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EXISTENCE RESULTS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

Abstract. We discuss the existence of solutions to the implicit fractional differential equation ${}^c\!D^\alpha u = f(t, u, u', {}^c\!D^\beta u, {}^c\!D^\alpha u)$ satisfying nonlocal boundary conditions. Here $1 < \beta < \alpha \leq 2$, f is continuous and ${}^c\!D$ is the Caputo fractional derivative. The existence results are proved by the Leray–Schauder degree method.*

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რეზიუმე. შესწავლილია არალოკალური სასაზღვრო ამოცანის ამოხსნადობის საკითხი არაცხადი წილად-წარმოებულიანი ${}^c\!D^\alpha u = f(t, u, u', {}^c\!D^\beta u, {}^c\!D^\alpha u)$ განტოლებისთვის, სადაც $1 < \beta < \alpha \leq 2$, f უწყვეტია და ${}^c\!D$ კაპუტოს წილადი წარმოებულია. ამონახსნის არსებობა დადგენილია ლერე-შაუდერის ხარისხოვანი მეთოდის გამოყენებით.

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1 Introduction

Let $T > 0$ be given, $J = [0, T]$, $\|x\| = \max\{|x(t)| : t \in J\}$ be the norm in $C(J)$, while $\|x\|_1 = \|x\| + \|x'\|$ is the norm in $C^1(J)$.

In accordance with [12, 13], let $\mathcal{M}$ be the set of (generally nonlinear) functionals $\phi : C(J) \to \mathbb{R}$ which are

(i) continuous, $\phi(0) = 0$,

(ii) increasing, that is, $x, y \in C(J)$, $x(t) < y(t)$ for $t \in J \Longrightarrow \phi(x) < \phi(y)$.

Examples of functionals belonging to the set $\mathcal{M}$ were given in [12, 13].

We are interested in the implicit fractional differential equation

$$ {}^c\!D^\alpha u(t) = f\big(t, u(t), u'(t), {}^c\!D^\beta u(t), {}^c\!D^\alpha u(t)\big), \tag{1.1}$$

where $1 < \beta < \alpha \le 2$, $f \in C(J \times \mathbb{R}^4)$ and ${}^c\!D$ denotes the Caputo fractional derivative. Further conditions on f will be specified later.

Together with (1.1), we consider the nonlocal boundary condition

$$ u(0) = u(T), \quad \phi(u) = 0, \ \ \phi \in \mathcal{M}. \tag{1.2}$$

Example 1.1. The special cases of (1.2) are the boundary conditions:

$x(0) = 0$, $x(T) = 0$;

$x(0) = -x(\xi) = x(T)$, where $\xi \in (0, T)$;

$x(0) = x(T)$, $\min\{x(t) : t \in J\} = 0$;

$x(0) = x(T) = -\max\{x(t) : t \in J\}$.

Definition 1.1. We say that $u : J \to \mathbb{R}$ is *a solution of equation* (1.1) if $u', {}^c\!D^\alpha u \in C(J)$ and u satisfies (1.1) for $t \in J$. A solution u of (1.1) satisfying condition (1.2) is called *a solution of problem* (1.1), (1.2).

If $x, {}^c\!D^\alpha x \in C(J)$, then it is not difficult to verify that ${}^c\!D^\beta x(t) = I^{\alpha-\beta}\, {}^c\!D^\alpha x(t)$ for $t \in J$. Hence, if u is a solution of equation (1.1), then the equality

$$ {}^c\!D^\alpha u(t) = f\big(t, u(t), u'(t), I^{\alpha-\beta}\, {}^c\!D^\alpha u(t), {}^c\!D^\alpha u(t)\big), \ \ t \in J,$$

holds, that is, $w = {}^c\!D^\alpha u$ satisfies the equality

$$ w(t) = f\big(t, u(t), u'(t), I^{\alpha-\beta} w(t), w(t)\big) \ \text{ for } t \in J. \tag{1.3}$$

The special case of equation (1.1) (for $\alpha = 2$, $a \in C(J)$, $f(t, x, y, v, z) = a(t)v + f_1(t, x, y, z)$) is the implicit generalized Bagley–Torvik fractional differential equation

$$ u''(t) = a(t)\, {}^c\!D^\beta u(t) + f\big(t, u(t), u'(t), u''(t)\big). \tag{1.4}$$

For more details on the generalized Bagley–Torvik fractional differential equation one can see [13–15] and the references therein.

We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative [8, 9, 11].

The Riemann–Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$ I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\, \mathrm{d}s,$$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative ${}^c\!D^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ is given as

$$ {}^c\!D^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \Big) \mathrm{d}s, $$

where $n = [\gamma] + 1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma \in \mathbb{N}$, then ${}^c\!D^\gamma x(t) = x^{(\gamma)}(t)$.

