D \to \mathbb{R}$ *is a continuous function.*

Proof. By definition, $x_1(c)$ describes the distance from 0 to the first zero of the function $f(c, \cdot)$. The existence of such a zero is stated in the second condition of the lemma. Therefore $x_1(c)$ is actually a function (its value is defined for every $c \in D$), but, perhaps, discontinuous. We intend to prove that $x_1(c)$ is a continuous function.

At every point $(c, x_1(c)) \in \widetilde{D}$, the function $f(c, x)$ fulfills the conditions of the Implicit Function Theorem. Therefore for any $\widetilde{c} \in D$ there exist rectangular neighborhoods $U \subset D$ of $\widetilde{c}$, $V \subset \mathbb{R}$ of $x_1(\widetilde{c})$, and a continuous function $g_{\widetilde{c}}(c) : U \to V$ such that for all $(c, x) \in U \times V$ the conditions $f(c, x) = 0$ and $x = g_{\widetilde{c}}(c)$ are equivalent.

It is clear that $x_1(\widetilde{c}) = g_{\widetilde{c}}(\widetilde{c})$, but we have to prove that $x_1(c) \equiv g_{\widetilde{c}}(c)$ in some neighborhood of $\widetilde{c}$. (We know that $f(c, g_{\widetilde{c}}(c)) = 0$, but the zeros of $f(c, \cdot)$ provided by $g_{\widetilde{c}}(c)$ may not be the zeros closest to the point $x = 0$.)

We will prove this by contradiction. Suppose that in any punctured neighborhood of some point $c^* \in D$ there exists a point c such that $g_{c^*}(c) \neq x_1(c)$. Then we have an infinite set $\{c_\alpha\}$ such that for every c_α the inequality $g_{c^*}(c_\alpha) \neq x_1(c_\alpha)$ holds. We can extract from $\{c_\alpha\}$ a sequence $\{c_n\}$ tending to the point c^*. The implicit function theorem for $f(c, x)$ takes place in a neighborhood $U \times V$ of the point $(c^*, x_1(c^*))$.

Now we look closely at the set $\{(c_n, x_1(c_n))\}$. It is a sequence in $\widetilde{D}$, which cannot enter $U \times V$, because otherwise the condition $f(c_n, x_1(c_n)) = 0$ inside $U \times V$ contradicts the very definition of $\{c_n\}$. At the same time, the points $(c_n, x_1(c_n))$ cannot be above the graph of $g_{c^*}(c)$ and above $U \times V$ by the definition of the function $x_1(c)$.

So, the sequence $\{x_1(c_n)\}$ is bounded by zero from below and by $\inf V < x_1(c^*)$ from above. Hence $\{x_1(c_n)\}$ has a limit inferior $x^* < x_1(c^*)$. We extract a subsequence $\{x_1(c_{n_i})\}$ tending to the above limit and then consider a sequence $\{(c_{n_i}, x_1(c_{n_i}))\}$. The function $f(c, x)$ is continuous, $f(c_{n_i}, x_1(c_{n_i})) = 0$, and $(c_{n_i}, x_1(c_{n_i})) \to (c^*, x^*)$ as $i \to \infty$. Therefore, $f(c^*, x^*) = 0$. But at the same time we have $x^* < x_1(c^*)$, and this contradicts the conditions of the lemma. Therefore, the point c^*, in fact, does not exist.

This means that for every point $\widetilde{c} \in D$ the equality $x_1(c) \equiv g_{\widetilde{c}}(c)$ is true in some neighborhood of $\widetilde{c}$. Every function $g_{\widetilde{c}}(c)$ is continuous near $\widetilde{c}$. Therefore, $x_1(c)$ is continuous at every point $c \in D$. □

3.1 Proof of the main result

Proof of Theorem 2.1. Consider a maximally extended solution $y(t)$ to (1.1) with initial data $y^{(i)}(a) = y_i$, $i \in \overline{0, n-1}$.

It follows from Lemmas 3.1–3.3 that if the inequalities

$$y(t_0) \geq 0,\ y'(t_0) > 0,\ y''(t_0) \geq 0,\ \ldots,\ y^{(n-1)}(t_0) \geq 0$$

hold at some point t_0, then there exists a point $t_1 > t_0$ such that

$$y(t_1) = 0,\ y'(t_1) < 0,\ y''(t_1) \leq 0,\ \ldots,\ y^{(n-1)}(t_1) \leq 0$$

and

$$t_1 - t_0 \le (\mu' y'(t_0))^{-\frac{k-1}{k+2}},$$

where $\mu' > 0$ and $Q > 1$ are constants depending only on k, m, and M.

