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VARIATION FORMULAS OF SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SEVERAL CONSTANT DELAYS AND THEIR APPLICATIONS IN OPTIMAL CONTROL PROBLEMS

Dedicated to the 90th birthday anniversary of academician Revaz Gamkrelidze

Abstract. For nonlinear functional differential equations with several constant delays, the theorems on the continuous dependence of solutions of the Cauchy problem on perturbations of the initial data and on the right-hand side of the equation are proved. Under the initial data we mean the collection of the initial moment, constant delays, initial vector and initial function. Perturbations of the initial data and of the right-hand side of the equation are small in a standard norm and in an integral sense, respectively. Variation formulas of a solution are derived for equations with a discontinuous initial and continuous initial conditions. In the variation formulas, the effects of perturbations of the initial moment and delays as well as the effects of continuous initial and discontinuous initial conditions are revealed. For the optimal control problems with delays, general boundary conditions and functional, the necessary conditions of optimality are obtained in the form of equality or inequality for the initial and final moments, for delays and an initial vector and also in the form of the integral maximum principle for the initial function and control.

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Key words and phrases. Delay functional differential equations, continuous dependence of solutions, variation formula of a solution, effect of initial moment perturbation, effect of the discontinuous initial condition, effect of the continuous initial condition, effect of constant delays perturbations, optimal control problem with delays, necessary conditions of optimality.

რეზიუმე. არაწრფივი ფუნქციონალურ-დიფერენციალური განტოლებებისათვის მუდმივი დაგვიანებებით დამტკიცებულია კოშის ამოცანის ამონახსნების უწყვეტად დამოკიდებულების თეორემები საწყისი მონაცემებისა და განტოლების მარჯვენა მხარის შეშფოთების მიმართ. საწყისი მონაცემების ქვეშ იგულისხმება საწყისი მომენტის, მუდმივი დაგვიანებების, საწყისი ვექტორისა და ფუნქციის ერთობლიობა. საწყისი მონაცემებისა და განტოლების მარჯვენა მხარის შეშფოთებები, შესაბამისად, მცირეა სტანდარტული ნორმით და ინტეგრალური აზით. გამოყვანილია ამონახსნის ვარიაციის ფორმულები განტოლებებისათვის, როგორც წყვეტილი, ასევე უწყვეტი საწყისი პირობით. ვარიაციის ფორმულებში გამოვლენილია საწყისი მომენტისა და დაგვიანებების შეშფოთების ეფექტები, წყვეტილი და უწყვეტი საწყისი პირობების ეფექტები. ოპტიმალური მართვის ამოცანებისთვის დაგვიანებებით, ზოგადი სასაზღვრო პირობებით და ფუნქციონალით მიღებულია ოპტიმალურობის აუცილებელი პირობები: უტოლობებისა და ტოლობების სახით საწყისი და საბოლოო მომენტებისათვის, დაგვიანებებისა და საწყისი ვექტორისათვის; ინტეგრალური მაქსიმუმის პრინციპის ფორმით საწყისი და მართვის ფუნქციებისთვის.

Introduction

As is known, real economical, biological, physical and majority of processes contain an information about their behavior in the past, i.e., the processes that contain effects with delayed action and which are described by functional differential equations with delays. To illustrate this, below we will consider two simplest models of the economic growth and the immune response with several constant delays.

The economic growth model. Let $N(t)$ be a quantity of a product produced at the moment t expressed in money units. The fundamental principle of the economic growth is of the form

$$N(t) = C(t) + I_{inv}(t), \tag{0.1}$$

where $C(t)$ is the so-called apply function and $I_{inv}(t)$ is a quantity induced investment. We consider the case where the functions $C(t)$ and $I(t)$ have the form

$$C(t) = \alpha_0 N(t), \quad \alpha_0 \in (0,1), \tag{0.2}$$

and

$$I_{inv}(t) = \sum_{i=1}^{s} \alpha_i N(t-\tau_i) + \alpha_{s+1}\dot{N}(t), \quad \tau_i > 0, \quad i = \overline{1,s}. \tag{0.3}$$

Formula (0.3) shows that the value of investment at the moment t depends on the quantity of money at the moments $t-\tau_i$, $i=\overline{1,s}$ (in the past), and on the velocity (production current) at the moment t. From the formulas (0.1)–(0.3) we get the equation with delays

$$\dot{N}(t) = \frac{1-\alpha_0}{\alpha_{s+1}} N(t) - \sum_{i=1}^{s} \frac{\alpha_i}{\alpha_{s+1}} N(t-\tau_i).$$

The immune response Marchuk's model [26]. A simple model about viruses attack on an organism and its immune response is the following functional differential equation:

$$\begin{cases} \dot{x}_1(t) = p_1 x_1(t) - p_2 x_1(t) x_3(t), \\ \dot{x}_2(t) = \sum\limits_{i=1}^{s} p_{i+2} x_1(t-\tau_i) x_3(t-\tau_i) - p_{s+3}(x_2(t) - x_2^*), \\ \dot{x}_3(t) = p_{s+4} x_2(t) - p_{s+5} x_3(t) - p_{s+6} x_1(t) x_3(t), \end{cases} \tag{0.4}$$

where $x_1(t)$ is the viruses concentration at time t; $x_2(t)$ is the plasma cells concentration producing antibodies. Plasma cells after a certain time period give the immune response which is characterized by the summand $\sum\limits_{i=1}^{s} p_{i+2} x_1(t-\tau_i) x_3(t-\tau_i)$, where $\tau_i > 0$ are delays of immune reactions, i.e., this expression supports reproduction of antibodies; $x_3(t)$ is the antibodies concentration which kills viruses. The first equation of system (0.4) describes changes of $x_1(t)$, here the first term $p_1 x_1(t)$ supports reproduction of viruses and the second term $p_2 x_1(t) x_3(t)$ characterizes the struggle between viruses and antibodies and do not supports reproduction of viruses. x_2^* is the physiological level of plasma cells, i.e., this concentration of plasma cells is always in the organism, and in the absence of viruses in the organism, the plasma cells remain at a constant level. Finally, $p_1, p_2, \dots, p_{s+6}$ are the positive constants.

A great deal of works (including, for example, [1–4, 12, 13, 19, 22]) are devoted to the investigation of functional differential equations with delay.

The present work consists of two parts, interconnected naturally in their meaning.

The first part considers the equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s)\big) \tag{0.5}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0. \tag{0.6}$$

The condition (0.6) is called a discontinuous initial condition since, in general, $x(t_0) \neq \varphi(t_0)$.

In the same part we study the continuous dependence of solutions of the problem (0.5), (0.6) on the initial data and on the right-hand side of the equation (0.5). Under the initial data we mean the collection of initial moment t_0, delays τ_i, $i = \overline{1, s}$, initial vector x_0 and initial function $\varphi(t)$. Moreover, we derive variation formulas of a solution (variation formulas) for the differential equation (0.5) with the discontinuous initial condition (0.6) and the continuous initial condition

$$x(t) = \varphi(t), \quad t \leq t_0. \tag{0.7}$$

The condition (0.7) is called a continuous initial condition since, always, $x(t_0) = \varphi(t_0)$. The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved in [6] for the ordinary differential equation.

In the second part, the optimization problems are investigated for the controlled equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t)\big),$$

and the necessary optimality conditions are obtained.

In Section 1, we prove a theorem on the continuous dependence of a solution in the case where the perturbation of f is small in the integral sense and initial data are small in the standard norm. Theorems on the continuous dependence of solutions of the Cauchy problem and the boundary value problems for various classes of ordinary differential equations and delay functional differential equations when perturbations of the right-hand side are small in the integral sense, are given in [6, 7, 18–21, 23, 24, 33–35, 39].

In Sections 2 and 3, we prove the variation formulas in which the effects of perturbations of the initial moment and several delays and also the effects of discontinuous and continuous initial conditions are detected. The variation formula of a solution plays a basic role in proving the necessary conditions of optimality for sensitivity analysis of mathematical models. Moreover, the variation formula allows one to get an approximate solution of the perturbed equation. The variation formulas for various classes of differential equations are given in [6, 7, 18–20, 36–42].

In Section 4, we extend the central result of the axiomatic theory of extremal problems (R. V. Gamkrelidze and G. L. Kharatishvili's theorem on the necessary criticality condition [7–9]) to the mappings defined on a finitely locally convex set. This is stipulated by the fact that it is more convenient to treat the optimal problems with delays as the problems of finding the mappings, defined and critical on a finitely locally convex set and on a quasi-convex filter, respectively. The proof of the necessary criticality condition given in Subsection 4.1, is performed according to the scheme presented in [7–9] with nonessential changes.

In Subsection 4.3, we prove the quasiconvexity of the filter arising in the optimal control problem with delays. The concept of quasiconvexity of a filter was introduced by R. V. Gamkrelidze, as a result of studying slide modes [10, 11]. Of special interest is the finding of control systems with a quasiconvex filter, since the necessary optimality conditions for these systems are deduced from the necessary criticality condition. In Subsection 4.4, we consider optimal control problems with a general functional and boundary conditions, the discontinuous initial condition and the continuous condition. The necessary conditions are obtained: for the initial and final moments in the form of inequalities and equalities, for delays in the form of inequalities and equalities, for the initial vector in the form of equality, and for the initial function and control function in the form of integral maximum principle. Optimal control problems for various classes of functional differential equations are investigated in [5, 15–18, 20, 25, 27–31].

1 Continuous dependence of solutions

1.1 Notation and auxiliary assertions

Let $I=[a,b]$ be a finite interval and $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^\top$ with $|x|^2=\sum\limits_{i=1}^{n}|x^i|^2$, where T is the sign of transposition. Let $\theta_{i2}>\theta_{i1}>0$, $i=\overline{1,s}$, be the given numbers; suppose that $O\subset\mathbb{R}^n$ is an open set, and E_f is a set of functions $f=(f^1,\dots,f^n)^\top$: $I\times O^{s+1}\to\mathbb{R}^n$ satisfying the following conditions: for each fixed $(x,x_1,\dots,x_s)\in O^{s+1}$, the function $f(t,x,x_1,\dots,x_s)$ is measurable; for each $f\in E_f$ and compact set $K\subset O$, there exist functions $m_{f,K}(t),L_{f,K}(t)\in L_1(I,R_+)$, $R_+=[0,\infty)$, such that for almost all $t\in I$

$$|f(t,x,x_1,\dots,x_s)|\le m_{f,K}(t)\ \ \forall\,(x,x_1,\dots,x_s)\in K^{s+1}$$

and

$$\big|f(t,x,x_1,\dots,x_s)-f(t,y,y_1,\dots,y_s)\big|\le L_{f,K}(t)\Big[|x-y|+\sum_{i=1}^{s}|x_i-y_i|\Big]$$
$$\forall\,(x,x_1,\dots,x_s)\in K^{s+1},\ \ \forall\,(y,y_1,\dots,y_s)\in K^{s+1}.$$

Two functions $f_1,f_2\in E_f$ are said to be equivalent, if for every fixed $(x,x_1,\dots,x_s)\in O^{s+1}$ and for almost all $t\in I$

$$f_1(t,x,x_1,\dots,x_s)-f_2(t,x,x_1,\dots,x_s)=0.$$

The equivalence classes of functions of the space E_f compose a vector space which is also denoted by E_f; these classes are called the functions and denoted by f again. In what follows, under $f\in E_f$ it is assumed any representative from the equivalence class of f.

Lemma 1.1 ([6, p. 56])**.** *Let $f\in E_f$. Then the function*

$$H(f;t',t'',x,x_1,\dots,x_s)=\Big|\int_{t'}^{t''}f(t,x,x_1,\dots,x_s)\,dt\Big|$$

is continuous in $(t',t'',x,x_1,\dots,x_s)\in I^2\times O^{s+1}$

Lemma 1.2 ([6, p. 41])**.** *Let $K_0\subset O$ and $K_1\subset O$ be compact sets with $K_0\subset\operatorname{int}K_1$. Then there exist a compact set $Q\subset O^{s+1}$ and a continuously differentiable function $\chi(x,x_1,\dots,x_s),(x,x_1,\dots,x_s)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ such that $K_0^{s+1}\subset Q\subset\operatorname{int}K_1^{s+1}$ and*

$$\chi(x,x_1,\dots,x_s)=\begin{cases}1, & (x,x_1,\dots,x_s)\in Q,\\ 0, & (x,x_1,\dots,x_s)\notin K_1^{s+1}.\end{cases}\tag{1.1}$$

Lemma 1.3. *Let $f\in E_f$. Then the function*

$$g(t,x,x_1,\dots,x_s)=\begin{cases}\chi(x,x_1,\dots,x_s)f(t,x,x_1,\dots,x_s), & t\in I,\ \ (x,x_1,\dots,x_s)\in K_1^{s+1},\\ 0, & t\in I,\ \ (x,x_1,\dots,x_s)\notin K_1^{s+1},\end{cases}\tag{1.2}$$

satisfies for almost all $t\in I$ the following conditions:

$$|g(t,x,x_1,\dots,x_s)|\le m_{f,K_1}(t)\ \ \forall\,(x,x_1,\dots,x_s)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n\tag{1.3}$$

and

$$\big|g(t,x,x_1,\dots,x_s)-g(t,y,y_1,\dots,y_s)\big|\le L_f(t)\Big[|x-y|+\sum_{i=1}^{s}|x_i-y_i|\Big]\tag{1.4}$$
$$\forall\,(x,x_1,\dots,x_s)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n,\ \ \forall\,(y,y_1,\dots,y_s)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n,$$

where

$$L_f(t) = L_{f,K_1}(t) + \alpha_0 m_{f,K_1}(t), \tag{1.5}$$

$$\alpha_0 = \sup\Big\{|\chi_x(x,x_1,\dots,x_s)| + \sum_{i=1}^{s}|\chi_{x_i}(x,x_1,\dots,x_s)| : \ (x,x_1,\dots,x_s)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n\Big\}.$$

Proof. The inequality (1.3) follows from the definition of the function g. Let

$$(x,x_1,\dots,x_s)\in K_1^{s+1} \ \text{ and } \ (y,y_1,\dots,y_s)\in K_1^{s+1},$$

then (see (1.2)) we have

$$\begin{aligned}
&\big|g(t,x,x_1,\dots,x_s)-g(t,y,y_1,\dots,y_s)\big|\\
&\quad=\big|\chi(x,x_1,\dots,x_s)f(t,x,x_1,\dots,x_s)-\chi(y,y_1,\dots,y_s)f(t,y,y_1,\dots,y_s)\big|\\
&\qquad=\Big|\chi(x,x_1,\dots,x_s)\big(f(t,x,x_1,\dots,x_s)-f(t,y,y_1,\dots,y_s)\big)\\
&\qquad\qquad+\big(\chi(x,x_1,\dots,x_s)-\chi(y,y_1,\dots,y_s)\big)f(t,y,y_1,\dots,y_s)\Big|\\
&\qquad\le L_{f,K_1}(t)\Big[|x-y|+\sum_{i=1}^{s}|x_i-y_i|\Big]+\big|\chi(x,x_1,\dots,x_s)-\chi(y,y_1,\dots,y_s))\big|m_{f,K_1}(t).
\end{aligned}\tag{1.6}$$

It is not difficult to see that

$$\begin{aligned}
&\big|\chi(x,x_1,\dots,x_s)-\chi(y,y_1,\dots,y_s))\big|\\
&\qquad=\Big|\int_0^1\frac{d}{d\xi}\,\chi\big(y+\xi(x-y),y_1+\xi(x_1-y_1),\dots,y_s+\xi(x_s-y_s)\big)\,d\xi\Big|\\
&\le\int_0^1\Big[\big|\chi_x\big(y+\xi(x-y),y_1+\xi(x_1-y_1),\dots,y_s+\xi(x_s-y_s)\big)\big|\,|x-y|+\sum_{i=1}^{s}|\chi_{x_i}(\,\cdot\,)|\,|x_i-y_i|\Big]\,d\xi\\
&\qquad\le\alpha_0\Big[|x-y|+\sum_{i=1}^{s}|x_i-y_i|\Big].
\end{aligned}$$

Taking this relation into account, from (1.6) we obtain (1.4). Let

$$(x,x_1,\dots,x_s)\in K_1^{s+1} \ \text{ and } \ (y,y_1,\dots,y_s)\notin K_1^{s+1},$$

then $\chi(y,y_1,\dots,y_s)=0$, i.e., $g(y,y_1,\dots,y_s)=0$, therefore we have

$$\begin{aligned}
&\big|g(t,x,x_1,\dots,x_s)-g(t,y,y_1,\dots,y_s)\big|\\
&\quad=|g(t,x,x_1,\dots,x_s)|=\big|\chi(x,x_1,\dots,x_s)-\chi(y,y_1,\dots,y_s))\big|\,|f(t,x,x_1,\dots,x_s)|\\
&\qquad\le\alpha_0 m_{f,K_1}(t)\Big[|x-y|+\sum_{i=1}^{s}|x_i-y_i|\Big]\le L_f(t)\Big[|x-y|+\sum_{i=1}^{s}|x_i-y_i|\Big].
\end{aligned}$$

It is easily seen that the latter inequality also holds in the case

$$(x,x_1,\dots,x_s)\notin K_1^{s+1} \ \text{ and } \ (y,y_1,\dots,y_s)\in K_1^{s+1}. \qquad\Box$$

Let $I_1=[\widehat{\tau},b]$, where $\widehat{\tau}=a-\max\{\theta_{12},\dots,\theta_{s2}\}$. By $\mathrm{PC}(I_1,\mathbb{R}^n)$ we denote the space of piecewise-continuous functions $\varphi: I_1\to\mathbb{R}^n$ with finitely many discontinuities of the first kind equipped with the norm $\|\varphi\|_{I_1}=\sup\{|\varphi(t)|:\ t\in I_1\}$. By $\Phi=\{\varphi\in\mathrm{PC}(I_1,\mathbb{R}^n):\ \mathrm{cl}\,\varphi(I_1)\subset O\}$ we denote a set of initial functions, where $\varphi(I_1)=\{\varphi(t):\ t\in I_1\}$.