In particular,

$$ \begin{aligned} {}^c\!D^\gamma x(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t \frac{(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} \left(x(s) - x(0) - x'(0)s \right) \mathrm{d}s \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{2-\gamma} (x(t) - x(0) - x'(0)t), \quad \gamma \in (1,2). \end{aligned} $$

It is well known that $I^\gamma : C(J) \to C(J)$ for $\gamma \in (0,1)$; $I^\gamma I^\mu x(t) = I^{\gamma+\mu} x(t)$ for $x \in C(J)$ and $\gamma, \mu \in (0,\infty)$; ${}^c\!D^\gamma I^\gamma x(t) = x(t)$ for $x \in C(J)$ and $\gamma > 0$; if $x, {}^c\!D^\gamma x \in C(J)$ and $\gamma \in (0,1)$, then $I^\gamma\, {}^c\!D^\gamma x(t) = x(t) - x(0)$.

The boundary value problems for implicit fractional differential equations were considered in the papers [1, 2, 4–6, 10] and the references therein. For instance, the problem

$$ \begin{gathered} {}^c\!D^\alpha u(t) = f\big(t, u(t), {}^c\!D^\alpha u(t)\big), \quad \alpha \in (0,1], \\ \sum_{k=1}^{n} a_k u(t_k) = u_0 \end{gathered} $$

was discussed in [6], while the problem

$$ \begin{gathered} {}^c\!D^\alpha u(t) = f\big(t, u(t), {}^c\!D^\alpha u(t)\big), \quad \alpha \in (1,2], \\ u(0) = u_0, \quad u(T) = u_1 \end{gathered} $$

was considered in [4].

The aim of this paper is to discuss the existence of solutions to problem (1.1), (1.2). The existence result is proved by the following procedure. We first show that for each $x \in C^1(J)$ there exists a unique solution $w \in C(J)$ of the equation $w = f(t, x(t), x'(t), I^{\alpha-\beta} w, w)$. Then we put $w = \mathcal{F}x$ and obtain an operator $\mathcal{F} : C^1(J) \to C(J)$ and prove that if u is a solution of the problem ${}^c\!D^\alpha u = \mathcal{F}u$, (1.2), then u is a solution of problem (1.1), (1.2). In order to prove that this problem has a solution, we introduce an operator $\mathcal{Q} : C^1(J) \times \mathbb{R} \to C^1(J) \times \mathbb{R}$ having the property that if (u, c) is its fixed point, then u is a solution of problem ${}^c\!D^\alpha u = \mathcal{F}u$, (1.2). The existence of a fixed point of $\mathcal{Q}$ is proved by the Leray–Schauder degree method [7].

We work with the following conditions on the function f in (1.1).

(H_1) There exist positive constants L_1 and L_2 such that

$$ \Delta = \frac{L_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + L_2 < 1 $$

and the estimate

$$ \big| f(t,x,y,v_1,z_1) - f(t,x,y,v_2,z_2) \big| \le L_1 |v_1 - v_2| + L_2 |z_1 - z_2| $$

is fulfilled for $t \in J$ and $x, y, v_j, z_j \in \mathbb{R}$.

(H_2) There exist $\rho, \mu \in (0,1)$ and $A > 0$ such that

$$ |f(t,x,y,0,0)| \le A\big(1 + |x|^\rho + |y|^\mu\big) \ \text{ for } t \in J,\ x, y \in \mathbb{R}. $$

(H_3) There exist positive constants A, B and C such that

$$|f(t,x,y,0,0)| \le A + B|x| + C|y| \text{ for } t \in J,\ x, y \in \mathbb{R}.$$

The paper is organized as follows. In Section 2, an operator $\mathcal{F}$ is introduced and its properties are given. In Section 3, the operators $\mathcal{Q}, \mathcal{K}$ and $\mathcal{H}$ are defined and their properties are stated. The main existence results for problem (1.1), (1.2) are given and proved in Section 4. Examples demonstrate our results.

2 Operator $\mathcal{F}$ and its properties

Keeping in mind (1.3), we need the following result.

Lemma 2.1. *Let (H_1) hold and let $x \in C^1(J)$. Then there exists a unique solution w of the equation*

$$w = f\big(t, x(t), x'(t), I^{\alpha-\beta}w, w\big) \tag{2.1}$$

in the set $C(J)$.