The analogous statement takes place if

$$y(t_0) \le 0,\ y'(t_0) < 0,\ y''(t_0) \le 0,\ \dots,\ y^{(n-1)}(t_0) \le 0.$$

Hence, if $y_0 = 0$ and $y_i > 0$ for $i \in \overline{1, n-1}$, then $y(t)$ is an oscillating solution, i.e., it has an infinite sequence of zeros $\{a, t_1, t_2, \dots\}$. In the sequel, $y_0 = 0$ and $y_i > 0$ for $i \in \overline{1, n-1}$.

We denote the distance between zeros by $L_i = t_i - t_{i-1}$. The distance from a to the $(S-1)$st zero is a function

$$L(y_1, y_2, \dots, y_{n-1}) = \sum_{j=1}^{S-1} L_j(y_1, y_2, \dots, y_{n-1}),$$

and its value depends on the initial data of the solution $y(t)$.

If $L(y_1, y_2, \dots, y_{n-1}) = b - a$, then the solution $y(t)$ has exactly S zeros on $[a, b]$. To prove the theorem we have to prove that for any b and a the last equation has at least one solution.

First, notice that L is a continuous function. If we rewrite (1.1) as a system of first-order ODEs, that system will satisfy the conditions of the continuous dependence on initial data theorem [8, §7, Theorem 6]. By $Y(t, a, y_0, y_1, y_2, \dots, y_{n-1})$ we denote a maximally extended solution to (1.1) with initial data $y^{(i)}(a) = y_i$, $i \in \overline{0, n-1}$. Therefore, $Y(t, a, y_0, y_1, y_2, \dots, y_{n-1})$ and n of its derivatives in t are continuous functions on their domains.

Are the conditions of Lemma 3.4 fulfilled? Put

$$D = \big\{(y_1, y_2, \dots, y_{n-1}) \mid \quad y_i > 0\big\} \subset \mathbb{R}^{n-1}.$$

For every such $(y_1, y_2, \dots, y_{n-1})$ we have already proved the existence of the first zero t_1, which satisfies $y'(t_1) \ne 0$. Further, there exists the second zero t_2, and for $\widetilde{D} \subset \mathbb{R}^n$ we take the area above $D \times \{0\}$ and under the graph of $t_2(y_1, y_2, \dots, y_{n-1})$. Obviously, $Y(a, a, y_0, y_1, y_2, \dots, y_{n-1}) = 0$, and $Y(t, a, y_0, y_1, y_2, \dots, y_{n-1})$ is defined on $\widetilde{D}$. (Here a is fixed and y_0 is equal to zero.)

The conditions of Lemma 3.4 are fulfilled, hence $t_1(y_1, y_2, \dots, y_{n-1})$, or L_1 is a continuous function on D. It is possible to prove by using Lemma 3.4 that all L_i, and therefore L are continuous. For L_2, for example, notice that $y(t_1(y_1, y_2, \dots, y_{n-1})), y'(t_1(y_1, y_2, \dots, y_{n-1})), \dots, y^{(n-1)}(t_1(y_1, y_2, \dots, y_{n-1}))$ are also continuous, because they are compositions of continuous functions $Y^{(i)}(\cdot, a, y_0, y_1, y_2, \dots, y_{n-1})$ and $t_1(y_1, \dots, y_{n-1})$.

Now we are to find an upper estimate of L. It is already proved that

$$L_1 \le (\mu' y_1)^{-\frac{k-1}{k+n-1}}.$$

It follows from Lemma 3.3 that

$$|y'(t_i)| \ge Q^i |y'(a)|.$$

Consider L_i. Since $-\frac{k-1}{k+n-1} < 0$, we have

$$L_i \le (\mu' Q^{i-1} y_1)^{-\frac{k-1}{k+n-1}} = (Q^{-\frac{k-1}{k+n-1}})^{i-1} (\mu' y_1)^{-\frac{k-1}{k+n-1}}.$$

Put $\widetilde{Q} = Q^{-\frac{k-1}{k+n-1}}$. Since $Q > 1$, $-\frac{k-1}{k+n-1} < 0$, and therefore $0 < \widetilde{Q} < 1$, the upper estimates of L_i form a decreasing geometric progression. Therefore,

$$L = L_1 + L_2 + \dots + L_{S-1} \le \frac{1 - \widetilde{Q}^S}{1 - \widetilde{Q}} (\mu' y'(a))^{-\frac{k-1}{k+n-1}} = c_1 y'(a)^{-\frac{k-1}{k+n-1}},$$

$$L < c_1 y'(a)^{-\frac{k-1}{k+n-1}}, \tag{3.1}$$

where c_1 is a constant depending on n, k, m, M, and S.