Let $\varphi_i \in \Phi$, $i = \overline{0,s}$, be fixed functions and let $t_\alpha \in (a,b)$, $\alpha = \overline{1,p}$, be discontinuity points of the function $\psi(t) = (\varphi_0(t), \varphi_1(t-\tau_1), \dots, \varphi_s(t-\tau_s))$, where $\tau_i \in [\theta_{i1}, \theta_{i2}]$, $i = \overline{1,s}$, are the given numbers.

We now introduce the notation

$$\varphi_{ij}(t) = \begin{cases} \varphi_i(t_{j-1} - \tau_i +), & t = t_{j-1}, \\ \varphi_i(t - \tau_i), & t \in (t_{j-1}, t_j), \\ \varphi_i(t_j - \tau_i -), & t = t_j, \end{cases} \tag{1.7}$$

where $i = \overline{0,s}$, $j = \overline{1,p+1}$, $t_0 = a$, $t_{p+1} = b$, $\tau_0 = 0$. Clearly, the function $\varphi_{ij}(t)$ is continuous on the interval $[t_{j-1}, t_j]$. Next, let k be a fixed natural number,

$$w_j(k;\psi) = \sup\Big\{ \sum_{i=0}^{s} |\varphi_{ij}(t') - \varphi_{ij}(t'')| : \ t', t'' \in [t_{j-1}, t_j], \ |t' - t''| \le \frac{t_j - t_{j-1}}{k} \Big\},$$
$$w(k;\psi) = \sup\big\{ w_j(k;\psi) : \ 1 \le j \le p+1 \big\}.$$

Lemma 1.4. *Let $\varphi_i \in \Phi$, $i = \overline{0,s}$, and let $\varphi_i(t) \in K$, where $K \subset O$ is a compact set. Then for an arbitrary $f \in E_f$ and a natural number k, the inequality*

$$\beta = \sup\bigg\{ \bigg| \int_{\xi_1}^{\xi_2} f\big(t, \varphi_0(t), \varphi_1(t-\tau_1), \dots, \varphi_s(t-\tau_s)\big)\, dt \bigg| : \ \xi_1, \xi_2 \in I \bigg\}$$
$$\le w(k;\psi) \int_I L_{f,K}(t)\, dt + k(p+1) H_0(f;K)$$

holds, where

$$H_0(f;K) = \sup\Big\{ H(f; t', t'', x, x_1, \dots, x_s) : \ (t', t'', x, x_1, \dots, x_s) \in I^2 \times K^{s+1} \Big\}$$

(*see Lemma* 1.1).

Proof. There exist the numbers $a_1, b_1 \in I$ such that

$$\beta = \bigg| \int_{a_1}^{b_1} f\big(t, \varphi_0(t), \varphi_1(t-\tau_1), \dots, \varphi_s(t-\tau_s)\big)\, dt \bigg|.$$

Let $a_1 \in [t_{l-1}, t_l)$ and $b_1 \in [t_{q-1}, t_q)$ with $1 \le l \le q \le p+1$. Divide each of the intervals $[a_1, t_l]$, $[t_{j-1}, t_j]$, $j = \overline{l+1, q-1}$, $[t_{q-1}, b_1]$, into k equal parts Δ_ρ^l, Δ_ρ^j, $j = \overline{l+1, q-1}$, Δ_ρ^q, $\rho = \overline{1,k}$, respectively. Obviously,

$$[a_1, b_1] = [a_1, t_l] \cup \Big(\bigcup_{j=l+1}^{q-1} [t_{j-1}, t_j] \Big) \cup [t_{q-1}, b_1] = \bigcup_{j=l}^{q} \bigcup_{\rho=1}^{k} \Delta_\rho^j.$$

Using this relation and the notation (1.7), we obtain

$$\beta \le \sum_{j=l}^{q} \sum_{\rho=1}^{k} \bigg| \int_{\Delta_\rho^j} f\big(t, \varphi_{0j}(t), \varphi_{1j}(t), \dots, \varphi_{sj}(t)\big)\, dt \bigg|.$$

Let $t_\rho^j \in \Delta_\rho^j$, $j = \overline{1,q}$, $\rho = \overline{1,k}$, be arbitrary fixed points. Then

$$\beta \le \sum_{j=l}^{q} \sum_{\rho=1}^{k} \int_{\Delta_\rho^j} \Big| f\big(t, \varphi_{0j}(t), \varphi_{1j}(t), \dots, \varphi_{sj}(t)\big) - f\big(t, \varphi_{0j}(t_\rho^j), \varphi_{1j}(t_\rho^j), \dots, \varphi_{sj}(t_\rho^j)\big) \Big|\, dt$$

$$+\sum_{j=l}^{q}\sum_{\rho=1}^{k}\bigg|\int\limits_{\Delta_\rho^j} f\big(t,\varphi_{0j}(t_\rho^j),\varphi_{1j}(t_\rho^j),\dots,\varphi_{sj}(t_\rho^j)\big)\,dt\bigg|$$

$$\le\sum_{j=l}^{q}\sum_{\rho=1}^{k}\int\limits_{\Delta_\rho^j}\Big[L_{f,K}(t)\sum_{i=0}^{s}\big|\varphi_{ij}(t)-\varphi_{ij}(t_\rho^j)\big|\Big]\,dt+k(q-l+1)H_0(f;K)$$

$$\le\sum_{j=l}^{q}\sum_{\rho=1}^{k}w_j(k;\psi)\int\limits_{\Delta_\rho^j}L_{f,K}(t)\,dt+k(p+1)H_0(f;K)$$

$$\le w(k;\psi)\int\limits_{I}L_{f,K}(t)\,dt+k(p+1)H_0(f;K).$$ □

Lemma 1.5. *Let $\varphi_i\in\Phi$, $i=\overline{0,s}$, and let $\varphi_i(t)\in K$, where $K\subset O$ is a compact set. Further, let the sequence $\delta f_i\in E_f$, $i=1,2,\dots$, satisfy the conditions*

$$\int\limits_I L_{\delta f_i,K}(t)\,dt\le\alpha_1=const,\ \ i=1,2,\dots,\ \ \text{and}\ \ \lim_{i\to\infty}H_0(\delta f_i;K)=0.$$

Then $\lim\limits_{i\to\infty}\beta_i=0$, where

$$\beta_i=\sup\bigg\{\bigg|\int\limits_{\xi_1}^{\xi_2}\delta f_i\big(t,\varphi_0(t),\varphi_1(t-\tau_1),\dots,\varphi_s(t-\tau_s)\big)\,dt\bigg|:\ \ \xi_1,\xi_2\in I\bigg\}.$$

Proof. Let $\varepsilon>0$ be an arbitrary number. By Lemma 1.4, we have

$$\beta_i\le w(k;\psi)\int\limits_I L_{\delta f_i,K}(t)\,dt+k(p+1)H_0(\delta f_i;K)\le\alpha_1 w(k;\psi)+k(p+1)H_0(\delta f_i;K).\tag{1.8}$$

The functions $\varphi_{ij}(t)$, $t\in[t_{j-1},t_j]$, are continuous. Therefore, $\lim\limits_{k\to\infty}w(k;\psi)=0$. There exist natural numbers k_0 and i_0 such that

$$w(k_0;\psi)\le\frac{\varepsilon}{2}\ \text{ and }\ k_0(p+1)H_0(\delta f_i;K)\le\frac{\varepsilon}{2}\,,\ \ i\ge i_0.\tag{1.9}$$

Taking into account the relations (1.9) in (1.8), we obtain $\beta_i\le\varepsilon$ for $i\ge i_0$. By the arbitrariness of ε, we can conclude that $\beta_i\to 0$, as $i\to\infty$. □

Lemma 1.6 ([6, p. 68])**.** *Let $m(t)\in L_1(I,\mathbb{R}_+)$. Then the formula*

$$\int\limits_a^t m(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}m(\xi_2)\,d\xi_2\cdots\int\limits_a^{\xi_{k-1}}m(\xi_k)\,d\xi_k=\frac{1}{k!}\bigg(\int\limits_a^t m(\xi)\,d\xi\bigg)^k$$

holds.

Lemma 1.7. *Let $f_1,f_2\in E_f$ be equivalent functions. Then for an arbitrary function $\varphi\in\Phi$, the relation*

$$\bigg|\int\limits_{\xi_1}^{\xi_2}\widehat{f}\big(t,\varphi(t),\varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big)\,dt\bigg|=0\ \ \forall\,\xi_1,\xi_2\in I\tag{1.10}$$

holds, where

$$\widehat{f}(t,x,x_1,\dots,x_s)=f_1(t,x,x_1,\dots,x_s)-f_2(t,x,x_1,\dots,x_s).$$

Proof. It is clear that for almost all $t \in I$,

$$\widehat{f}(t, x, x_1, \dots, x_s) = 0 \ \ \forall (x, x_1, \dots, x_s) \in O^{1+s}.$$

Therefore

$$H_0(\widehat{f}; K) = 0, \ \text{ where } \ K = \operatorname{cl} \varphi(I_1) \subset O.$$

Using Lemma 1.3, for an arbitrary natural number k and $\xi_1, \xi_2 \in I$, we get

$$\bigg| \int\limits_{\xi_1}^{\xi_2} \widehat{f}\big(t, \varphi(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)\big)\, dt \bigg| \le w(k; \psi) \int\limits_I L_{\widehat{f};K}(t)\, dt,$$

where $\psi(t) = (\varphi(t), \varphi(t-\tau_1), \dots, \varphi_s(t-\tau_s))$ and $w(k;\psi) \to 0$, as $k \to \infty$. Thus the relation (1.10) is valid. □

Lemma 1.8. *Let $f \in E_f$. Then the mapping*

$$\varphi \longrightarrow \int\limits_a^t f\big(\xi, \varphi(\xi), \varphi(\xi-\tau_1), \dots, \varphi(\xi-\tau_s)\big)\, d\xi, \ \ \varphi \in \Phi,$$

is uniquely defined (*see Lemma* 1.7).

Let X be a metric space, ϱ be a distance function on X, and let

$$F(\,\cdot\,;\mu) : X \to X \tag{1.11}$$

be a family of mappings depending on the parameter $\mu \in \Lambda$, where Λ is a topological space. The family of the mappings (1.11) is said to be uniformly contractive if there exists a number $\alpha \in (0,1)$ independent of μ such that the inequality

$$\varrho\big(F(y_1;\mu), F(y_2;\mu)\big) \le \alpha \varrho(y_1, y_2) \ \ \forall\, y_1, y_2 \in X$$

holds for each $\mu \in \Lambda$.

Define the iteration of the mapping (1.11):

$$F^k(y;\mu) = F(F^{k-1}(y;\mu);\mu), \ \ k = 1, 2, \dots, \ \ F^0(y;\mu) = y.$$

Obviously,

$$F^k(\,\cdot\,;\mu) : X \to X \ \ \forall\, \mu \in \Lambda. \tag{1.12}$$

Theorem 1.1 ([6, p. 61]; [14, p. 608])**.** *Let X be a complete metric space. If a certain iteration* (1.12) *is a uniformly contractive family, then for every $\mu \in \Lambda$ the mapping* (1.11) *has a unique fixed point $y_\mu \in X$, i.e., $F(y_\mu;\mu) = y_\mu$. Moreover, if for fixed $\mu_0 \in \Lambda$, a certain iteration $F^k(y_{\mu_0};\,\cdot\,) : \Lambda \to X$ is continuous at the point μ_0, then the mapping $y_\mu : \Lambda \to X$ is likewise continuous at the point μ_0.*

1.2 Theorems on continuous dependence of solutions

To each element

$$\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda = [a, b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times E_f$$

we assign the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s)\big) \tag{1.13}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{1.14}$$

Definition 1.1. Let $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (1.13) with the initial condition (1.14), or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (1.14) and on the interval $[t_0, t_1]$ satisfies the integral equation

$$x(t) = x_0 + \int\limits_{t_0}^{t} f\big(\xi, x(\xi), x(\xi - \tau_1), \dots, x(\xi - \tau_s)\big)\, d\xi$$

(see Lemma 1.7).

Obviously, the function $x(t;\mu)$, $t \in [t_0, t_1]$, is absolutely continuous and satisfies the equation (1.13) almost everywhere on $[t_0, t_1]$. If $t_1 - t_0$ is sufficiently small, then there exists a solution corresponding to μ [3, 13, 22].

In the space E_f, we introduce a family of subsets

$$\Re = \big\{V_{K,\delta} :\ \ K \subset O,\ \ \delta > 0\big\}.$$

Here, $K \subset O$ is a compact set, $\delta > 0$ is an arbitrary number, and

$$V_{K,\delta} = \big\{\delta f \in E_f :\ \ H_0(\delta f; K) \le \delta\big\}.$$

The family $\Re$ can be taken as a basis of neighborhoods of zero in the space E_f [32]. Hence it defines a locally convex Hausdorff vector topology with which E_f becomes a topological vector space. Everywhere in what follows, we will assume that the space E_f is endowed precisely with that topology.

We introduce the set

$$W(K;\alpha) = \bigg\{\delta f \in E_f :\ \ \exists\, m_{\delta f,K}(t), L_{\delta f,K}(t) \in L_1(I, R_+),\ \ \int\limits_I \big[m_{\delta f,K}(t) + L_{\delta f,K}(t)\big]\, dt \le \alpha\bigg\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number independent of δf.

Let $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, f_0) \in \Lambda$ be a fixed element,

$$B(t_{00};\delta) = \big\{t_0 \in I :\ \ |t_0 - t_{00}| < \delta\big\},\quad B(\tau_{i0};\delta) = \big\{\tau_i \in [\theta_{i1}, \theta_{i2}] :\ \ |\tau_i - \tau_{i0}| < \delta\big\},\ \ i = \overline{1, s},$$
$$B(x_{00};\delta) = \big\{x_0 \in O :\ \ |x_0 - x_{00}| < \delta\big\},\quad B(\varphi_0;\delta) = \big\{\varphi \in \Phi :\ \ \|\varphi - \varphi_0\|_{I_1} < \delta\big\},$$

Theorem 1.2. *Let $x_0(t)$ be a solution corresponding to $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, f_0) \in \Lambda$ and defined on $[\widehat{\tau}, t_{10}]$, where $t_{10} < b$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \operatorname{cl}\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.1. *There exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\begin{aligned}\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f_0 + \delta f) &\in V(\mu_0; K_1, \delta_0, \alpha)\\ &= B(t_{00};\delta_0) \times B(\tau_{10};\delta_0) \times \cdots \times B(\tau_{s0};\delta_0) \times B(x_{00};\delta_0) \times B(\varphi_0;\delta_0)\\ &\quad\times \big[f_0 + (W(K_1;\alpha) \cap V_{K_1,\delta_0})\big]\end{aligned}$$

there corresponds the solution $x(t;\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t;\mu) \in K_1$.

1.2. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$|x(t;\mu) - x(t;\mu_0)| \le \varepsilon\ \ \forall\, t \in [\theta, t_{10} + \delta_1],\ \ \theta = \max\{t_0, t_{00}\}$$

holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$.