Proof. Let an operator $\mathcal{S} : C(J) \to C(J)$ be defined as

$$(\mathcal{S}w)(t) = f\big(t, x(t), x'(t), I^{\alpha-\beta}w(t), w(t)\big).$$

We show that $\mathcal{S}$ is a contractive operator. To this end, let $w_1, w_2 \in C(J)$. Then

$$\begin{aligned}
\big|(\mathcal{S}w_1)(t) - (\mathcal{S}w_2)(t)\big| &= \Big|f\big(t, x(t), x'(t), I^{\alpha-\beta}w_1(t), w_1(t)\big) - f\big(t, x(t), x'(t), I^{\alpha-\beta}w_2(t), w_2(t)\big)\Big| \\
&\le L_1\big|I^{\alpha-\beta}(w_1(t) - w_2(t))\big| + L_2|w_1(t) - w_2(t)| \\
&\le \frac{L_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\,\|w_1 - w_2\| + L_2\|w_1 - w_2\| \\
&\le \Delta\|w_1 - w_2\|, \quad t \in J.
\end{aligned}$$

In particular,

$$\|\mathcal{S}w_1 - \mathcal{S}w_2\| \le \Delta\|w_1 - w_2\|.$$

Due to $\Delta < 1$, the operator $\mathcal{S}$ is contractive and therefore there exists a unique fixed point w of $\mathcal{S}$. It is clear that w is a unique solution of (2.1) in $C(J)$. ☐

By Lemma 2.1, for each $x \in C^1(J)$ there exists a unique solution $w \in C(J)$ of equation (2.1). We put $w = \mathcal{F}x$ and obtain an operator $\mathcal{F} : C^1(J) \to C(T)$ satisfying

$$(\mathcal{F}x)(t) = f\big(t, x(t), x'(t), I^{\alpha-\beta}(\mathcal{F}x)(t), (\mathcal{F}x)(t)\big) \text{ for } t \in J \text{ and } x \in C^1(J). \tag{2.2}$$

The properties of $\mathcal{F}$ are collected in the following result.

Lemma 2.2. *Let (H_1) hold. Then $\mathcal{F} : C^1(J) \to C(J)$ is a continuous operator and*

$$\|\mathcal{F}x\| \le \frac{1}{1-\Delta}\max\big\{|f(t, x(t), x'(t), 0, 0)| :\ t \in J\big\}, \quad x \in C^1(J). \tag{2.3}$$

Proof. Let $\{x_n\} \subset C^1(J)$ be a convergent sequence and let $x \in C^1(J)$ be its limit. Let (for $t \in J$, $n \in \mathbb{N}$)

$$d_n(t) = f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)\big) - f\big(t, x(t), x'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)\big).$$

Then $\lim_{n\to\infty} \|d_n\| = 0$. It follows from the relation (see (2.2))

$$\begin{aligned}
\big|\mathcal{F}x_n(t) - \mathcal{F}x(t)\big| &\le \Big|f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x_n(t), \mathcal{F}x_n(t)\big) - f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x_n(t)\big)\Big| \\
&\quad + \Big|f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x_n(t)\big) - f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)\big)\Big| \\
&\qquad + |d_n(t)| \\
&\le L_1\big|I^{\alpha-\beta}(\mathcal{F}x_n(t) - \mathcal{F}x(t))\big| + L_2\big|\mathcal{F}x_n(t) - \mathcal{F}x(t)\big| + |d_n(t)| \\
&\le \Big(\frac{L_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + L_2\Big)\|\mathcal{F}x_n - \mathcal{F}x\| + \|d_n\|, \;\; t \in J, \; n \in \mathbb{N},
\end{aligned}$$

that

$$\|\mathcal{F}x_n - \mathcal{F}x\| \le \Delta\|\mathcal{F}x_n - \mathcal{F}x\| + \|d_n\|, \;\; n \in \mathbb{N}.$$

Therefore

$$\|\mathcal{F}x_n - \mathcal{F}x\| \le \frac{\|d_n\|}{1-\Delta}, \;\; n \in \mathbb{N},$$

and so $\lim_{n\to\infty} \|\mathcal{F}x_n - \mathcal{F}x\| = 0$. Hence $\mathcal{F}$ is continuous.