To get a lower estimate of L it is sufficient to make a lower estimation of L_1. Consider a point $t_0' \in [a, t_1]$ such that $y'(t_0') = 0$. On the segment $[t_0', t_1]$, the derivatives y', y'' are non-positive. Therefore,

$$Qy'(a) < |y'(t_1)| = |y'(t_1)| - |y'(t_0')| = \int\limits_{t_0'}^{t_1} |y''(\xi)|\,d\xi < |t_1 - t_0'| \max_{[t_0', t_1]} |y''|.$$

We must get an upper estimate of $\max\limits_{[t_0', t_1]} |y''|$. Notice the behaviour of the derivatives of $y(t)$ as t goes from a to t_1. On the segment $[a, t_1]$, the inequality $y(t) > 0$ holds. First, near a, every derivative, except $y^{(n)}$, is positive. It follows that $y^{(n-1)}$ is decreasing and after some point the inequality $y^{(n-1)} < 0$ holds, when $y^{(n)}$ is still negative. Hence, now $y^{(n-2)}$ starts to decrease, and we can repeat the same steps, until the solution y intersects the $0 - t$-axis, i.e., when we move t from a to t_1, the derivatives change their signs in order and higher-order derivatives change sign before low-order ones. Therefore, on $[t_0', t_1]$, the second derivative of the solution y is negative, because on the segment $[t_0', t_1]$ the first derivative $y'(t) < 0$.

Denote $|y|^k \operatorname{sgn} y$ by $|y|^k_{\pm}$. All initial data are positive, hence

$$\begin{aligned} 0 > y''(t) &= y_2 + y_3(t-a) + y_4\frac{(t-a)^2}{2!} + \cdots + y_{n-1}\frac{(t-a)^{n-3}}{(n-3)!} \\ &\qquad - \int\limits_a^t \cdots \int\limits_a^t p(t, y, \ldots, y^{(n-1)})|y|^k_{\pm}(d\,t)^{n-2} \\ &> -\int\limits_a^t \cdots \int\limits_a^t p(t, y, \ldots, y^{(n-1)})|y|^k_{\pm}(d\,t)^{n-2} > -M|t-a|^{n-2}\max_{[a,t_1]}|y|^k, \end{aligned}$$

whence

$$\max_{[t_0', t_1]} |y''| < M|t_1 - a|^{n-2} \max_{[a,t_1]} |y|^k.$$

Now we get an upper estimation of $\max\limits_{[a,t_1]} |y|^k$. The inequality $y(t) > 0$ holds on $[a, t_1]$, whence

$$\begin{aligned} y(t) &= y_1(t-a) + y_2\frac{(t-a)^2}{2!} + \cdots + y_{n-1}\frac{(t-a)^{n-1}}{(n-1)!} - \int\limits_a^t \cdots \int\limits_a^t p(\xi, y, \ldots, y^{(n-1)})|y|^k_{\pm}\ (d\xi)^n \\ &< y_1(t-a) + y_2\,\frac{(t-a)^2}{2!} + \cdots + y_{n-1}\,\frac{(t-a)^{n-1}}{(n-1)!}\,. \end{aligned}$$

Therefore,

$$\max_{[a,t_1]} |y(t)|^k < \Big(y_1(t_1 - a) + \cdots + y_{n-1}\,\frac{(t_1-a)^{n-1}}{(n-1)!}\Big)^k.$$

Combining both estimates, we get

$$Qy_1 < M|t_1 - t_0'||t_1 - a|^{n-2}\Big(y_1(t_1 - a) + \cdots + y_{n-1}\,\frac{(t_1-a)^{n-1}}{(n-1)!}\Big)^k.$$

By definition, $t_1 - a = L_1$ and $|t_1 - t_0'| < L_1$, hence

$$Qy_1 < ML_1^{n-1}\Big(y_1 L_1 + \cdots + y_{n-1}\,\frac{L_1^{n-1}}{(n-1)!}\Big)^k.$$

Suppose $y_1 = y_2 = \cdots = y_{n-1}$ and y_1 is a variable. In this case,

$$ML_1^{n-1}\Big(L_1 + \cdots + \frac{L_1^{n-1}}{(n-1)!}\Big)^k > Qy_1^{1-k}.$$

In the left-hand side of the inequality we have the function of L_1 which is defined for every $L_1 > 0$, is equal to zero when $L_1 = 0$, and is monotonically increasing. The value of the right-hand side may be arbitrarily large as y_1 is arbitrarily small. Hence, for any $\lambda > 0$, we can choose initial data providing $L > \lambda$.

But, due to (3.1), for any $\lambda > 0$ we can choose initial data providing $0 < L < \lambda$. Therefore, the value of $L(y_1, y_2, \dots, y_{n-1})$ may be arbitrarily large, arbitrarily small, and, at the same time, $L(y_1, y_2, \dots, y_{n-1})$ is proven to be continuous. Thus, we conclude that the range of values of $L(y_1, y_2, \dots, y_{n-1})$ is $(0, +\infty)$. Therefore, the equation

$$L(y_1, y_2, \dots, y_{n-1}) = b - a$$

can be resolved for any $b > a$. This proves the theorem. $\square$

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