1.3. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|\, dt \le \varepsilon$$

holds for any $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$.

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10}+\delta_1]$.

In the space $E_{\delta\mu} = E_\mu - \mu_0$ with the elements $\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f)$, where $E_\mu = \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}^n\times \mathrm{PC}(I_1,\mathbb{R}^n)\times E_f$, we introduce the set of variations

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_{\delta\mu} :$$

$$|\delta t_0| \le \gamma,\ \ |\delta\tau_i| \le \gamma,\ \ i=\overline{1,s},\ \ |\delta x_0| \le \gamma,\ \ \|\delta\varphi\|_I \le \gamma,\ \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,\ \ |\lambda_i| \le \gamma,\ \ i=\overline{1,k}\Big\},$$

where $\gamma > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i=\overline{1,k}$, are fixed functions.

Theorem 1.3. *Let $x_0(t)$ be a solution corresponding to $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, f_0) \in \Lambda$ and defined on $[\widehat{\tau}, t_{10}]$ with $t_{00}, t_{10} \in (a,b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i=\overline{1,s}$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

1.4. *There exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0,\varepsilon_1]\times\Im$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10}+\delta_1] \subset I_1$ corresponds to this element. Moreover, $x(t;\mu_0+\varepsilon\delta\mu) \in K_1$.*

1.5. *The following relations hold:*

$$\lim_{\varepsilon\to 0} \sup\Big\{|x(t;\mu_0+\varepsilon\delta\mu) - x(t;\mu_0)| :\ \ t \in [\theta, t_{10}+\delta_1]\Big\} = 0,$$

$$\lim_{\varepsilon\to 0} \int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t;\mu_0+\varepsilon\delta\mu) - x(t;\mu_0)|\, dt = 0$$

uniformly in $\delta\mu \in \Im$, where $\theta = \max\{t_{00}, t_{00}+\varepsilon\delta t_0\}$.

Theorem 1.2 is the corollary of Theorem 1.1.

Let $E_u(I)$ be the space of measurable functions $u(t) \in \mathbb{R}^r$, $t \in I$, satisfying the condition: $\mathrm{cl}\, u(I)$ is a compact set in $\mathbb{R}^r$. Let $U_0 \subset \mathbb{R}^r$ be an open set and $\Omega(I,U_0) = \{u \in E_u(I) :\ \mathrm{cl}\, u(I) \subset U_0\}$.

To each element $w = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, u) \in \Lambda_1 = [a,b)\times[\theta_{11},\theta_{12}]\times\cdots\times[\theta_{s1},\theta_{s2}]\times O\times\Phi\times\Omega(I,U_0)$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = \phi\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t)\big) \tag{1.15}$$

with the discontinuous initial condition (1.14). Here the function $\phi(t,x,x_1,\dots,x_s,u)$ is defined on $I\times O^{s+1}\times U_0$ and satisfies the following conditions: for each fixed $(x,x_1,\dots,x_s,u) \in O^{s+1}\times U_0$ the function $\phi(\,\cdot\,,x,x_1,\dots,x_s,u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist the functions $m_{K,U}(t), L_{K,U}(t) \in L_1(I,R_+)$ such that for almost all $t \in I$,

$$|\phi(t,x,x_1,\dots,x_s,u)| \le m_{K,U}(t)\ \ \forall\,(x,x_1,\dots,x_s,u) \in K^{s+1}\times U,$$

$$\big|\phi(t,x,x_1,\dots,x_s,u_1) - \phi(t,y,y_1,\dots,y_s,u_2)\big| \le L_{f,K}(t)\Big[|x-y| + \sum_{i=1}^{s}|x_i-y_i| + |u_1-u_2|\Big]$$

$$\forall\,(x,x_1,\dots,x_s) \in K^{s+1},\ \ \forall\,(y,y_1,\dots,y_s) \in K^{s+1}\ \text{ and }\ \forall\,(u_1,u_2) \in U^2.$$

Definition 1.2. Let $w = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, u) \in \Lambda_1$. A function $x(t) = x(t;w) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (1.15) with the initial condition (1.14), or a solution corresponding to the element w and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (1.14) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies the equation (1.15) almost everywhere (a.e.) on $[t_0,t_1]$.

Theorem 1.4. *Let $x_0(t)$ be a solution corresponding to $w_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ and defined on $[\widehat{\tau}, t_{10}]$, with $t_{10} < b$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \mathrm{cl}\,\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.6. *There exist the numbers* $\delta_i > 0$, $i = 0, 1$, *such that to each element*

$$w = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, u) \in \widehat{V}(w_0; \delta_0)$$
$$= B(t_{00}; \delta_0) \times B(\tau_{10}; \delta_0) \times \cdots \times B(\tau_{s0}; \delta_0) \times B(x_{00}; \delta_0) \times B(\varphi_0; \delta_0) \times B(u_0; \delta_0)$$

there corresponds a solution $x(t; w)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ *and satisfying the condition* $x(t; w) \in K_1$*; here* $B(u_0; \delta_0) = \{u \in \Omega(I, U_0) : \ \|u - u_0\|_I < \delta_0\}$.

1.7. *For an arbitrary* $\varepsilon > 0$, *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ *such that the inequality*

$$|x(t; w) - x(t; w_0)| \le \varepsilon \ \ \forall\, t \in [\theta, t_{10} + \delta_1], \ \ \theta = \max\{t_0, t_{00}\},$$

holds for any $w \in \widehat{V}(w_0; \delta_2)$.

1.8. *For an arbitrary* $\varepsilon > 0$, *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ *such that the inequality*

$$\int_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t; w) - x(t; w_0)|\, dt \le \varepsilon$$

holds for any $w \in \widehat{V}(w_0; \delta_3)$.

Due to the uniqueness, the solution $x(t; w_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

In the space $E_{\delta w} = E_w - w_0$ with the elements $\delta w = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta u)$, where $E_w = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E_u(I)$, we introduce the set of variations

$$\Im_1 = \Big\{\delta w = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta u) \in E_{\delta w} :$$
$$|\delta t_0| \le \gamma, \ \ |\delta\tau_i| \le \gamma, \ \ i = \overline{1, s}, \ \ |\delta x_0| \le \beta, \ \ \|\delta\varphi\|_{I_1} \le \gamma, \ \ \|\delta u\|_I \le \gamma\Big\},$$

where $\gamma > 0$ is a fixed number.

Theorem 1.5. *Let* $x_0(t)$ *be a solution corresponding to* $w_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ *and defined on* $[\widehat{\tau}, t_{10}]$ *with* $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, *and let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* K_0. *Then the following conditions hold:*

1.9. *There exist numbers* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that for an arbitrary* $(\varepsilon, \delta w) \in [0, \varepsilon_1] \times \Im_1$ *we have* $w_0 + \varepsilon\delta w \in \Lambda_1$ *and the solution* $x(t; w_0 + \varepsilon\delta w)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ *corresponds to this element. Moreover,* $x(t; w_0 + \varepsilon\delta w) \in K_1$.

1.10. *The following relations hold:*

$$\lim_{\varepsilon \to 0} \sup \Big\{|x(t; w_0 + \varepsilon\delta w) - x(t; w_0)| : \ \ t \in [\theta, t_{10} + \delta_1]\Big\} = 0,$$

$$\lim_{\varepsilon \to 0} \int_{\widehat{\tau}}^{t_{10}+\delta_1} \big|x(t; w_0 + \varepsilon\delta w) - x(t; w_0)\big|\, dt = 0$$

uniformly in $\delta w \in \Im_1$, *where* $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 1.5 is the corollary of Theorem 1.4.

Let $I_2 = [a, \widehat{\tau}_1]$, where $\widehat{\tau}_1 = b + \max\{\theta_{12}, \dots, \theta_{s2}\}$. By $\Phi_1 = \{\varphi \in \mathrm{PC}(I_2, \mathbb{R}^n) : \ \mathrm{cl}\,\varphi(I_2) \subset O\}$ we denote a set of initial functions for the functional differential equation with advanced arguments. To each element

$$\vartheta = (t_1, \tau_1, \dots, \tau_s, x_1, \varphi, f) \in \Lambda_2 = (a, b] \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi_1 \times E_f$$

we assign the functional differential equation with the advanced argument

$$\dot{x}(t) = f\big(t, x(t), x(t+\tau_1), \dots, x(t+\tau_s)\big)$$

with the discontinuous initial condition

$$x(t_1) = x_1, \ \ x(t) = \varphi(t), \ \ t \in (t_1, \widehat{\tau}_1].$$

Definition 1.3. Let $\vartheta = (t_1, \tau_1, \dots, \tau_s, x_1, \varphi, f) \in \Lambda_2$. A function $x(t) = x(t;\vartheta) \in O$, $t \in [t_0, \widehat{\tau}_1]$, $t_0 \in [a, t_1)$, is called a solution corresponding to the element ϑ and defined on the interval $[t_0, \widehat{\tau}_1]$ if it satisfies the initial condition and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the integral equation

$$x(t) = x_1 + \int\limits_t^{t_1} f\big(\xi, x(\xi), x(\xi+\tau_1), \dots, x(\xi+\tau_s)\big)\, d\xi.$$

Theorem 1.6. *Let $x_0(t)$ be a solution corresponding to $\vartheta_0 = (t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{10}, \varphi_0, f_0) \in \Lambda_2$ and defined on $[t_{00}, \widehat{\tau}_2]$, where $t_{00} > a$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl} \varphi_0(I_2) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.11. *There exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\begin{aligned}&\vartheta = (t_1, \tau_1, \dots, \tau_s, x_1, \varphi, f_0 + \delta f) \in V(\vartheta_0; K_1, \delta_0, \alpha)\\ &= B(t_{10};\delta_0) \times B(\tau_{10};\delta_0) \times \dots \times B(\tau_{s0};\delta_0) \times B(x_{10};\delta_0) \times B_1(\varphi_0;\delta_0) \times \big[f_0 + (W(K_1;\alpha) \cap V_{K_1,\delta_0})\big]\end{aligned}$$

there corresponds the solution $x(t;\vartheta)$ defined on the interval $[t_{00} - \delta_1, \widehat{\tau}_2] \subset I_2$ and satisfying the condition $x(t;\vartheta) \in K_1$.

1.12. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$|x(t;\vartheta) - x(t;\vartheta_0)| \le \varepsilon \ \ \forall\, t \in [t_{00} - \delta_1, \theta], \ \ \theta = \min\{t_1, t_{10}\}$$

holds for any $\vartheta \in V(\vartheta_0; K_1, \delta_2, \alpha)$.

1.13. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$\int\limits_{t_{00}-\delta_1}^{\widehat{\tau}_2} |x(t;\vartheta) - x(t;\vartheta_0)|\, dt \le \varepsilon$$

holds for any $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$.

Here $B_1(\varphi_0;\delta) = \big\{\varphi \in \Phi_1 : \ \ \|\varphi - \varphi_0\|_{I_2} < \delta\big\}$.

Theorem 1.6 is proved analogously to Theorem 1.2.

1.3 Proof of Theorem 1.2 (on the continuous dependence of a solution for a class of functional differential equations)

To each element $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda$ we assign the functional differential equation

$$\dot{y}(t) = f(t_0, \tau_1, \dots, \tau_s, \varphi, y)(t) = f\big(t, y(t), h(t_0, \varphi, y)(t-\tau_1), \dots, h(t_0, \varphi, y)(t-\tau_s)\big) \tag{1.16}$$

with the initial condition

$$y(t_0) = x_0, \tag{1.17}$$

where $h : I \times \Phi \times C(I, \mathbb{R}^n) \to \mathrm{PC}(I_1, \mathbb{R}_n)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t) & \text{for } \ t \in [\widehat{\tau}, t_0), \\ y(t) & \text{for } t \in [t_0, b], \end{cases} \tag{1.18}$$

and $C(I, \mathbb{R}^n)$ is the space of continuous function $y : I \to \mathbb{R}^n$ equipped with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$.

Definition 1.4. An absolutely continuous function $y(t) = y(t;\mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (1.16) with the initial condition (1.17), or a solution corresponding to the element $\mu \in \Lambda$ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$ and it satisfies the equation (1.16) a.e. on the interval $[r_1, r_2]$.

Remark 1.1. Let $y(t;\mu)$, $t \in [r_1, r_2]$, $\mu \in A$, be a solution of the equation (1.16) with the initial condition (1.17). Then, as is easily seen, the function

$$x(t;\mu) = h(t_0, \varphi, y(\,\cdot\,;\mu))(t), \;\; t \in [\widehat{\tau}, r_2],$$

is the solution of the equation (1.13) with the initial condition (1.14).

Theorem 1.7. *Let $y_0(t) = y(t;\mu_0)$, $\mu_0 \in A$, be a solution defined on $[r_1, r_2] \subset (a, b)$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \operatorname{cl}\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

1.14. *There exist the numbers $\delta_i > 0$, $i = 0, 1$, such that a solution $y(t;\mu)$ defined on $[r_1-\delta_1, r_2+\delta_1] \subset I$ corresponds to each element*

$$\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover,

$$\varphi(t) \in K_1, \;\; t \in I_1; \quad y(t;\mu) \in K_1, \;\; t \in [r_1 - \delta_1, r_2 + \delta_1],$$

for arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$.

1.15. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the inequality*

$$|y(t;\mu) - y(t;\mu_0)| \le \varepsilon \;\; \forall\, t \in [r_1 - \delta_1, r_2 + \delta_1] \tag{1.19}$$

holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$.

Proof. Let $\varepsilon_0 > 0$ be insomuch small that a closed ε_0-neighborhood of the set K_0

$$K(\varepsilon_0) = \big\{x \in \mathbb{R}^n : \;\; \exists \widehat{x} \in K_0, \;\; |x - \widehat{x}| \le \varepsilon_0\big\}$$

lies in $\operatorname{int} K_1$. By Lemma 1.2, there exist a compact set $Q : K_0^{s+1}(\varepsilon_0) \subset Q \subset K_1^{s+1}$ and a continuously differentiable function $\chi : \mathbb{R}^{n(s+1)} \to [0, 1]$ of the form (1.1).

To each element $\mu \in \Lambda$, we assign the functional differential equation

$$\dot{z}(t) = g(t_0, \tau_1, \dots, \tau_s, \varphi, z)(t) = g\big(t, z(t), h(t_0, \varphi, z)(t - \tau_1), \dots, h(t_0, \varphi, z)(t - \tau_s)\big) \tag{1.20}$$

with the initial condition

$$z(t_0) = x_0, \tag{1.21}$$

where $g = \chi f$. The function $g(t, x, x, x_1, \dots, x_s)$ satisfies the conditions (1.3) and (1.4).

The solution of the equation (1.20) with the initial condition (1.21) depends on the parameter

$$\mu \in \Lambda_0 = [a, b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times \Big(f_0 + W(K_1, \alpha)\Big) \subset E_\mu.$$

The topology in Λ_0 is inherited from the vector space E_μ.

On the complete metric space $C(I, \mathbb{R}^n)$ we introduce a family

$$F(\,\cdot\,;\mu) : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n) \tag{1.22}$$

of mapping depending on the parameter μ by the formula

$$\zeta(t) = \zeta(t; z, \mu) = x_0 + \int_{t_0}^{t} g(t_0, \tau_1, \dots, \tau_s, \varphi, z)(\xi)\, d\xi.$$

Clearly, every fixed point $z(t;\mu)$, $t\in I$, of the mapping (1.22) is a solution of the equation (1.20) with the initial condition (1.21).