It remains to prove that estimate (2.3) is valid. Let $x \in C^1(J)$. Then (2.2) and (H_1) give

$$\begin{aligned}
|\mathcal{F}x(t)| &\le \big|f(t, x(t), x'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)) - f(t, x(t), x'(t), 0, \mathcal{F}x(t))\big| \\
&\quad + \big|f(t, x(t), x'(t), 0, \mathcal{F}x(t) - f(t, x(t), x'(t), 0, 0)| + |f(t, x(t), x'(t), 0, 0)\big| \\
&\le L_1\big|I^{\alpha-\beta}\mathcal{F}x(t)\big| + L_2|\mathcal{F}x(t)| + |f(t, x(t), x'(t), 0, 0)| \\
&\le \Delta\|\mathcal{F}x\| + |f(t, x(t), x'(t), 0, 0)|, \;\; t \in J.
\end{aligned}$$

In particular,

$$\|\mathcal{F}x\| \le \Delta\|\mathcal{F}x\| + \max\big\{|f(t, x(t), x'(t), 0, 0)| : \; t \in J\big\}$$

and (2.3) follows. □

3 Auxiliary results

We investigate the fractional differential equation

$$^{c}D^{\alpha}u(t) = (\mathcal{F}u)(t). \tag{3.1}$$

The following result gives the relation between the solutions of problems (3.1), (1.2) and (1.1), (1.2).

Lemma 3.1. *Let (H_1) hold. If u is a solution of problem* (3.1), (1.2)*, then u is a solution of problem* (1.1), (1.2).

Proof. Let u be a solution of problem (3.1), (1.2). In view of (2.2), we see that

$$^{c}D^{\alpha}u(t) = f\big(t, u(t), u'(t), I^{\alpha-\beta}\,{}^{c}D^{\alpha}u(t), {}^{c}D^{\alpha}u(t)\big) \;\text{ for } t \in J.$$

Hence u is a solution of equation (1.1), because $I^{\alpha-\beta}\,{}^{c}D^{\alpha}u = {}^{c}D^{\beta}u$. Since u satisfies the boundary condition (1.2), u is a solution of problem (1.1), (1.2). □

In order to prove that problem (3.1), (1.2) has a solution, we introduce an operator $\mathcal{Q} : C^1(J) \times \mathbb{R} \to C^1(J) \times \mathbb{R}$ by the formula

$$Q(x, c) = \Big(c + I^{\alpha}(\mathcal{F}x)(t) - \frac{t}{T}\, I^{\alpha}(\mathcal{F}x)(t)\Big|_{t=T}, \; c + \phi(x)\Big),$$

where ϕ is from (1.2).

Lemma 3.2. *Let* (H_1) *hold. If* (x, c) *is a fixed point of the operator* $\mathcal{Q}$*, then* x *is a solution of problem* $(3.1), (1.2)$ *and* $c = x(0)$*.*

Proof. Let (x, c) be a fixed point of $\mathcal{Q}$. Then

$$x(t) = c + I^\alpha(\mathcal{F}x)(t) - \frac{t}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}, \quad t \in J, \tag{3.2}$$

$$\phi(x) = 0. \tag{3.3}$$

It follows from (3.2) that $x(0) = c$, $x(T) = c$, $x \in C^1(J)$ and

$$^{c}\!D^\alpha x(t) = {}^{c}\!D^\alpha I^\alpha(\mathcal{F}x)(t) = (\mathcal{F}x)(t), \quad t \in J.$$

These facts together with (3.3) imply that x is a solution of (3.1), (1.2) and $c = x(0)$, □

Lemmas 3.1 and 3.2 show that for the solvability of problem $(1.1), (1.2)$ we need to prove that the operator $\mathcal{Q}$ admits a fixed point. Really, if (x, c) is a fixed point of $\mathcal{Q}$, then x is a solution of (1.1), (1.2). To this end, we first define an operator $\mathcal{K} : C^1(J) \times \mathbb{R} \times [0, 1] \to C^1(J) \times \mathbb{R}$ as

$$\mathcal{K}(x, c, \lambda) = \big(c, c + \phi(x) + (\lambda - 1)\phi(-x)\big).$$

Let

$$\Omega_1 = \big\{(x, c) \in C^1(J) \times \mathbb{R} : \ \|x\|_1 < M, \ |c| < M\big\}.$$

where M is a positive constant.

Lemma 3.3. *The relation*

$$\deg(\mathcal{I} - \mathcal{K}(\,\cdot\,, \cdot\,, 1), \Omega_1, 0) \neq 0$$

*is valid, where "*deg*" stands for the Leray–Schauder degree and* $\mathcal{I}$ *is the identical operator on* $C^1(J) \times \mathbb{R}$*.*

Proof. It is not difficult to show that $\mathcal{K}$ is a completely continuous operator and since

$$\mathcal{K}(-x, -c, 0) = (-c, -c + \phi(-x) - \phi(x)) = -(c, c + \phi(x) - \phi(-x)) = -\mathcal{K}(x, c, 0)$$

for $x \in C^1(J)$ and $c \in \mathbb{R}$, $\mathcal{K}(\,\cdot\,, \cdot\,, 0)$ is an odd operator.

Assume that $\mathcal{K}(x, c, \lambda) = (x, c)$ for some $(x, c) \in C^1(J) \times \mathbb{R}$ and $\lambda \in [0, 1]$. Then

$$x(t) = c, \quad t \in J, \tag{3.4}$$

$$\phi(x) + (\lambda - 1)\phi(-x) = 0. \tag{3.5}$$

In view of (3.4), it follows from (3.5) that $\phi(c) + (\lambda - 1)\phi(-c) = 0$. If $c \neq 0$, then properties (i) and (ii) of $\phi \in \mathcal{M}$ give $\phi(c)\phi(-c) < 0$, which contradicts $\phi(c) + (\lambda - 1)\phi(-c) = 0$. Hence $c = 0$, and so $x = 0$. We have proved that $\mathcal{K}(x, c, \lambda) \neq (x, c)$ for $(x, c) \in \partial\Omega_1$ and $\lambda \in [0, 1]$. By the Borsuk antipodal theorem and the homotopy property,

$$\deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,, \cdot\,, 0), \Omega_1, 0\big) \neq 0,$$

$$\deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,, \cdot\,, 0), \Omega_1, 0\big) = \deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,, \cdot\,, 1), \Omega_1, 0\big).$$

Combining these relations we give the conclusion of Lemma 3.3. □

Finally, let an operator $\mathcal{H} : C^1(J) \times \mathbb{R} \times [0, 1] \to C^1(J) \times \mathbb{R}$ be defined as

$$\mathcal{H}(x, c, \lambda) = (\mathcal{H}_1(x, c, \lambda), \mathcal{H}_2(x, c)),$$

where $\mathcal{H}_1 : C^1(J) \times \mathbb{R} \times [0, 1] \to C^1(J)$, $\mathcal{H}_2(x, c) : C^1(J) \times \mathbb{R} \to \mathbb{R}$,

$$\mathcal{H}_1(x, c, \lambda)(t) = c + \lambda\Big(I^\alpha(\mathcal{F}x)(t) - \frac{t}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big),$$

$$\mathcal{H}_2(x, c) = c + \phi(x).$$

It is clear that

$$\mathcal{H}(x, c, 0) = \mathcal{K}(x, c, 1), \quad \mathcal{H}(x, c, 1) = \mathcal{Q}(x, c) \tag{3.6}$$

for $(x, c) \in C^1(J) \times \mathbb{R}$.

The following result states that $\mathcal{H}$ is completely continuous.

Lemma 3.4. *Let (H_1) hold. Then $\mathcal{H}$ is a completely continuous operator.*

Proof. Step 1. $\mathcal{H}$ is continuous.

Let $\{x_n\} \subset C^1(J)$, $\{c_n\} \subset \mathbb{R}$, $\{\lambda_n\} \subset [0,1]$ be convergent sequences and let $\lim_{n\to\infty} \|x_n - x\|_1 = 0$, $\lim_{n\to\infty} c_n = c$, $\lim_{n\to\infty} \lambda_n = \lambda$, where $x \in C^1(J)$, $c \in \mathbb{R}$, $\lambda \in [0,1]$.

By Lemma 2.2, $\lim_{n\to\infty} \|\mathcal{F}x_n - \mathcal{F}x\| = 0$. Since

$$\Big| I^\alpha(\mathcal{F}x_n)(t) - \frac{t}{T}\, I^\alpha(\mathcal{F}x_n)(t)\Big|_{t=T} - I^\alpha(\mathcal{F}x)(t) + \frac{t}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big|$$
$$\le \|\mathcal{F}x_n - \mathcal{F}x\| \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \Big)\,\mathrm{d}s \le \frac{2T^\alpha}{\Gamma(\alpha+1)}\, \|\mathcal{F}x_n - \mathcal{F}x\|$$

and

$$\Big| I^{\alpha-1}(\mathcal{F}x_n)(t) - \frac{1}{T}\, I^\alpha(\mathcal{F}x_n)(t)\Big|_{t=T} - I^{\alpha-1}(\mathcal{F}x)(t) + \frac{1}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big|$$
$$\le \|\mathcal{F}x_n - \mathcal{F}x\| \Big(\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s + \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \Big)\,\mathrm{d}s \le \frac{T^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \frac{1}{\alpha}\Big) \|\mathcal{F}x_n - \mathcal{F}x\|,$$

it is easy to verify that $\lim_{n\to\infty} \|\mathcal{H}_1(x_n, c_n, \lambda_n) - \mathcal{H}_1(x, c, \lambda)\|_1 = 0$. This fact together with $\lim_{n\to\infty} \mathcal{H}_2(x_n, c_n) = \mathcal{H}_2(x, c)$ gives $\lim_{n\to\infty} \mathcal{H}(x_n, c_n, \lambda_n) = \mathcal{H}(x, c, \lambda)$ in $C^1(J) \times \mathbb{R}$. Hence $\mathcal{H}$ is continuous.