Define the kth iteration $F^k(z;\mu)$ by

$$\zeta_k(t)=\zeta_k(t;z,\mu)=x_0+\int\limits_{t_0}^{t} g(t_0,\tau_1,\dots,\tau_s,\varphi,\zeta_{k-1})(\xi)\,d\xi,\ \ k=1,2,\dots,\ \ \zeta_0(t)=z(t).$$

Let us now prove that for a sufficiently large k, the family of mappings $F^k(z;\mu)$ is uniformly contractive. For this purpose, we estimate the difference

$$\begin{aligned}|\zeta_k'(t)-\zeta_k''(t)|&=\big|\zeta_k(t;z',\mu)-\zeta_k(t;z'',\mu)\big|\\&\le\int\limits_a^t\big|g(t_0,\tau_1,\dots,\tau_s,\varphi,\zeta_{k-1}')(\xi)-g(t_0,\tau_1,\dots,\tau_s,\varphi,\zeta_{k-1}'')(\xi)\big|\,d\xi\\&\le\int\limits_a^t L_f(\xi)\Big[|\zeta_{k-1}'(\xi)-\zeta_{k-1}''(\xi)|+\sum_{j=1}^{s}\big|h(t_0,\varphi,\zeta_{k-1}')(t-\tau_j)-h(t_0,\varphi,\zeta_{k-1}'')(t-\tau_j)\big|\Big]\,d\xi,\\&\qquad k=1,2,\dots\end{aligned}\tag{1.23}$$

(see (1.4)), where the function $L_f(\xi)$ is of the form (1.5). Here it is assumed that $\zeta_0'=z'(t)$ and $\zeta_0''=z''(t)$. It follows from the definition of the operator $h(\,\cdot\,)$ (see (1.18)) that

$$h(t_0,\varphi,\zeta_{k-1}')(\xi-\tau_j)-h(t_0,\varphi,\zeta_{k-1}'')(\xi-\tau_j)=h(t_0,0,\zeta_{k-1}'-\zeta_{k-1}'')(\xi-\tau_j).$$

Hence, for $\xi\in[a,t_0+\tau_j)$, we have

$$h(t_0,0,\zeta_{k-1}'-\zeta_{k-1}'')(\xi-\tau_j)=0.\tag{1.24}$$

Let $t_0+\tau_j<b$; then for $\xi\in[t_0+\tau_j,b]$ we obtain

$$\begin{aligned}\big|h(t_0,0,\zeta_{k-1}'-\zeta_{k-1}'')(\xi-\tau_j)\big|&=\big|\zeta_{k-1}'(\xi-\tau_j)-\zeta_{k-1}''(\xi-\tau_j)\big|\\&\le\sup\Big\{\big|\zeta_{k-1}'(t-\tau_j)-\zeta_{k-1}''(t-\tau_j)\big|:\ t\in[t_0+\tau_j,\xi]\Big\}\\&\le\sup\Big\{\big|\zeta_{k-1}'(t)-\zeta_{k-1}''(t)\big|:\ t\in[a,\xi]\Big\}.\end{aligned}\tag{1.25}$$

If $t_0+\tau_j>b$, then (1.24) holds on the whole interval I. The relation (1.23), together with (1.24) and (1.25), implies that

$$\begin{aligned}|\zeta_k'(t)-\zeta_k''(t)|&\le\sup\big\{|\zeta_k'(\xi)-\zeta_k{}''(\xi)|:\ \xi\in[a,t]\big\}\\&\le(s+1)\int\limits_a^t L_f(\xi_1)\sup\big\{|\zeta_{k-1}'(\xi)-\zeta_{k-1}''(\xi)|:\ \xi\in[a,\xi_1]\big\}\,d\xi_1,\ \ k=1,2,\dots.\end{aligned}$$

Therefore,

$$|\zeta_k'(t)-\zeta_k''(t)|\le(s+1)^2\int\limits_a^t L_f(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}L_f(\xi_2)\sup\big\{|\zeta_{k-2}'(\xi)-\zeta_{k-2}''(\xi)|:\ \xi\in[a,\xi_2]\big\}\,d\xi_2.$$

Continuing this procedure, we obtain

$$|\zeta_k'(t)-\zeta_k''(t)|\le(s+1)^k\alpha_k(t)\|z'-z''\|_I,$$

where

$$\alpha_k(t)=\int\limits_a^t L_f(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}L_f(\xi_2)\,d\xi_2\cdots\int\limits_a^{\xi_{k-1}}L_f(\xi_k)\,d\xi_k=\frac{1}{k!}\bigg(\int\limits_a^t L_f(\xi)\,d\xi\bigg)^k$$

(see Lemma 1.6). Thus

$$d\big(F^k(z';\mu),F^k(z'';\mu)\big)=\|\zeta_k'-\zeta_k''\|_I\le\frac{(s+1)^k}{k!}\bigg(\int\limits_a^b L_f(\xi)\,d\xi\bigg)^k\|z'-z''\|_I=\alpha_k(b)\|z'-z''\|_I.$$

Let us prove the existence of a number $\alpha_2>0$ such that

$$\int\limits_I L_f(t)\,dt\le\alpha_2\quad\forall\, f\in f_0+W(K_1;\alpha).$$

Indeed, let $(x,x_1,\dots,x_s)\in K_1^{s+1}$ and $f\in f_0+W(K_1;\alpha)$, then

$$|f(t,x,x_1,\dots,x_2)|\le m_{f_0,K_1}(t)+m_{\delta f,K_1}(t):=m_{f,K_1}(t),\ \ t\in I.$$

Further, let $x'x_i',x'',x_i''\in K_1$, $i=\overline{1,s}$, then

$$\begin{aligned}
&\big|f(t,x'.x_1',\dots,x_s')-f(t,x'',x_1'',\dots,x_s'')\big|\\
&\quad\le\big|f_0(t,x',x_1',\dots,x_s')-f_0(t,x'',x_1'',\dots,x_s'')\big|+\big|\delta f(t,x',x_1',\dots,x_s')-\delta f(t,x'',x_1'',\dots,x_s'')\big|\\
&\qquad\le\big(L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)\big)\Big[|x'-x''|+\sum_{i=1}^s|x_i'-x_i''|\Big]\\
&\qquad\qquad=L_{f,K_1}(t)\Big[|x'-x''|+\sum_{i=1}^s|x_i'-x_i''|\Big],
\end{aligned}$$

where $L_{f,K_1}(t)=L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)$.

By (1.5),

$$\begin{aligned}
\int\limits_I L_f(t)\,dt&=\int\limits_I\big(L_{f,K_1}(t)+\alpha_0 m_{f,K_1}(t)\big)\,dt\\
&=\int\limits_I\big[L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)+\alpha_0(m_{f_0,K_1}(t)+m_{\delta f,K_1}(t))\big]\,dt\\
&\le\alpha(\alpha_0+1)+\int\limits_I\big[L_{f_0,K_1}(t)+\alpha_0 m_{f_0,K_1}(t)\big]\,dt:=\alpha_2.
\end{aligned}$$

Taking into account this estimate, we obtain $\alpha_k(b)\le((s+1)\alpha_2)^k/k!$. Consequently, there exists a positive integer k_1 such that $\alpha_{k_1}(b)<1$. Therefore, the k_1st iteration of the family (1.22) is contracting. By Theorem 1.1, the mapping (1.22) has a unique fixed point for each μ. Hence it follows that the equation (1.20) with the initial condition (1.21) has a unique solution $z(t;\mu)$, $t\in I$.

Let us prove that the mapping $F^k(z(\,\cdot\,;\mu_0);\,\cdot\,):\Lambda_0\to C(I,\mathbb{R}^n)$ is continuous at the point $\mu=\mu_0$ for an arbitrary $k=1,2,\dots$. Towards this end, it suffices to show that if the sequence $\mu_i=(t_{0i},\tau_{1i},\dots,\tau_{si},x_{0i},\varphi_i,f_i)\in A_0$, $i=1,2,\dots$, where $f_i=f_0+\delta f_i$, converges to $\mu_0=(t_{00},\tau_{10},\dots,\tau_{s0},x_{00},\varphi_0,f_0)$, i.e. if

$$\lim_{i\to\infty}\Big(|t_{0i}-t_{00}|+\sum_{j=1}^s|\tau_{ji}-\tau_{j0}|+|x_{0i}-x_{00}|+\|\varphi_i-\varphi_0\|_{1_1}+H_0(\delta f_i;K_1)\Big)=0,$$

then

$$\lim_{i\to\infty}F^k(z(\,\cdot\,;\mu_0);\mu_i)=F^k(z(\,\cdot\,;\mu_0);\mu_0)=z(\,\cdot\,;\mu_0).\tag{1.26}$$

We now prove the relation (1.26) by induction. Let $k=1$, then we have

$$\begin{aligned}
&|\zeta_1^i(t)-z_0(t)|\le|x_{0i}-x_{00}|\\
&\quad+\bigg|\int\limits_{t_{0i}}^t g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,z_0)(\xi)\,d\xi-\int\limits_{t_{00}}^t g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\,d\xi\bigg|\le\alpha_1^i+\alpha_2^i(t),
\end{aligned}\tag{1.27}$$

where

$$\zeta_1^i(t)=\zeta_1(t;z_0,\mu_i),\quad z_0(t)=z_0(t;\mu_0),\quad g_i=\chi f_i=g_0+\delta g_i,\quad g_0=\chi f_0,\quad \delta g_i=\chi\delta f_i;$$

$$\alpha_1^i=|x_{0i}-x_{00}|+\Bigg|\int\limits_{t_{0i}}^{t_{00}}\big|g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\big|\,d\xi\Bigg|,$$

$$\alpha_2^i(t)=\Bigg|\int\limits_{t_{0i}}^{t}\big[g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,z_0)(\xi)-g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\big]\,d\xi\Bigg|.$$

According to (1.3),

$$\alpha_1^i\le|x_{0i}-x_{00}|+\Bigg|\int\limits_{t_{0i}}^{t_{00}}m_{f_0,K_1}(t)\,dt\Bigg|,$$

therefore,

$$\lim_{i\to\infty}\alpha_1^i=0. \tag{1.28}$$

After elementary transformation we obtain

$$\begin{aligned}
\alpha_2^i(t)\le&\Bigg|\int\limits_{t_{0i}}^{t}\big[g_0(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,z_0)(\xi)-g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\big]\,d\xi\Bigg|\\
&+\Bigg|\int\limits_{t_0}^{t}\big[\delta g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,z_0)(\xi)-\delta g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_0,z_0)(\xi)\big]\,d\xi\Bigg|\\
&+\Bigg|\int\limits_{t_{0i}}^{t}\delta g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_0,z_0)(\xi)\,d\xi\Bigg|\\
\le&\sum_{j=1}^{s}(\alpha_{2j}^i+\alpha_{3j}^i)+\alpha_4^i(t),
\end{aligned} \tag{1.29}$$

where

$$\alpha_{2j}^i=\int\limits_I L_{f_0}(\xi)\big|h(t_{0i},\varphi_i,z_0)(\xi-\tau_{ji})-h(t_{00},\varphi_0,z_0)(\xi-\tau_{j0})\big|\,d\xi,$$

$$\alpha_{3j}^i=\int\limits_I L_{\delta f_i}(\xi)\big|h(t_{0i},\varphi_i,z_0)(\xi-\tau_{ji})-h(t_{0i},\varphi_0,z_0)(\xi-\tau_{j0})\big|\,d\xi,$$

$$\alpha_4^i(t)=\Bigg|\int\limits_{t_{0i}}^{t}\delta g_i(t_{0i},\tau_{0i},\dots,\tau_{si},\varphi_0,z_0)(\xi)\,d\xi\Bigg|,\quad \delta g_i=g_i-g_0.$$

We now estimate α_{2j}^i, α_{3j}^i and $\alpha_4^i(t)$. We have

$$\begin{aligned}
\alpha_{2j}^i\le&\int\limits_I L_{f_0}(\xi)\big|h(t_{0i},\varphi_i,z_0)(\xi-\tau_{ji})-h(t_{0i},\varphi_0,z_0)(\xi-\tau_{ji})\big|\,d\xi\\
&+\int\limits_I L_{f_0}(t)\big|h(t_{0i},\varphi_0,z_0)(\xi-\tau_{ji})-h(t_{00},\varphi_0,z_0)(\tau_{j0}(t))\big|\,d\xi\\
\le&\int\limits_I L_{f_0}(\xi)\big|h(t_{0i},\varphi_i-\varphi_0,0)(\xi-\tau_{ji})\big|\,d\xi
\end{aligned}$$

$$+\int\limits_I L_{f_0}(\xi)\big|h(t_{0i},\varphi_0,z_0)(\xi-\tau_{ji})-h(t_{00},\varphi_0,z_0)(\xi-\tau_{ji})\big|\,d\xi$$
$$+\int\limits_I L_{f_0}(\xi)\big|h(t_{00},\varphi_0,z_0)(\xi-\tau_{ji})-h(t_{00},\varphi_0,z_0)(\xi-\tau_{j0})\big|\,d\xi$$
$$\le \|\varphi_i-\varphi_0\|_{I_1}\int\limits_I L_{f_0}(\xi)\,d\xi+\alpha^i_{21j}+\alpha^i_{22j}.$$

Introduce the notation

$$\xi_{0ji}=\min\{t_{00}+\tau_{ji},t_{0i}+\tau_{ji}\},\quad \xi_{1ji}=\max\{t_{00}+\tau_{ji},t_{0i}+\tau_{ji}\}.$$

It is easy to see that

$$\alpha^i_{21j}=\int\limits_{\xi_{0ji}}^{\xi_{1ji}} L_{f_0}(\xi)\big|h(t_{0i},\varphi_0,z_0)(\xi-\tau_{ji})-h(t_{00},\varphi_0,z_0)(\xi-\tau_{ji})\big|\,d\xi$$

and

$$\lim_{i\to\infty}(\xi_{1ji}-\xi_{0ji})=0.$$

Consequently, $\alpha^i_{21j}\to 0$.

Introduce the notation

$$\nu_{0ji}=\min\{t_{00}+\tau_{ji},t_{00}+\tau_{j0}\},\quad \nu_{1ji}=\max\{t_{00}+\tau_{ji},t_{00}+\tau_{j0}\}.$$

For α^i_{22j}, we have

$$\alpha^i_{22j}=\int\limits_{\nu_{0ji}}^{\nu_{1ji}} L_{f_0}(\xi)\big|h(t_{00},\varphi_0,z_0)(\xi-\tau_{ji})-h(t_{00},\varphi_0,z_0)(\xi-\tau_{j0})\big|\,d\xi.$$

Thus, $\alpha^i_{22j}\to 0$. Consequently,

$$\alpha^i_{2j}\to 0. \tag{1.30}$$

Further,

$$\alpha^i_{3j}\le\int\limits_I L_{\delta f_i}(\xi)\big|\varphi_i(\xi-\tau_{ji})-\varphi_0(\xi-\tau_{ji})\big|\,d\xi\le\|\varphi_i-\varphi_0\|_{I_1}\int\limits_I L_{\delta f_i}(\xi)\,d\xi\longrightarrow 0. \tag{1.31}$$

We now estimate $\alpha^i_4(t)$. The function $\varphi_0(\xi)$, $\xi\in I_1$, is piecewise-continuous with a finite number of discontinuity points of the first kind, i.e., there exist subintervals (θ_p,θ_{p+1}), $p=\overline{1,m}$, where the function $\varphi_0(\xi)$ is continuous, with

$$\theta_1=\widehat{\tau},\ \ \theta_{m+1}=b,\ \ I_1=\bigcup_{p=1}^{m-1}[\theta_p,\theta_{p+1})\cup[\theta_m,\theta_{m+1}].$$

On the interval I_1, we define the continuous functions $z_i(\xi)$, $i=\overline{1,m+1}$, as follows:

$$z_1(\xi)=\varphi_{01}(\xi),\dots,z_m(\xi)=\varphi_{0m}(\xi),\quad z_{m+1}(\xi)=\begin{cases} z_0(a), & \xi\in[\widehat{\tau},a),\\ z_0(\xi), & \xi\in I,\end{cases}$$

where

$$\varphi_{0p}(\xi)=\begin{cases}\varphi_0(\theta_p+), & \xi\in[\widehat{\tau},\theta_p],\\ \varphi_0(\xi), & \xi\in(\theta_p,\theta_{p+1}),\\ \varphi_0(\theta_{p+1}-), & \xi\in[\theta_{p+1},b],\end{cases}\quad p=\overline{1,m}.$$

One can readily see that $\alpha_4^i(t)$ satisfies the following estimation:

$$\begin{aligned}
\alpha_4^i(t) &\le \sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\max_{t',t''\in I}\Big|\int_{t'}^{t''}\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{1i}),\dots,z_{m_s}(\xi-\tau_{si})\big)\,d\xi\Big| \\
&\le \sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\max_{t',t''\in I}\Big|\int_{t'}^{t''}\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{10}),\dots,z_{m_s}(\xi-\tau_{s0})\big)\,d\xi\Big| \\
&\quad+\sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\max_{t',t''\in I}\Big|\int_{t'}^{t''}\Big|\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{1i}),\dots,z_{m_s}(\xi-\tau_{si})\big) \\
&\qquad\qquad-\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{10}),\dots,z_{m_s}(\xi-\tau_{s0})\big)\Big|\,d\xi\Big| \\
&\le \sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\max_{t',t''\in I}\Big|\int_{t'}^{t''}\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{10}),\dots,z_{m_s}(\xi-\tau_{s0})\big)\,d\xi\Big| \\
&\quad+\sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\int_I L_{\delta f_i,K_1}(\xi)\sum_{j=1}^{s}\big|z_{m_j}(\xi-\tau_{ji})-z_{m_j}(\xi-\tau_{j0})\big|\,d\xi \\
&\le \sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\max_{t',t''\in I}\Big|\int_{t'}^{t''}\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{10}),\dots,z_{m_s}(\xi-\tau_{s0})\big)\,d\xi\Big| \\
&\quad+\sum_{m_1=1}^{m+1}\cdots\sum_{m_s=1}^{m+1}\sum_{j=1}^{s}\max_{\xi\in I}\big|z_{m_j}(\xi-\tau_{ji})-z_{m_j}(\xi-\tau_{j0})\big|\int_I L_{\delta g_i,K_1}(\xi)\,d\xi. \qquad (1.32)
\end{aligned}$$

Obviously,

$$H_0(\delta g_i;K_1)=H_0(\chi\delta f_i;K_1)\le H_0(\delta f_i;K_1)$$

(see (1.1)). Since $H_0(\delta f_i;K_1)\to 0$, as $i\to\infty$, we have

$$\lim_{i\to\infty}H_0(\delta g_i,K_1)=0.$$

This allows us to use Lemma 1.5 which, in its turn, implies that

$$\lim_{i\to\infty}\max_{t',t''\in I}\Big|\int_{t'}^{t''}\delta g_i\big(\xi,z_0(\xi),z_{m_1}(\xi-\tau_{10}),\dots,z_{m_s}(\xi-\tau_{s0})\big)\,d\xi\Big|=0\ \ \forall\, m_k\in\{1,m+1\},\ \ k=\overline{1,s}.$$

Moreover, it is clear that

$$\lim_{i\to\infty}\max_{t\in I}\big|z_{m_j}(\tau_{ji}(\xi))-z_{m_j}(\tau_{j0}(\xi))\big|=0.$$

The right-hand side of the inequality (1.32) consists of finitely many summands and, therefore,

$$\lim_{i\to\infty}\alpha_4^i(t)=0 \qquad (1.33)$$

uniformly in $t\in I$.