Step 2. $\mathcal{H}$ takes bounded sets into bounded sets.

Let $\mathcal{U} \subset C^1(J)$ and $\mathcal{V} \subset \mathbb{R}$ be bounded, $\|x\|_1 \le V$ for $x \in \mathcal{U}$, $|c| \le V$ for $c \in \mathcal{V}$, where V is a positive constant. Then $M_1 = \sup\{|f(t, x(t), x'(t), 0, 0)| : t \in J,\ x \in \mathcal{U}\} < \infty$. In view of (2.3), we have $\|\mathcal{F}x\| \le M$ for $x \in \mathcal{U}$, where $M = M_1/(1-\Delta)$. Hence (for $u \in \mathcal{U}$, $c \in \mathcal{V}$, $\lambda \in [0,1]$, $t \in J$)

$$|\mathcal{H}_1(x, c, \lambda)(t)| \le V + M\Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s \Big) \le V + \frac{2MT^\alpha}{\Gamma(\alpha+1)}\,,$$
$$\Big| \frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{H}_1(x, c, \lambda)(t) \Big| \le M\Big(\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s + \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s \Big) \le \frac{MT^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \frac{1}{\alpha}\Big),$$

and therefore

$$\|\mathcal{H}_1(x, c, \lambda)\|_1 \le V + \frac{MT^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \frac{2T+1}{\alpha}\Big). \tag{3.7}$$

Due to the properties (i) and (ii) of ϕ and $-V \le x(t) \le V$ for $t \in J$, $x \in \mathcal{U}$, we see that $\phi(-V) \le \phi(x) \le \phi(V)$, and therefore

$$|\mathcal{H}_2(x, c)| = |c + \phi(x)| \le W \ \text{ for } u \in \mathcal{U},\ c \in \mathcal{V}, \tag{3.8}$$

where $W = V + \max\{|\phi(-V)|, \phi(V)\}$.

From (3.7) and (3.8) we conclude that $\mathcal{H}$ maps $\mathcal{U} \times \mathcal{V} \times [0,1]$ into a bounded set in $C^1(J) \times \mathbb{R}$.

Step 3. For each bounded $\mathcal{U} \subset C^1(J)$ the family $\{I^{\alpha-1}(\mathcal{F}x) : x \in \mathcal{U}\}$ is equicontinuous on J.

Let $\mathcal{U}$ be a bounded set in $C^1(J)$. As in Step 2, $\|\mathcal{F}x\| \le M$ for $x \in \mathcal{U}$, where $M > 0$. Let $x \in \mathcal{U}$

and $0 \le t_1 < t_2 \le T$. Then

$$\Big|I^{\alpha-1}(\mathcal{F}x)(t)\big|_{t=t_2} - I^{\alpha-1}(\mathcal{F}x)(t)\big|_{t=t_1}\Big| = \Big|\int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s - \int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s\Big|$$

$$= \Big|\int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s + \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s\Big|$$

$$\le M\Big(\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s + \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s\Big)$$

$$= \frac{M}{\Gamma(\alpha)}\big(t_1^{\alpha-1} + 2(t_2-t_1)^{\alpha-1} - t_2^{\alpha-1}\big) < \frac{2M}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}.$$

Since $t^{\alpha-1}$ is a continuous function on J, we see that the family $\{I^{\alpha-1}(\mathcal{F}x):\ x\in\mathcal{U}\}$ is equicontinuous on J.

To summarize, $\mathcal{H}$ is continuous by Step 1 and it follows from Steps 2 and 3 and the Arzelà-Ascoli theorem that $\mathcal{H}_1$ is relatively compact in $C^1(J)$. Besides, (3.8) implies that $\mathcal{H}_2$ is relatively compact in $\mathbb{R}$. Consequently, $\mathcal{H}$ is completely continuous. □

The following two results give bounds for fixed points of $\mathcal{H}$.

Lemma 3.5. *Let (H_1) and (H_2) hold. Then there exists $S>0$ such that the estimate*

$$\|x\|_1 < S,\ \ |c| < S, \tag{3.9}$$

holds for fixed points (x,c) of the operator $\mathcal{H}(\,\cdot\,,\,\cdot\,,\lambda)$ with $\lambda\in[0,1]$.