The conditions (1.30), (1.31) and (1.33) yield

$$\lim_{i\to\infty}\alpha_2^i(t)=0 \qquad (1.34)$$

uniformly in $t\in I$ (see (1.29)).

Taking into account (1.28) and (1.34), we see that (1.27) implies

$$\|\zeta_1^i-z_0\|_I=0.$$

The relation (1.26) is proved for $k=1$.

Let (1.26) hold for a certain $k>1$; we will prove it for $k+1$. Elementary transformations yield

$$\begin{aligned}
&|\zeta^i_{k+1}(t)-z_0(t)| \\
&\quad\le |x_{0i}-x_{00}|+\bigg|\int\limits_{t_{0i}}^{t} g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,\zeta^i_k)(\xi)\,d\xi-\int\limits_{t_{00}}^{t} g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\,d\xi\bigg| \\
&\quad\le |x_{0i}-x_{00}|+\bigg|\int\limits_{t_{0i}}^{t_{00}} g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\,d\xi\bigg| \\
&\quad+\bigg|\int\limits_{t_{0i}}^{t}\Big[g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,z_0)(\xi)-g_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,z_0)(\xi)\Big]\,d\xi\bigg| \\
&\quad+\bigg|\int\limits_{t_{0i}}^{t}\Big|g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,\zeta^i_k)(\xi)-g_i(t_{0i},\tau_{1i},\dots,\tau_{si},\varphi_i,z_0)(\xi)\Big|\,d\xi\bigg|=\alpha^i_1+\alpha^i_2(t)+\alpha^i_{4k}.
\end{aligned}$$

The quantities α^i_1 and $\alpha^i_2(t)$ have been estimated previously, and it remains to estimate α^i_{4k}. We have

$$\begin{aligned}
\alpha^i_{4k}&\le\int\limits_I L_{f_i}(\xi)\Big[\zeta^i_k(\xi)-z_0(\xi)|+\sum_{j=1}^{s}|h(t_{0i},0,\zeta^i_k-z_0)(\xi-\tau_{ji})|\Big]\,d\xi \\
&\le (s+1)\|\zeta^i_k-z_0\|_I\int\limits_I L_{f_i}(\xi)\,d\xi\le (s+1)\alpha_2\|\zeta^i_k-z_0\|_I.
\end{aligned}$$

Since

$$\lim_{i\to\infty}\|\zeta^i_k-z_0\|_I=0,$$

it follows that

$$\lim_{i\to\infty}\alpha^i_{4k}=0. \tag{1.35}$$

According to (1.28), (1.34) and (1.35), we have

$$\lim_{i\to\infty}\|\zeta^i_{k+1}-z_0\|_I=0.$$

The relation (1.26) is proved for every $k=1,2,\dots$.

Let the number $\delta_1>0$ be insomuch small that $[r_1-\delta_1,r_2+\delta_1]\subset I$ and $|z(t;\mu_0)-z(r_1;\mu_0)|\le\varepsilon_0/2$ for $t\in[r_1-\delta_1,r_1]$ and $|z(t;\mu_0)-z(r_2;\mu_0)|\le\varepsilon_0/2$ for $t\in[r_2,r_2+\delta_1]$.

From the uniqueness of the solution $z(t;\mu_0)$ we can conclude that $z(t;\mu_0)=y_0(t)$ for $t\in[r_1,r_2]$. Taking into account the above inequalities, we have

$$\Big(z(t;\mu_0),h\big(t_{00},\varphi_0,z(\,\cdot\,;\mu_0)\big)(t-\tau_{10}),\dots,h\big(t_{00},\varphi_0,z(\,\cdot\,;\mu_0)\big)(t-\tau_{s0})\Big)\in K^{s+1}\Big(\frac{\varepsilon_0}{2}\Big)\subset Q,\quad t\in[r_1-\delta_1,r_2+\delta_1].$$

Hence

$$\chi\Big(z_0(t),h\big(t_{00},\varphi_0,z(\,\cdot\,;\mu_0)\big)(t-\tau_{10}),\dots,h\big(t_{00},\varphi_0,z(\,\cdot\,;\mu_0)\big)(t-\tau_{s0})\Big)=1,\ \ t\in[r_1-\delta_1,r_2+\delta_1],$$

and the function $z(t;\mu_0)$ satisfies the equation

$$\dot y(t)=f_0\big(t,z_0(t),h(t_{00},\varphi,y)(t-\tau_{10}),\dots,h(t_{00},\varphi,y)(t-\tau_{s0})\big),\ \ t\in[r_1-\delta_1,r_2+\delta_1],$$

and the initial condition

$$y(t_{00})=x_{00}.$$

Therefore,

$$y(t;\mu_0)=z(t;\mu_0),\ \ t\in[r_1-\delta_1,r_2+\delta_1].$$

According to the fixed point Theorem 1.1, for $\varepsilon_0/2$ there exists a number $\delta_0\in(0,\varepsilon_0)$ such that a solution $z(t;\mu)$ satisfying the condition

$$|z(t;\mu)-z(t;\mu_0)|\le\frac{\varepsilon_0}{2}\,,\ \ t\in I,$$

corresponds to each element $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$.

Therefore, for $t\in[r_1-\delta_1,r_2+\delta_1]$,

$$z(t;\mu)\in K(\varepsilon_0)\ \ \forall\,\mu\in V(\mu_0;K_1,\delta_0,\alpha).$$

Taking into account that $\varphi(t)\in K(\varepsilon_0)$, we can see that for $t\in[r_1-\delta_1,r_2+\delta_1]$,

$$\chi\Big(z(t;\mu),h\big(t_0,\varphi,z(\,\cdot\,;\mu)\big)(t-\tau_1),\dots,h\big(t_0,\varphi,z(\,\cdot\,;\mu)\big)(t-\tau_s)\Big)=1\ \ \forall\,\mu\in V(\mu_0;K_1,\delta_0,\alpha).$$

Hence the function $z(t;\mu)$ satisfies the equation (1.16) and the condition (1.17), i.e.,

$$y(t;\mu)=z(t;\mu)\in K_1,\ \ t\in[r_1-\delta_1,r_2+\delta_1],\ \ \mu\in V(\mu_0;K_1,\delta_0,\alpha). \tag{1.36}$$

The first part of Theorem 1.7 is proved. By Theorem 1.1, for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that for each $\mu\in V(\mu_0;K_1,\delta_2,\alpha)$,

$$|z(t;\mu)-z(t;\mu_0)|\le\varepsilon,\ \ t\in I,$$

whence, using (1.36), we obtain (1.19). □

Proof of Theorem 1.2. In Theorem 1.7, let $r_1=t_{00}$ and $r_2=t_{10}$. Obviously, the solution $x_0(t)=x(t;\mu_0)$ on the interval $[t_{00},t_{10}]$ satisfies the following equation:

$$\dot y(t)=f_0(t_0,\tau_{10},\dots,\tau_{s0},\varphi_0,y)(t).$$

Therefore, in Theorem 1.7, in the capacity of the solution $y_0(t)=y(t;\mu_0)$ we can take the function $x_0(t)$, $t\in[t_{00},t_{10}]$.

By Theorem 1.7, there exist the numbers $\delta_i>0$, $i=0,1$, and for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that the solution $y(t;\mu)$, $t\in[t_{00}-\delta_1,t_{10}+\delta_1]$, corresponds to each $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$. Moreover, the following conditions hold:

$$\begin{cases}\varphi(t)\in K_1, & t\in I_1;\ \ y(t;\mu)\in K_1,\\ |y(t:\mu)-y(t;\mu_0)|\le\varepsilon, & t\in[t_{00}-\delta_1,t_{10}+\delta_1],\\ \mu\in V(\mu_0;K_1,\delta_2,\alpha).\end{cases} \tag{1.37}$$

For an arbitrary $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$, the function

$$x(t;\mu)=\begin{cases}\varphi(t), & t\in[\widehat\tau,t_0),\\ y(t;\mu), & t\in[t_0,t_1+\delta_1],\end{cases}$$

is the solution corresponding to μ. Moreover, if $t\in[\theta,t_{10}+\delta_1]$, then $x(t;\mu_0)=y(t;\mu_0)$ and $x(t;\mu)=y(t;\mu)$. Taking into account (1.37), we see that this implies 1.1 and 1.2. It is not difficult to note that for an arbitrary $\mu\in V(\mu_0;K_1,\delta_2,\alpha)$, we have

$$\begin{aligned}\int\limits_{\widehat\tau}^{t_{10}+\delta_1}|x(t;\mu)-x(t;\mu_0)|\,dt&=\int\limits_{\widehat\tau}^{\theta_0}|\varphi(t)-\varphi_0(t)|\,dt+\int\limits_{\theta_0}^{\theta}|x(t;\mu)-x(t;\mu_0)|\,dt+\int\limits_{\theta}^{t_{10}+\delta_1}|x(t;\mu)-x(t;\mu_0)|\,dt\\ &\le\|\varphi-\varphi_0\|_{I_1}(b-\widehat\tau)+N|t_0-t_{00}|+\max_{t\in[\theta,t_{10}+\delta_1]}|x(t;\mu)-x(t;\mu_0|((b-\widehat\tau),\end{aligned}$$

where $\theta_0=\min\{t_0,t_{00}\}$, $N=\sup\{|x'-x''|:x',x''\in K_1\}$.

By 1.1 and 1.2, this inequality implies 1.3. □

1.4 Proof of Theorem 1.4

To each element $w \in \Lambda_1$ we put in correspondence the functional differential equation

$$\dot y(t) = \phi(t_0, \varphi, \tau_1, \dots, \tau_s, y, u)(t) = \phi\big(t, y(t), h(t_0, \varphi, y)(t - \tau_1), \dots, h(t_0, \varphi, y)(t - \tau_s), u(t)\big) \tag{1.38}$$

with the initial condition (1.17).

Theorem 1.8. *Let $y_0(t) = y(t; w_0)$, $w_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ be defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

1.16. *There exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\begin{aligned} w = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, u) &\in \widehat V(w_0; \delta_0) \\ &= B(t_{00}; \delta_0) \times V(\tau_{10}; \delta_0) \times \dots \times B(\tau_{s0}; \delta_0) \times B(x_{00}; \delta_0) \times B(\varphi_0; \delta_0) \times V_2(u_0; \delta_0) \end{aligned}$$

there corresponds the solution $y(t; w)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and satisfying the condition $y(t; w) \in K_1$.

1.17. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$|y(t; w) - y(t; w_0)| \le \varepsilon \ \ \forall t \in [r_1 - \delta_1, r_2 + \delta_1]$$

holds for any $w \in \widehat V(w_0; \delta_2)$.

Proof. We rewrite the equation (1.38) in the form

$$\dot y(t) = \phi_0(t_0, \varphi, \tau_1, \dots, \tau_s, y)(t) + \delta\phi_u(t_0, \varphi, \tau_1, \dots, \tau_s, y)(t),$$

where

$$\begin{aligned} \phi_0(t, x, x_1, \dots, x_s) &= \phi(t, x, x_1, \dots, x_s, u_0(t)) \in E_f, \\ \delta\phi_u(t, x, x_1, \dots, x_s) &= \phi(t, x, x_1, \dots, x_s, u(t)) - \phi_0(t, x, x_1, \dots, x_s) \in E_f. \end{aligned}$$

Let $\widehat\delta_0 > 0$ be a number insomuch small that $B(u_0; \widehat\delta_0) \subset \Omega$. There exists a compact set $\widehat U \subset U_0$ such that any function from the neighborhood $B(u_0; \widehat\delta_0)$ takes its values in $\widehat U$.

Let $K \subset O$ be a compact set. There exists a function $L_K(t) \in L_1(I, \mathbb{R}_+)$ such that for almost all $t \in I$, the inequality

$$\begin{gathered} \big|\phi(t, x', x_1', \dots, x_s'', u') - \phi(t, x'', x_1', \dots, x_s'', u'')\big| \le L_K(t)\Big[|x' - x''| + \sum_{i=1}^{s} |x_i' - x_i''| + |u' - u''|\Big] \\ \forall x', x'' \in K, \ \ \forall x_i', x_i'' \in K, \ \ i = \overline{1, s}, \ \ \forall u', u'' \in \widehat U \end{gathered}$$

holds. Hence

$$\begin{gathered} |\delta\phi_u(t, x, x_1, \dots, x_s)| \le L_K(t)|u(t) - u_0(t)| \le \widehat\delta_0 L_K(t) \ \ \forall x_i \in K, \ \ i = \overline{1, s}, \ \ \forall u \in B(u_0; \widehat\delta_0), \\ \big|\delta\phi_u(t, x', x_1', \dots, x_s') - \delta\phi_u(t, x'', x_1'', \dots, x_s'')\big| \le 2L_K(t)\Big[|x' - x''| + \sum_{i=1}^{s} |x_i' - x_i''|\Big] \\ \forall x', x'' \in K, \ \ \forall x_i', x_i'' \in K, \ \ i = \overline{1, s}. \end{gathered}$$

It is easy to see that the inclusions $\{\delta\phi_u(t, x, x_1, \dots, x_s) : \ u \in B(u_0; \delta)\} \subset W(K; \alpha)$ and $\{\delta\phi_u(t, x, x_1, \dots, x_s) : \ u \in B(u_0; \delta)\} \subset V_{K, \widehat\delta_1}$ hold for $\delta \in (0, \widehat\delta_0]$, where

$$\alpha = (2 + \widehat\delta_0) \int_I L_K(t)\, dt, \quad \widehat\delta_1 = \delta \int_I L_f(t)\, dt.$$

We can now apply Theorem 1.7 which, in its turn, proves Theorem 1.8. □

Proof of Theorem 1.4. In Theorem 1.8, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t) = x(t; w_0)$ satisfies on the interval $[t_{00}, t_{10}]$ the following equation:

$$\dot{y}(t) = \phi(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, y, u_0)(t).$$

Therefore, in Theorem 1.8, we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$ as the solution $y_0(t) = y(t; w_0)$. Then the proof of the theorem completely coincides with that of Theorem 1.2; for this purpose, it suffices to replace everywhere the element μ by the element w and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\widehat{V}(w_0; \delta_0)$. □

2 Variation formulas of solutions for equations with the discontinuous initial condition

2.1 Auxiliary assertions

Consider the set of functions $f = (f^1, \dots, f^n)^\top : I \times O^{s+1} \to \mathbb{R}^n$ satisfying the following conditions: for almost all $t \in I$, the function $f(t, \cdot) : O^{s+1} \to \mathbb{R}^n$ is continuously differentiable; for every $(x, x_1, \dots, x_s) \in O^{s+1}$, the functions $f(t, x, x_1, \dots, x_s), f_x(t, \cdot), f_{x_i}(t, \cdot)$, $i = \overline{1, s}$, where $x = (x^1, \dots, x^n)^\top$, $x_i = (x_i^1, \dots, x_i^n)^\top$, are measurable on I; for any such function f and any compact set $K \subset O$, there exists a function $m_{f,K}(t) \in L_1(I, \mathbb{R}_+)$ such that for any $(x, x_1, \dots, x_s) \in K^{s+1}$ and for almost all $t \in I$,

$$|f(t, x, x_1, \dots, x_s)| + |f_x(t, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, \cdot)| \le m_{f,K}(t).$$

The classes of such equivalent functions compose a vector space, which will be denoted by $E_f^{(1)}$; these classes are also called the functions and they will likewise be denoted by f.

Lemma 2.1 ([6, p. 80])**.** *Let $K \subset O$ be a compact set and let $f \in E_f^{(1)}$. Then*

$$\sup\Big\{|f(t, x, x_1, \dots, x_s)| + |f_x(t, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, \cdot)| : \ (x, x_1, \dots, x_s) \in K^{s+1}\Big\} \in L_1(I, \mathbb{R}_+).$$

Lemma 2.2. *The inclusion*

$$E_f^{(1)} \subset E_f \tag{2.1}$$

holds.

Proof. Let $f \in E_f^{(1)}$ and let $K_0 \subset O$ be an arbitrary compact set. To prove the inclusion (2.1), it suffices to show that there exists a function $L_{f,K_0}(t) \in L_1(I, \mathbb{R}_+)$ such that for almost all $t \in I$,

$$\big|f(t, x', x'_1, \dots, x'_s) - f(t, x'', x''_1, \dots, x''_s)\big| \le L_{f,K_0}(t)\Big\{|x' - x''| + \sum_{i=1}^{s} |x'_i - x''_i|\Big\}$$
$$\forall\, x', x'' \in K_0, \ \ \forall\, x'_i, x''_i \in K_0, \ \ i = \overline{1, s}.$$

Introduce the function $g = (g^1, \dots, g^n)^\top = \chi f = (\chi f^1, \dots, \chi f^n)^\top$ (see (1.1) and (1.2)). Clearly, for $(x, x_1, \dots, x_s) \notin K_1^{s+1}$, we have

$$|g_x(t, x, x_1, \dots, x_s)| + \sum_{i=1}^{s} |g_{x_i}(t, \cdot)| = 0, \ \ i = \overline{1, s}. \tag{2.2}$$

Let $(x, x_1, \dots, x_s) \in K_1^{s+1}$. It is not difficult to see that the relations

$$|g_x| = \Big[\sum_{k,j=1}^{n} |g^k_{x^j}|^2\Big]^{\frac{1}{2}} \le \sum_{k,j=1}^{n} \Big|(\chi f^k)_{x^j}\Big|,$$
$$|g_{x_i}| = \Big[\sum_{k,j=1}^{n} |g^k_{x_i^j}|^2\Big]^{\frac{1}{2}} \le \sum_{k,j=1}^{n} \Big|(\chi f^k)_{x_i^j}\Big|,$$

where
$$(\chi f^k)_{x^j} = \frac{\partial}{\partial x^j}\,(\chi f^k),$$
are valid. We have
$$|(\chi f^k)_{x^j}| \le |f^k|\,|\chi_{x^j}| + |f^k_{x^j}| \le |f|\,|\chi_{x^j}| + |f_x| \le m_{f,K_1}(t)(\alpha_0+1),$$
$$|(\chi f^k)_{x_i^j}| \le |f^k|\,|\chi_{x_i^j}| + |f^k_{x_i^j}| \le |f|\,|\chi_{x_i^j}| + |f_{x_i}| \le m_{f,K_1}(t)(\alpha_0+1),\ \ i=\overline{1,s},$$

where
$$\alpha_0 = \sup\Big\{|\chi_x(x,x_1,\dots,x_s)| + \sum_{i=1}^{s}|\chi_{x_i}(\,\cdot\,)| :\ \ (x,x_1,\dots,x_s)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n\Big\},$$
$$m_{f,K_1}(t) = \sup\Big\{|f(t,x,x_1,\dots,x_s)| + |f_x(t,\,\cdot\,)| + \sum_{i=1}^{s} f_{x_i}(t,\,\cdot\,)| :\ \ (x,x_1,\dots,x_s)\in K_1^{s+1}\Big\}$$
(see (2.1)). Thus, for $\forall\,(t,x,x_1,\dots,x_s)\in I\times K_1^{s+1}$, we get
$$|g_x(t,x,x_1,\dots,x_s)| + \sum_{i=1}^{s}|g_{x_i}(t,\,\cdot\,)| \le m_{g,K_1}(t),\ \ i=\overline{1,s}, \tag{2.3}$$
where
$$m_{g,K_1}(t) = n^2(s+1)(\alpha_0+1)m_{f,K_1}(t).$$
It is clear that (2.3) is valid for $(t,x,x_1,\dots,x_s)\in I\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n$, as well (see (2.2)). Let $(x',x_1',\dots,x_s')$ and $(x'',x_1'',\dots,x_s'')$ be arbitrary points in K_0^{s+1}. Then (see (1.1)) we have
$$\begin{aligned}
\big|f(t,x',x_1',\dots,x_s') - f(t,x'',x_1'',\dots,x_s'')\big| &= \big|g(t,x',x_1',\dots,x_s') - g(t,x'',x_1'',\dots,x_s'')\big|\\
&\le \Big|\int_0^1 \frac{d}{d\theta}\,g\big(t,x''+\theta(x'-x''),x_1''+\theta(x_1'-x_1''),\dots,x_s''+\theta(x_s'-x_s'')\big)\,d\theta\Big|\\
&\le \int_0^1 \Big[\big|g_x\big(t,x''+\theta(x'-x''),x_1''+\theta(x_1'-x_1''),\dots,x_s''+\theta(x_s'-x_s'')\big)\big|\,|x'-x''|\\
&\quad+\sum_{i=1}^{s}\big|g_{x_i}\big(t,x''+\theta(x'-x''),x_1''+\theta(x_1'-x_1''),\dots,x_s''+\theta(x_s'-x_s'')\big)\big|\,|x_i{}''-x_i''|\Big]\,d\theta\\
&\le m_{g,K_1}(t)\Big[|x'-x''| + \sum_{i=1}^{s}|x_i{}''-x_i''|\Big].
\end{aligned}$$
Therefore, as $L_{f,K_0}(t)$ we can take $m_{g,K_1}(t)$. □

Consider now the linear delay functional differential equation
$$\dot x(t) = A(t)x(t) + \sum_{i=1}^{s} A_i(t)x_i(t-\tau_i) + f(t) \tag{2.4}$$
with the discontinuous initial condition
$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0. \tag{2.5}$$
Here, $A(t)$, $A_i(t)$, $i=\overline{1,s}$, are the integrable matrix functions of dimension $n\times n$; $t_0\in[a,b)$, $\tau_i\in[\theta_{i1},\theta_{i2}]$, $i=\overline{1,s}$, are fixed numbers; $\varphi\in\mathrm{PC}(I_1,\mathbb{R}^n)$ is a fixed initial function and $x_0\in\mathbb{R}^n$ is a fixed initial vector.

The equation (2.4) with the initial condition (2.5) has a unique solution $x(t)$ defined on $[\widehat{\tau}, b]$ (see Definition 1.1).

For every $t \in (a, b]$, on the interval $[a, t]$, let us consider the following matrix functional differential equation with the advanced arguments:

$$Y_\xi(\xi; t) = -Y(\xi; t)A(\xi) - \sum_{i=1}^{s} Y(\xi + \tau_i; t)A_i(\xi + \tau_i), \ \ \xi \in [a, t], \tag{2.6}$$

with the initial condition

$$Y(t; t) = \Upsilon, \ \ Y(\xi; t) = \Theta, \ \ \xi \in (t, \widehat{\tau}_1], \tag{2.7}$$

where Υ is the identity matrix and Θ is the zero matrix.

For every $t \in (a, b]$, the equation (2.6) with the discontinuous initial condition (2.7) has a unique solution $Y(\xi; t)$ defined on $[a, \widehat{\tau}_1]$ (see Definition 1.3).

Lemma 2.3 (Cauchy formula)**.** *The solution of the equation* (2.4) *with the initial condition* (2.5) *can be represented on the interval* $[t_0, b]$ *by the following formula:*

$$x(t) = Y(t_0; t)x_0 + \sum_{i=1}^{s} \int_{t_0-\tau_i}^{t_0} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)\varphi(\xi)\, d\xi + \int_{t_0}^{t} Y(\xi; t)f(\xi)\, d\xi, \tag{2.8}$$

where $Y(\xi; t)$ *is a solution of the equation* (2.6) *with the initial condition* (2.7)*.*

Proof. On the interval $[t_0, t]$, where $\xi \in (t_0, b]$, consider the equation

$$\dot{x}(\xi) = A(\xi)x(\xi) + \sum_{i=1}^{s} A_i(\xi)x_i(\xi - \tau_i) + f(\xi) \tag{2.9}$$

with the initial condition

$$x(\xi) = \varphi(\xi), \ \ \xi \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0.$$

Multiplying the equation (2.9) by the matrix function $Y(\xi; t)$ and integrating in $\xi \in [t_0, t]$, we obtain

$$\int_{t_0}^{t} Y(\xi; t)\dot{x}(\xi)\, d\xi = \int_{t_0}^{t} Y(\xi; t)\Big[A(\xi)x(\xi) + \sum_{i=1}^{s} A_i(\xi)x(\xi - \tau_i)\Big]\, d\xi + \int_{t_0}^{t} Y(\xi; t)f(\xi)\, d\xi. \tag{2.10}$$

The integration by pats on the left-hand side of (2.10) with regard for $Y(t; t) = \Upsilon$ yields

$$\int_{t_0}^{t} Y(\xi; t)\dot{x}(\xi)\, d\xi = x(t) - Y(t_0; t)x_0 - \int_{t_0}^{t} Y_\xi(\xi; t)x(\xi)\, d\xi. \tag{2.11}$$

Further,

$$\begin{aligned}\int_{t_0}^{t} Y(\xi; t)A_i(\xi)x(\xi - \tau_i)\, d\xi &= \int_{t_0-\tau_i}^{t-\tau_i} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)x(\xi)\, d\xi \\ &= \int_{t_0-\tau_i}^{t_0} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)\varphi(\xi)\, d\xi + \int_{t_0}^{t} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)x(\xi)\, d\xi \end{aligned} \tag{2.12}$$

(see (2.7)). Taking into account (2.11) and (2.12), from (2.10) we find that

$$\begin{aligned} x(t) &= Y(t_0; t)x_0 + \sum_{i=1}^{s} \int_{t_0-\tau_i}^{t_0} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)\varphi(\xi)\, d\xi \\ &+ \int_{t_0}^{t} \Big[Y_\xi(\xi; t) + A(\xi)x(\xi) + \sum_{i=1}^{s} Y(\xi + \tau_i; t)A_i(\xi + \tau)\Big]x(\xi)\, d\xi + \int_{t_0}^{t} Y(\xi; t)f(\xi) d\xi. \end{aligned}$$

$Y(\xi; t)$ satisfies the equation (2.6) and, therefore, the latter relation implies the formula (2.8). □

Lemma 2.4 (Gronwall–Bellman's inequality)**.** *Let $v(t) \geq 0, t \in [t_0, b]$, be a continuous scalar-valued function, $m(t) \in L_1(I, \mathbb{R}_+)$, and let the inequality*

$$v(t) \leq c + \int_{t_0}^{t} m(\xi)v(\xi)\, d\xi,$$

where $c \geq 0$, hold. Then

$$v(t) \leq c \exp\bigg(\int_{t_0}^{t} m(\xi)\, d\xi\bigg), \quad t \in [t_0, b].$$

Lemma 2.5. *Let $t' \in (a, b]$. For an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that the inequality*

$$|Y(\xi; t') - Y(\xi; t'')| \leq \varepsilon \ \ \forall \xi \in [a, \underline{t}],$$

holds for arbitrary $t'' \in [t' - \delta, t' + \delta] \cap I$, where $\underline{t} = \min\{t', t''\}$.

Lemma 2.5 is a simple consequence of Theorem 1.6.

Lemma 2.6. *The matrix function $Y(\xi; t)$ is continuous on the set $\Pi = \{(\xi, t) : \ a < \xi < t, \ t \in (a, b)\}$.*

Proof. Let $(\xi, t) \in \Pi$ be a fixed point. There exists $\delta_1 > 0$ such that $\xi + \Delta\xi < \min\{t + \Delta t, t\}$ and $t + \Delta t < b$ for $|\Delta\xi| \leq \delta_1, |\Delta t| \leq \delta_1$, i.e., $(\xi + \Delta\xi, t + \Delta t) \in \Pi$.

Using Lemma 2.5, we see that for each $\varepsilon > 0$, there exists $\delta_2 \in (0, \delta_1)$ such that for arbitrary $\Delta\xi$ and Δt satisfying the conditions $|\Delta\xi| \leq \delta_2$ and $|\Delta t| \leq \delta_2$, the inequality

$$\big|Y(\xi + \Delta\xi; t + \Delta t) - Y(\xi + \Delta\xi; t)\big| \leq \frac{\varepsilon}{2}$$

holds.

On the other hand, the function $Y(\varsigma; t)$ is continuous with respect to $\varsigma \in [a, t]$, i.e., there exists a number $\delta_3 \in (0, \delta_1)$ such that

$$\big|Y(\xi + \Delta\xi; t) - Y(\xi; t)\big| \leq \frac{\varepsilon}{2}, \quad |\Delta\xi| \leq \delta_3.$$

Hence, for $|\Delta\xi| \leq \delta$, $|\Delta t| \leq \delta$, $\delta = \{\delta_2, \delta_3\}$, we have

$$\begin{aligned} &\big|Y(\xi + \Delta\xi; t + \Delta t) - Y(\xi; t)\big| \\ &\qquad \leq \big|Y(\xi + \Delta\xi; t + \Delta t) - Y(\xi + \Delta\xi; t)\big| + \big|Y(\xi + \Delta\xi; t) - Y(\xi; t)\big| \leq \varepsilon. \end{aligned}$$ □

2.2 Formulation of main results

To each element

$$\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)} = [a, b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times E_f^{(1)}$$

we assign the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_s)\big) \tag{2.13}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{2.14}$$

Definition 2.1. Let $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (2.13) with the initial condition (2.14), or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (2.14) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (2.13) a.e. on $[t_0, t_1]$.

Let $x_0(t)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda^{(1)}$ and defined on the interval $[\widehat{\tau}, t_{10}]$, where $t_{00}, t_{10} \in (a, b)$, $t_{00} < t_{10}$ and $\tau_{i0} \in (\theta_{1i}, \theta_{2i})$, $i = \overline{1, s}$.