Proof. Let $\mathcal{H}(x,c,\lambda)=(x,c)$ for some $(x,c)\in C^1(J)\times\mathbb{R}$ and $\lambda\in[0,1]$. Then

$$x(t) = c + \lambda\Big(I^\alpha(\mathcal{F}x)(t) - \frac{t}{T}I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big),\ \ t\in J, \tag{3.10}$$

$$\phi(x) = 0. \tag{3.11}$$

By (H_2),

$$\big|f(t,x(t),x'(t),0,0)\big| \le A\big(1+|x(t)|^\rho + |x'(t)|^\mu\big) \le A\big(1+\|x\|_1^\rho + \|x\|_1^\mu\big),\ \ t\in J,$$

and therefore (see (2.3)))

$$\|\mathcal{F}x\| \le \frac{A(1+\|x\|_1^\rho+\|x\|_1^\mu)}{1-\Delta}. \tag{3.12}$$

Due to (3.11), we have $x(\xi)=0$ for some $\xi\in J$ [12]. Hence (3.10) gives

$$c = -\lambda\Big(I^\alpha(\mathcal{F}x)(t)\Big|_{t=\xi} - \frac{\xi}{T}I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big),$$

and therefore

$$x(t) = \lambda\Big(I^\alpha(\mathcal{F}x)(t) - I^\alpha(\mathcal{F}x)(t)\Big|_{t=\xi} - \frac{t-\xi}{T}I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big),\ \ t\in J.$$

Then

$$|x(t)| \le \|\mathcal{F}x\|\Big(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \int_0^\xi\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s\Big) \le \frac{3T^\alpha}{\Gamma(\alpha+1)}\|\mathcal{F}x\|,$$

$$|x'(t)| \le \|\mathcal{F}x\|\Big(\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s + \frac{1}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s\Big) \le \frac{T^{\alpha-1}}{\Gamma(\alpha)}\Big(1+\frac{1}{\alpha}\Big)\|\mathcal{F}x\|,\ \ t\in J.$$

In particular,

$$\|x\| \le \frac{3T^\alpha}{\Gamma(\alpha+1)}\|\mathcal{F}x\|, \quad \|x'\| \le \frac{T^{\alpha-1}}{\Gamma(\alpha)}\Big(1+\frac{1}{\alpha}\Big)\|\mathcal{F}x\|.$$

Hence

$$\|x\|_1 \le \frac{T^{\alpha-1}}{\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big)\|\mathcal{F}x\| \tag{3.13}$$

and (see (3.12)))

$$\|x\|_1 \le K\big(1+\|x\|_1^\rho+\|x\|_1^\mu\big), \tag{3.14}$$

where

$$K = \frac{AT^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big).$$

Since (note that $\rho,\mu \in (0,1)$) $\lim\limits_{v\to\infty}\frac{v}{K(1+v^\rho+v^\mu)} = \infty$, there exists $S>0$ such that

$$v > K(1+v^\rho+v^\mu) \ \text{ for all } v \ge S.$$

The last inequality together with (3.14) gives $\|x\|_1 < S$. In view of $c = x(0)$, we get $|c| < S$. Since S is independent of x, c, λ, estimate (3.9) follows. □

Lemma 3.6. *Let* (H_1) *and* (H_3) *hold and let*

$$W = \frac{(B+C)T^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big) < 1.$$

Then the estimate

$$\|x\|_1 \le S_1, \ \ |c| \le S_1,$$

holds for fixed points (x,c) *of the operator* $\mathcal{H}(\,\cdot\,,\,\cdot\,,\lambda)$ *with* $\lambda\in[0,1]$, *where*

$$S_1 = \frac{AT^{\alpha-1}}{(1-\Delta)(1-W)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big).$$

Proof. Let $\mathcal{H}(x,c,\lambda) = (x,c)$ for some $(x,c)\in C^1(J)\times\mathbb{R}$ and $\lambda\in[0,1]$. Analysis similar to that in the proof of Lemma 3.5 shows that $c = x(\xi)$ for some $\xi\in J$ and estimate (3.13) is valid. From (H_3) and (2.3) we have

$$\|\mathcal{F}x\| \le \frac{A+B\|x\|+C\|x'\|}{1-\Delta} \le \frac{A+(B+C)\|x\|_1}{1-\Delta}\,,$$

and therefore

$$\|x\|_1 \le \frac{T^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big)\big(A+(B+C)\|x\|_1\big) = \frac{AT^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big)+W\|x\|_1.$$

Hence

$$(1-W)\|x\|_1 \le \frac{AT^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big),$$

which implies $\|x\|_1 \le S_1$ and $|c| \le S_1$ because $c = x(0)$. □

4 The main results and examples

Theorem 4.1. *Let* (H_1) *and* (H_2) *hold. Then problem* (1.1), (1.2) *has at least one solution.*

Proof. Let $S>0$ be from Lemma 3.5 and let

$$\Omega = \big\{(x,c)\in C^1(J)\times\mathbb{R}:\ \|x\|_1 < S,\ |c| < S\big\}.$$