In the space $E^{(1)}_{\delta\mu} = E^{(1)}_{\mu} - \mu_0$ with the elements $\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f)$, where $E^{(1)}_{\mu} = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E^{(1)}_f$, we introduce the set of variations

$$\Im^{(1)} = \Big\{ \delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E^{(1)}_{\delta\mu} : \ |\delta t_0| \le \gamma, \ |\delta\tau_i| \le \gamma, \ i = \overline{1, s},$$
$$|\delta x_0| \le \gamma, \ \|\delta\varphi\|_I \le \gamma, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \le \gamma, \ i = \overline{1, k} \Big\},$$

where $\gamma > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = \overline{1, k}$, are fixed functions.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(1)}$, to the element $\mu_0 + \varepsilon\delta\mu$ there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ (see Lemma 2.8).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t) = \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t) \ \ \forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im^{(1)}. \tag{2.15}$$

Theorem 2.1. *Let the following conditions hold:*

2.1. $\tau_{s0} > \cdots > \tau_{10}$ *and* $t_{00} + \tau_{s0} < t_{10}$*;*

2.2. *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot{\varphi}_0(t)$ *is bounded;*

2.3. *the function* $f_0(w)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{s+1}$, *is bounded;*

2.4. *there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^-, \ \ w \in (a, t_{00}] \times O^{s+1},$$

where $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$*;*

2.5. *there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w^0_{1i}, w^0_{2i})} [f_0(w_{1i}) - f_0(w_{2i})] = f_{0i},$$

where $w_{1i}, w_{2i} \in (a, b) \times O^{s+1}$, $i = \overline{1, s}$,

$$w^0_{1i} = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$
$$x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big),$$
$$w^0_{2i} = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$
$$\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big).$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im^{(1)}_-$, *where* $\Im^{(1)}_- = \{\delta\mu \in \Im^{(1)} : \ \delta t_0 \le 0\}$, *we have*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu). \tag{2.16}$$

Here

$$\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0^- \delta t_0 + \beta(t; \delta\mu) \tag{2.17}$$

and

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta x_0-\Big[\sum_{i=1}^{s}Y(t_{00}+\tau_{i0};t)f_{0i}\Big]\delta t_0$$
$$-\sum_{i=1}^{s}\bigg[Y(t_{00}+\tau_{i0};t)f_{0i}+\int_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i$$
$$+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]\,d\xi,$$

where it is assumed that

$$\int_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi=\int_{t_{00}}^{t_{00}+\tau_{i0}}Y(\xi;t)f_{0x_i}[\xi]\dot{\varphi}_0(\xi-\tau_{i0})\,d\xi+\int_{t_{00}+\tau_{i0}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi.$$

Next, $Y(\xi;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t)=-Y(\xi;t)f_{0x}[\xi]-\sum_{i=1}^{s}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}],\ \ \xi\in[t_{00},t], \tag{2.18}$$

and the condition

$$Y(t;t)=\Upsilon,\ \ Y(\xi;t)=\Theta,\ \ \xi>t; \tag{2.19}$$

where

$$f_{0x_i}[\xi]=f_{0x_i}\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big),$$
$$\delta f[\xi]=\delta f\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big).$$

Some comments. The expression (2.17) is called the variation formula of the solution.

The addend

$$-\Big[Y(t_{00};t)f_0^-+\sum_{i=1}^{s}Y(t_{00}+\tau_{i0};t)f_{0i}\Big]\delta t_0$$

in the formula (2.17) is the effect of the discontinuous initial condition (2.14) and perturbation of the initial moment t_{00}.

The addend

$$-\sum_{i=1}^{s}\bigg[Y(t_{00}+\tau_{i0};t)f_{0i}+\int_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i$$

in the formula (2.17) is the effect of perturbations of the delays τ_{i0}, $i=\overline{1,s}$. The expression

$$Y(t_{00};t)\delta x_0+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$

in the formula (2.17) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$.

The addend

$$\int_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]\,d\xi$$

in the formula (2.17) is the effect of perturbation of the right-hand side of the equation

$$\dot{x}(t) = f_0\big(t, x(t), x(t-\tau_{10}), \dots, x(t-\tau_{s0})\big).$$

Next, it is clear that

$$\delta x(t;\delta\mu) = \delta x_0(t;\delta\mu) - \sum_{i=1}^{s} \delta x_i(t;\delta\mu), \ \ t \in [t_{10}-\delta_2, t_{10}+\delta_2],$$

where

$$\begin{aligned}\delta x_0(t;\delta\mu) &= Y(t_{00};t)[\delta x_0 - f_0^- \delta t_0] + \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi \\ &\quad + \int_{t_{00}}^{t} Y(\xi;t)\Big[-\sum_{i=1}^{s} f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\delta\tau_i + \delta f[\xi]\Big]\,d\xi, \\ \delta x_i(t;\delta\mu) &= Y(t_{00}+\tau_{i0};t) f_{0i}(\delta t_0 + \delta\tau_i).\end{aligned}$$

On the basis of the Cauchy formula (see Lemma 2.3), the function

$$\delta x_0(t) = \begin{cases} \delta\varphi(t), & t \in [\widehat{\tau}, t_{00}), \\ \delta x_0(t,\delta\mu), & t \in [t_{00}, t_{10}+\delta_2], \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\delta x(t-\tau_i) - \sum_{i=1}^{s} f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\delta\tau_i + \delta f[\xi]$$

with the initial condition

$$\delta x(t) = \delta\varphi(t), \ \ t \in [\widehat{\tau}, t_{00}), \ \ \delta x(t_{00}) = \delta x_0 - f_0[t]\delta t_0,$$

and the function

$$\delta x_i(t) = \begin{cases} 0, & t \in [\widehat{\tau}, t_{00}+\tau_{i0}), \\ Y(t_{00}+\tau_{i0};t) f_{0i}(\delta t_0 + \delta\tau_i), & t \in [t_{00}+\tau_{i0}, t_{10}+\delta_2], \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\delta x(t-\tau_i)$$

with the initial condition

$$\delta x(t) = 0, \ \ t \in [\widehat{\tau}, t_{00}+\tau_{i0}), \ \ \delta x(t_{00}+\tau_{i0}) = f_{0i}(\delta t_0 + \delta\tau_i).$$

Theorem 2.2. *Let the conditions* 2.1–2.3 *and* 2.5 *of Theorem* 2.1 *hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0} f_0(w) = f_0^+, \ \ w \in [t_{00}, b) \times O^{s+1}. \tag{2.20}$$

Then there exist numbers $\varepsilon_2 \in (0,\varepsilon_1)$ *and* $\delta_2 \in (0,\delta_1)$, *with* $t_{10}-\delta_2 > t_{00}+\tau_{s0}$ *such that for arbitrary* $(t,\varepsilon,\delta\mu) \in [t_{10}-\delta_2, t_{10}+\delta_2] \times (0,\varepsilon_2) \times \Im_+^{(1)}$, *where* $\Im_+^{(1)} = \{\delta\mu \in \Im^{(1)} : \ \delta t_0 \geq 0\}$, *the formula* (2.16) *holds. Here,*

$$\delta x(t;\delta\mu) = -Y(t_{00};t) f_0^+ \delta t_0 + \beta(t;\delta\mu).$$

Theorem 2.3. *Let the conditions* 2.1–2.5 *of Theorem* 2.1 *and the condition* (2.20) *hold. Moreover, $f_0^- = f_0^+ := \widehat{f}_0$. Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$, with $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im^{(1)}$, the formula* (2.16) *holds, where*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t)\widehat{f}_0 \delta t_0 + \beta(t; \delta\mu).$$

Theorem 2.3 is a corollary to Theorems 2.1 and 2.2.

Theorem 2.4. *Let the conditions* 2.1–2.4 *of Theorem* 2.1 *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}^-, \;\; w_{1i}, w_{2i} \in (a, t_{00} + \tau_{i0}] \times O^{s+1}, \;\; i = \overline{1, s},$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$, with $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_{--}^{(1)}$, where $\Im_{--}^{(1)} = \{\delta\mu \in \Im^{(1)} : \; \delta t_0 \leq 0, \; \delta\tau_i \leq 0, \; i = \overline{1, s}\}$, the formula (2.16) *holds. Here*

$$\delta x(t; \delta\mu) = -\Big[Y(t_{00}; t) f_0^- + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}^-\Big] \delta t_0 - \sum_{i=1}^{s} \big[Y(t_{00} + \tau_{i0}; t) f_{0i}^-\big] \delta\tau_i + \beta_1(t; \delta\mu),$$

where

$$\begin{aligned} \beta_1(t; \delta\mu) = Y(t_{00}; t)\delta x_0 - \sum_{i=1}^{s} \bigg[\int_{t_{00}}^{t} Y(\xi; t) f_{0x_i}[\xi] \dot{x}_0(\xi - \tau_{i0})\, d\xi\bigg] \delta\tau_i \\ + \sum_{i=1}^{s} \int_{t_{00} - \tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \delta\varphi(\xi)\, d\xi + \int_{t_{00}}^{t} Y(\xi; t) \delta f[\xi]\, d\xi. \end{aligned}$$

Theorem 2.4 can be proved by analogy to Theorem 2.1.

Theorem 2.5. *Let the conditions* 2.1–2.3 *of Theorem* 2.1 *and the condition* (2.20) *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}^+, \;\; w_{1i}, w_{2i} \in [t_{00} + \tau_{i0}, b) \times O^{s+1}, \;\; i = \overline{1, s},$$

Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$, with $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_{++}^{(1)}$, where $\Im_{++}^{(1)} = \{\delta\mu \in \Im^{(1)} : \; \delta t_0 \geq 0, \; \delta\tau_i \geq 0, \; i = \overline{1, s}\}$, the formula (2.16) *holds. Here*

$$\delta x(t; \delta\mu) = -\Big[Y(t_{00}; t) f_0^+ + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}^+\Big] \delta t_0 - \sum_{i=1}^{s} \big[Y(t_{00} + \tau_{i0}; t) f_{0i}^+\big] \delta\tau_i + \beta_1(t; \delta\mu).$$

Theorem 2.5 can be proved by analogy to Theorem 2.2.

2.3 Lemma on estimation of the increment of a solution with respect to the variation set $\Im_{-}^{(1)}$

To each element $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$, we assign the functional differential equation

$$\dot{y}(t) = f(t_0, \tau_1, \dots, \tau_s, \varphi, y)(t) \tag{2.21}$$

with the initial condition

$$y(t_0) = x_0 \tag{2.22}$$

(see (1.16)).

Definition 2.2. Let $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$. An absolutely continuous function $y(t) = y(t;\mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (2.21) with the initial condition (2.22), or a solution corresponding to the element μ and defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$, and the function $y(t)$ satisfies the equation (2.21) a.e. on $[r_1, r_2]$.

Remark 2.1. Let $y(t;\mu)$, $t \in [r_1, r_2]$, be a solution corresponding to the element $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$. Then the function

$$x(t;\mu) = h(t_0, \varphi, y(\,\cdot\,;\mu))(t), \quad t \in [\widehat{\tau}, r_2], \tag{2.23}$$

is a solution of the equation (2.13) with the initial condition (2.14) (see Definition 2.1 and (1.18)).

Lemma 2.7. *Let $y_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda^{(1)}$ and defined on $[r_1, r_2] \subset (a,b)$; let $t_{00} \in [r_1, r_2), \tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1,s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(1)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(1)}$. In addition, to this element there corresponds a solution $y(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$. Moreover,*

$$\begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t) \in K_1, & t \in I_1, \\ y(t;\mu_0 + \varepsilon\delta\mu) \in K_1, & t \in [r_1 - \delta_1, r_2 + \delta_1], \end{cases} \tag{2.24}$$

and

$$\lim_{\varepsilon \to 0} y(t;\mu_0 + \varepsilon\delta\mu) = y(t;\mu_0)$$

uniformly for $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(1)}$.

This lemma is a consequence of Theorem 1.7.

Lemma 2.8. *Let $x_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda^{(1)}$ and defined on $[\widehat{\tau}, t_{10}]$ (see Definition 1.1), let $t_{00}, t_{10} \in (a,b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1,s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(1)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(1)}$. In addition, to this element there corresponds a solution $x(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover,*

$$x(t;\mu_0 + \varepsilon\delta\mu) \in K_1, \quad t \in [\widehat{\tau}, t_{10} + \delta_1]. \tag{2.25}$$

It is easy to see that if in Lemma 2.7 one puts $r_1 = t_{00}$, $r_2 = t_{10}$, then $x_0(t) = y_0(t)$, $t \in [t_{00}, t_{10}]$, and

$$x(t;\mu_0 + \varepsilon\delta\mu) = h\big(t_0, \varphi, y(\,\cdot\,;\mu_0 + \varepsilon\delta\mu)\big)(t), \quad (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im^{(1)}$$

(see (2.23)). Thus, Lemma 2.8 is a simple corollary of Lemma 2.6 (see (2.24)).

Due to the uniqueness, the solution $y(t;\mu_0)$ on the interval $[r_1 - \delta_1, r_2 + \delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1 - \delta_1, r_2 + \delta_1]$.

Lemma 2.7 allows one to define the increment of the solution $y_0(t) = y(t;\mu_0)$:

$$\begin{aligned} \Delta y(t) = \Delta y(t;\varepsilon\delta\mu) &= y(t;\mu_0 + \varepsilon\delta\mu) - y(t;\mu_0) \\ &= y(t) - y_0(t) \quad \forall\, (t, \varepsilon, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times (0, \varepsilon_1) \times \Im^{(1)}. \end{aligned} \tag{2.26}$$

Obviously,

$$\lim_{\varepsilon \to 0} \Delta y(t;\varepsilon\delta\mu) = 0 \tag{2.27}$$

uniformly with respect to $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(1)}$ (see Lemma 2.7).

Lemma 2.9. *Let* $\tau_{s0} > \cdots > \tau_{10}$ *and* $t_{00} + \tau_{s0} \le r_2$. *Moreover, the conditions* 2.2–2.4 *of Theorem* 2.1 *hold. Then there exists a number* $\varepsilon_2 \in (0, \varepsilon_1)$ *such that*

$$\max_{t \in [t_{00}, r_2+\delta_1]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{2.28}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_-^{(1)}$. *Moreover,*

$$\Delta y(t_{00}) = \varepsilon[\delta x_0 - f_0^- \delta t_0] + o(\varepsilon\delta\mu). \tag{2.29}$$

Proof. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_-^{(1)}$ the inequalities

$$t_0 + \tau_i > t_{00}, \ \ i = \overline{1, s}, \tag{2.30}$$

hold, where $t_0 = t_{00} + \varepsilon\delta t_0$, $\tau_i = \tau_{i0} + \varepsilon\delta\tau_i$. On the interval $[t_{00}, r_2 + \delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$, where $y(t) = y(t; \mu_0 + \varepsilon\delta\mu)$, satisfies the equation

$$\dot{\Delta} y(t) = a(t; \varepsilon\delta\mu) + \varepsilon b(t; \varepsilon\delta\mu), \tag{2.31}$$

where

$$\begin{aligned} a(t; \varepsilon\delta\mu) &= f_0(t_0, \tau_1, \dots, \tau_s, \varphi, y_0 + \Delta y)(t) - f_0(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, y_0)(t), \\ b(t; \varepsilon\delta\mu) &= \delta f(t_0, \tau_1, \dots, \tau_s, \varphi, y_0 + \Delta y)(t). \end{aligned} \tag{2.32}$$

We rewrite the equation (2.31) in the integral form

$$\Delta y(t) = \Delta y(t_{00}) + \int_{t_{00}}^{t} \big[a(\xi; \varepsilon\delta\mu) + \varepsilon b(\xi; \varepsilon\delta\mu)\big]\, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_{00})| + a_1(t; t_{00}, \varepsilon\delta\mu) + \varepsilon b_1(t_{00}; \varepsilon\delta\mu), \tag{2.33}$$

where

$$a_1(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} |a(\xi; \varepsilon\delta\mu)|\, d\xi, \quad b_1(t_{00}; \varepsilon\delta\mu) = \int_{t_{00}}^{r_2+\delta_1} |b(\xi; \varepsilon\delta\mu)|\, d\xi.$$

Let us prove the formula (2.29). We have

$$\begin{aligned} \Delta y(t_{00}) &= y(t_{00}; \mu_0 + \varepsilon\delta\mu) - y_0(t_{00}) \\ &= x_{00} + \varepsilon\delta x_0 + \int_{t_0}^{t_{00}} \big[f_0(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)) + \varepsilon b(t; \varepsilon\delta\mu)\big]\, dt - x_{00} \end{aligned} \tag{2.34}$$

(see (2.30)). It is clear that if $t \in [t_0, t_{00}]$, then

$$\lim_{\varepsilon\to 0} \big(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)\big) = \lim_{t\to t_{00}-} \big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})\big) = w_0$$

(see (2.27)). Consequently,

$$\lim_{\varepsilon\to 0} \sup_{t\in[t_0, t_{00}]} \big|f_0(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)) - f_0^-\big| = 0.$$

This relation implies that

$$\begin{aligned} &\int_{t_0}^{t_{00}} f_0\big(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)\big)\, dt \\ &= -\varepsilon f_0^- \delta t_0 + \int_{t_0}^{t_{00}} \big[f_0(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)) - f_0^-\big]\, dt = -\varepsilon f_0^- \delta t_0 + o(\varepsilon\delta\mu). \end{aligned} \tag{2.35}$$

Further, we have

$$\int_{t_0}^{t_{00}} |b(t;\varepsilon\delta\mu)|\,dt \le \int_{t_0}^{t_{00}} \sum_{i=1}^{k} |\lambda_i|\,\big|\delta f_i(t_0,\tau_1,\dots,\tau_s,\varphi,y_0+\Delta y)(t)\big|\,dt \le \gamma \sum_{i=1}^{k} \int_{t_0}^{t_{00}} m_{\delta f_i,K_1}(t)\,dt. \tag{2.36}$$

From (2.34), by virtue of (2.35), (2.36), we obtain (2.29).