By Lemma 3.4, the restriction of $\mathcal{H}$ to $\overline{\Omega} \times [0,1]$ is a compact operator and Lemma 3.5 shows that $\mathcal{H}(x,c,\lambda) \neq (x,c)$ for $(x,c) \in \partial\Omega$ and $\lambda \in [0,1]$. Hence it follows from the homotopy property that

$$\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,0),\Omega,0\big) = \deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big).$$

In view of (3.6) and Lemma 3.3 (for $M = S$ in Ω_1), we have

$$\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,0),\Omega,0\big) = \deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big) \neq 0,$$
$$\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big) = \deg\big(\mathcal{I} - \mathcal{Q}(\,\cdot\,,\,\cdot\,),\Omega,0\big),$$

and so

$$\deg\big(\mathcal{I} - \mathcal{Q}(\,\cdot\,,\,\cdot\,),\Omega,0\big) \neq 0. \tag{4.1}$$

Consequently, there exists a fixed point (u,c) of $\mathcal{Q}$ and, by Lemmas 3.1 and 3.2, u is a solution of problem (1.1), (1.2). □

Theorem 4.2. *Let* (H_1) *and* (H_3) *hold and let* $W < 1$, *where* W *is from Lemma* 3.6. *Then problem* (1.1), (1.2) *has at least one solution.*

Proof. Let S_1 be from Lemma 3.6 and let

$$\Omega = \big\{(x,c) \in C^1(J) \times \mathbb{R} :\ \|x\|_1 < S_1 + 1,\ |c| < S_1 + 1\big\}.$$

By Lemma 3.6, $\mathcal{H}(x,c,\lambda) \neq (x,c)$ for $(x,c) \in \partial\Omega$ and $\lambda \in [0,1]$. Analysis similar to that in the proof of Theorem 4.1 shows that relation (4.1) holds. Hence there exists a fixed point (u,c) of $\mathcal{Q}$ and u is a solution of problem (1.1), (1.2). □

Example 4.1. Let $r \in C(J)$, $\rho,\mu \in (0,1)$ and $k > \sqrt{2T^{\alpha-\beta}/\Gamma(\alpha-\beta+1)}$. Then the function

$$f(t,x,y,v,z) = r(t) + |x|^\rho + |y|^\mu \arctan y + \frac{1}{k+|v|} + \frac{(x+y)\ln(1+|z|)}{2+x^2+y^2}$$

satisfies condition (H_1) for $L_1 = 1/k^2$, $L_2 = 1/2$ and condition (H_2) for $A = \max\{\|r\|, \pi/2, 1/k\}$. By Theorem 4.1 there exists at least one solution u of the equation

$${}^cD^\alpha u = r(t) + |u|^\rho + |u'|^\mu \arctan u' + \frac{1}{k + {}^cD^\beta u} + \frac{(u+u')\ln(1+|{}^cD^\alpha u|)}{2+u^2+(u')^2} \tag{4.2}$$

satisfying the boundary condition (1.2).

For instance, if $\phi(u) = \min\{u(t) :\ t \in J\}$, then there exists at least one solution u of (4.2) fulfilling

$$u(0) = u(T), \quad \min\{u(t) :\ t \in J\} = 0.$$

Example 4.2. Let $T = 1$, $\alpha = 3/2$, $\beta \in (1,3/2)$, $|k| < \Gamma(5/2-\beta)/4$ and $r, r_1, r_2 \in C[0,1]$, $\|r_1\| + \|r_2\| < 3\sqrt{\pi}/44$. Then the function

$$f(t,x,y,v,z) = r(t) + r_1(t)x + r_2(t)y + kv + \frac{y\ln(1+|z|)}{1+4y^2}$$

satisfies condition (H_1) for $L_1 = |k|$, $L_2 = 1/4$ (note that $\Delta < 1/2$) and condition (H_3) for $A = \|r\|$, $B = \|r_1\|$, $C = \|r_2\|$. Since

$$W = \frac{(B+C)T^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1 + \frac{3T+1}{\alpha}\Big) = \frac{22(\|r_1\| + \|r_2\|)}{3(1-\Delta)\sqrt{\pi}} \le \frac{44(\|r_1\| + \|r_2\|)}{3\sqrt{\pi}} < 1,$$

by Theorem 4.2 there exists at least one solution u of the equation

$${}^cD^{3/2}u = r(t) + r_1(t)u + r_2(t)u' + k\,{}^cD^\beta u + \frac{u'\ln(1+|{}^cD^{3/2}u|)}{1+4(u')^2}$$

satisfying the boundary condition (1.2).

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