Let us now prove the inequality (2.28). Let

$$\rho_{i,1} = \min\{t_0+\tau_i, t_{00}+\tau_{i0}\}, \quad \rho_{i,2} = \max\{t_{00}+\tau_i, t_{00}+\tau_{i0}\}, \ \ i=\overline{1,s}.$$

It is easy to see that

$$\rho_{i,2} \ge \rho_{i,1} > t_{00} \ \text{ and } \ \rho_{i,2}-\rho_{i,1} = O(\varepsilon\delta\mu).$$

Let ε_2 be insomuch small that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2)\times\Im_-^{(1)}$ the inequalities

$$\rho_{i,1} < \rho_{i+1,1}, \ \ i=\overline{1,s-1}, \ \ \rho_{s,2} < r_2+\delta_1$$

hold. Now we estimate $a_1(t;t_{00},\varepsilon\delta\mu)$, $t\in[t_{00}, r_2+\delta_1]$. Let $t\in[t_{00},\rho_{1,1}]$. Obviously,

$$a_1(t;t_{00},\varepsilon\delta\mu) \le \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t;t_{00},\varepsilon\delta\mu), \tag{2.37}$$

where

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|h(t_0,\varphi,y_0+\Delta y)(\xi-\tau_i) - h(t_{00},\varphi_0,y_0)(\xi-\tau_{i0})\big|\,d\xi$$

(see Lemma 2.2). It is clear that if $t\in[t_{00},\rho_{11})$, then for $\xi\in[t_{00},t]$ and any $i=\overline{1,s}$, we have $\xi-\tau_i<t_0$ and $\xi-\tau_{i0}<t_{00}$, therefore,

$$\begin{aligned} a_{2i}(t;t_{00},\varepsilon\delta\mu) &= \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi \\ &\le O(\varepsilon\delta\mu) + \int_{t_{00}}^{b} L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi, \ \ i=\overline{1,s}. \end{aligned}$$

The boundedness of the function $\dot\varphi_0(t)$, $t\in I_1$, yields

$$|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})| = \bigg|\int_{\xi-\tau_{i0}}^{\xi-\tau_i} \dot\varphi_0(t)\,dt\bigg| = O(\varepsilon\delta\mu).$$

Thus, for $t\in[t_{00},\rho_{1,1}]$, we have

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \ \ i=\overline{1,s}.$$

Consequently, for $t\in[t_{00},\rho_{1,1}]$, we get

$$a_1(t;t_{00},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi. \tag{2.38}$$

Let $t\in[\rho_{1,1},\rho_{1,2}]$. Then

$$a_1(t;t_{00},\varepsilon\delta\mu) = a_1(\rho_{1,1};t_{00},\varepsilon\delta\mu) + a_1(t;\rho_{1,1},\varepsilon\delta\mu).$$

By the condition of Theorem 2.1, the function $|a(\xi;\varepsilon\delta\mu)|$, $\xi\in[t_{00},r_2+\delta_1]$, is bounded, i.e.,

$$\big|a_1(t;\rho_{1,1},\varepsilon\delta\mu)\big|\le O(\varepsilon\delta\mu),\ \ t\in[\rho_{1,1},\rho_{1,2}].$$

Therefore, for $t\in[\rho_{1,1},\rho_{1,2}]$, we have

$$\begin{aligned}a_1(t;t_{00},\varepsilon\delta\mu)&\le a_1(\rho_{1,1};t_{00},\varepsilon\delta\mu)+O(\varepsilon\delta\mu)\\&\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{\rho_{1,1}}L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}L_{f_0,K_1}(\xi)|\,\Delta y(\xi)|\,d\xi.\end{aligned}$$

Thus on the interval $[t_{00},\rho_{1,2}]$, the formula (2.38) is valid.

Let $t\in[\rho_{1,2},\rho_{2,1}]$, then $t-\tau_1>t_0$, $t-\tau_{10}>t_{00}$ and $t-\tau_i<t_0$, $t-\tau_{i0}<t_{00}$, $i=\overline{2,s}$. For this case we have

$$\begin{aligned}&a_1(t;t_{00},\varepsilon\delta\mu)=a_1(\rho_{1,2};t_{00},\varepsilon\delta\mu)+a_1(t;\rho_{1,2},\varepsilon\delta\mu)\\&\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{\rho_{1,2}}L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi+\int\limits_{\rho_{1,2}}^{t}L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi+\int\limits_{\rho_{1,2}}^{t}L_{f_0,K_1}(\xi)|\Delta y(\xi-\tau_1)|\,d\xi\\&\qquad+\int\limits_{\rho_{1,2}}^{t}L_{f_0,K_1}(\xi)\big|y_0(\xi-\tau_1)-y_0(\xi-\tau_{10})\big|\,d\xi+\sum_{i=2}^{s}a_{2i}(t;\rho_{12},\varepsilon\delta\mu)\end{aligned}$$

(see (2.37)). It is clear that

$$\big|y_0(\xi-\tau_1)-y_0(\xi-\tau_{10})\big|\le\bigg|\int\limits_{\xi-\tau_1}^{\xi-\tau_{10}}\big|f_0(t_{00},\tau_{10},\dots,\tau_{s0},y_0)(t)\big|\,dt\bigg|\le O(\varepsilon\delta\mu)$$

and

$$\begin{aligned}a_{2i}(t;\rho_{1,2},\varepsilon\delta\mu&=\int\limits_{\rho_{1,2}}^{t}L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi\\&\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{b}L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi\le O(\varepsilon\delta\mu),\ \ i=\overline{2,s}.\end{aligned}$$

Thus, for $t\in[t_{00},\rho_{2,1}]$,

$$\begin{aligned}a_1(t;t_{00},\varepsilon\delta\mu)&\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi+\int\limits_{\rho_{1,2}-\tau_1}^{t-\tau_1}L_{f_0,K_1}(\xi+\tau_1)|\Delta y(\xi)|\,d\xi\\&\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi)+\chi_1(\xi+\tau_1)L_{f_0,K_1}(\xi+\tau_1)\Big]\,|\Delta y(\xi)|\,d\xi,\ \ \rho_{1,2}-\tau_1\ge t_{00},\end{aligned}$$

where $\chi_1(\xi)$ is the characteristic function of the interval I. Continuing this process for $t\in[t_{00},\rho_{s,2}]$, we can prove that

$$a_1(t;t_{00},\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi)+\sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi.\tag{2.39}$$

Let $t \in [\rho_{s,2}, r_2+\delta_1]$, then

$$\begin{aligned}
a_1(t;t_{00},\varepsilon\delta\mu) &= a_1(\rho_{s,2};t_{00},\varepsilon\delta\mu) + a_1(t;\rho_{s,2},\varepsilon\delta\mu)\\
&\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{\rho_{s,2}} \Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi\\
&\quad + \int\limits_{\rho_{s,2}}^{t} L_{f_0,K_1}(\xi)\Big[|\Delta y(\xi)| + \sum_{i=1}^{s}\big|y_0(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big| + \sum_{i=1}^{s}|\Delta y(\xi-\tau_i)|\,d\xi\Big]\\
&\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \int\limits_{t_{00}}^{t}\Big[\sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi\\
&\quad + \sum_{i=1}^{s}\int\limits_{\rho_{s,2}-\tau_i}^{t-\tau_i} \chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi\\
&\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi.
\end{aligned}$$

Consequently, for $t \in [t_{00}, r_2+\delta_1]$, we have

$$a_1(t;t_{00},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi \tag{2.40}$$

(see (2.39)). Obviously,

$$b_1(t_{00},\varepsilon\delta\mu) \le \gamma\int\limits_{t_{00}}^{r_2+\delta_1}\sum_{i=1}^{k}\big|\delta f_i(t_0,\tau_1,\dots,\tau_s,y_0+\Delta y)(t)\big|\,dt \le \gamma\sum_{i=1}^{k}\int\limits_{I} m_{\delta f_i,K_1}(t)\,dt. \tag{2.41}$$

According to (2.29), (2.40) and (2.41), the inequality (2.33) directly implies

$$\begin{aligned}
|\Delta y(t)| &\le O(\varepsilon\delta\mu)\\
&\quad + \int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi,\quad t\in[t_{00},r_2+\delta_1].
\end{aligned}$$

By the Gronwall–Bellman inequality, from the above we obtain (2.28). □

2.4 Proof of Theorem 2.1

Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.8, then

$$x_0(t) = \begin{cases}\varphi_0(t), & t\in[\widehat{\tau}, t_{00}),\\ y_0(t), & t\in[t_{00}, t_{10}],\end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1)\times\Im_{-}^{(1)}$,

$$x(t;\mu_0+\varepsilon\delta\mu) = \begin{cases}\varphi(t) := \varphi_0(t)+\varepsilon\delta\varphi(t), & t\in[\widehat{\tau}, t_0),\\ y(t;\mu_0+\varepsilon\delta\mu), & t\in[t_0, t_{10}+\delta_1]\end{cases}$$

(see (2.23)). Note that $\delta\mu \in \Im_{-}^{(1)}$, i.e., $t_0 < t_{00}$, therefore we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0 + \varepsilon\delta\mu) - \varphi_0(t) & \text{for } t \in [t_0, t_{00}), \\ \Delta y(t) & \text{for } t \in [t_{00}, t_{10} + \delta_1] \end{cases}$$

(see (2.15) and (2.26)). By Lemma 2.9, we have

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall\,(t,\varepsilon,\delta\mu) \in [t_{00}, t_{10} + \delta_1] \times (0,\varepsilon_2) \times \Im_{-}^{(1)}, \tag{2.42}$$

$$\Delta x(t_{00}) = \varepsilon[\delta x_0 - f_0^{-}\delta t_0] + o(\varepsilon\delta\mu). \tag{2.43}$$

The function $\Delta x(t)$ satisfies the equation

$$\begin{aligned} \dot{\Delta} x(t) &= f_0[t, x_0 + \Delta x] + \varepsilon\delta f[t, x_0 + \Delta x] - f_0[t] \\ &= f_{0x}[t]\Delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t - \tau_{i0}) + \varepsilon\delta f[t] + \sum_{i=1}^{2} \vartheta_i(t;\varepsilon\delta\mu) \end{aligned} \tag{2.44}$$

on the interval $[t_{00}, t_{10} + \delta_1]$, where

$$\begin{aligned} f_0[t, x_0 + \Delta x] &= f_0\Big(t, x_0(t) + \Delta x(t), x_0(t-\tau_1) + \Delta x(t-\tau_1), \dots, x_0(t-\tau_s) + \Delta x(t-\tau_s)\Big), \\ f_0[t] &= f_0\big(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0})\big), \\ \delta f[t, x_0 + \Delta x] &= \delta f\Big(t, x_0(t) + \Delta x(t), x_0(t-\tau_1) + \Delta x(t-\tau_1), \dots, x_0(t-\tau_s) + \Delta x(t-\tau_s)\Big), \\ \delta f[t] &= \delta f\big(t, x_0(t), x_0(t-\tau_1), \dots, x_0(t-\tau_s)\big), \end{aligned}$$

$$\vartheta_1(t;\varepsilon\delta\mu) = f_0[t, x_0 + \Delta x] - f_0[t] - f_{0x}[t]\Delta x(t) - \sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t - \tau_{i0}), \tag{2.45}$$

$$\vartheta_2(t;\varepsilon\delta\mu) = \varepsilon\big[\delta f([t, x_0 + \Delta x]) - \delta f[t]\big]. \tag{2.46}$$

By using the Cauchy formula, one can represent the solution of the equation (2.44) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi + \sum_{p=0}^{2} R_p(t;t_{00},\varepsilon\delta\mu), \ \ t \in [t_{00}, t_{10} + \delta_1], \tag{2.47}$$

where

$$\begin{cases} R_0(t;t_{00},\varepsilon\delta\mu) = \displaystyle\sum_{i=1}^{s} R_{i0}(t;t_{00},\varepsilon\delta\mu), \\ R_{i0}(t;t_{00},\varepsilon\delta\mu) = \displaystyle\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0};t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi)\,d\xi, \\ R_p(t;t_{00},\varepsilon\delta\mu) = \displaystyle\int_{t_{00}}^{t} Y(\xi;t)\vartheta_p(\xi;\varepsilon\delta\mu)\,d\xi, \qquad p = 1,2, \end{cases} \tag{2.48}$$

and $Y(\xi;t)$ is the matrix function satisfying the equation (2.18) and the condition (2.19). Let $\delta_2 \in (0,\delta_1)$ be insomuch small that the inequalities

$$t_{00} - \delta_2 > a, \ \ t_{00} + \tau_{s0} < t_{10} - \delta_2$$

hold. The function $Y(\xi;t)$ is continuous on the set $\{(\xi,t) :\ \xi \in [t_{00}-\delta_2, t_{00}],\ t \in [t_{10}-\delta_2, t_{10}+\delta_2]\} \subset \Pi$, by Lemma 2.6. Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)[\delta x_0 - f_0^{-}\delta t_0] + o(t;\varepsilon\delta\mu) \tag{2.49}$$

(see (2.29)). One can readily see that

$$R_{i0}(t;t_{00},\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\tau_{i0}}^{t_0}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi$$

$$=\varepsilon\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_0+\tau_{i0}}^{t_{00}+\tau_{i0}}Y(\xi;t)f_{0x_i}[\xi]\Delta x(\xi-\tau_{i0})d\xi+o(t;\varepsilon\delta\mu).$$

Thus

$$R_0(t;t_{00},\varepsilon\delta\mu)=\varepsilon\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$
$$+\sum_{i=1}^{s}\int\limits_{t_0+\tau_{i0}}^{t_{00}+\tau_{i0}}Y(\xi;t)f_{0x_i}[\xi]\Delta x(\xi-\tau_{i0})\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.50}$$

Let

$$\varrho_{i,1}=\min\{t_0+\tau_i,t_{00}+\tau_{i0}\},\quad \varrho_{i,2}=\max\{t_0+\tau_i,t_{00}+\tau_{i0}\},\ \ i=\overline{1,s},$$

and let a number $\varepsilon_2\in(0,\varepsilon_1)$ be insomuch small that

$$t_{00}<\varrho_{1,1},\ \ \varrho_{i,2}<\varrho_{i+1,1},\ \ i=\overline{1,s-1},\ \ \varrho_{s,2}<t_{10}-\delta_2.$$

For $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$, we have

$$R_1(t;t_{00},\varepsilon\delta\mu)=\sum_{i=0}^{s}w_i(t;\varepsilon\delta\mu),$$

where

$$w_0(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{00}+\tau_{10}}\vartheta_{11}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad w_i(t;\varepsilon\delta\mu)=\int\limits_{t_{00}+\tau_{i0}}^{t_{00}+\tau_{i+10}}\vartheta_{11}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad i=\overline{1,s-1},$$

$$w_s(t;\varepsilon\delta\mu)=\int\limits_{t_{00}+\tau_{s0}}^{t}\vartheta_{11}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad \vartheta_{11}(\xi;t,\varepsilon\delta\mu)=Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)$$

(see (2.45)). Let $\varrho_{1,1}=t_0+\tau_1$ and $t_0+\tau_1<t_0+\tau_{10}$, then we have

$$w_0(t;\varepsilon\delta\mu)=w_{01}(t;\varepsilon\delta\mu)+w_{02}(t;\varepsilon\delta\mu).$$

Here,

$$w_{01}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_0+\tau_1}\vartheta_{11}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$w_{02}(t;\varepsilon\delta\mu)=\int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}}Y(\xi;t)\big\{f_0[\xi,x_0+\Delta x]-f_0[\xi]\big\}\,d\xi$$
$$-\int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}}Y(\xi;t)\Big[f_{0x}[\xi]\Delta x(\xi)+\sum_{i=2}^{s}f_{0x_i}[\xi]\Delta x(\xi-\tau_{i0})\Big]\,d\xi$$

$$-\int\limits_{t_0+\tau_1}^{t_0+\tau_{10}} Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi-\tau_{10})\,d\xi-\int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi-\tau_{10})\,d\xi.$$

We introduce the notations:

$$f_0[\xi;\theta,\varepsilon\delta\mu]=f_0\Big(\xi,x_0(\xi)+\theta\Delta x(\xi),x_0(\xi-\tau_{10})+\theta\big(x_0(\xi-\tau_1)-x_0(\xi-\tau_{10})+\Delta x(\xi-\tau_1)\big),\dots,$$
$$x_0(\xi-\tau_{s0})+\theta\big(x_0(\xi-\tau_s)-x_0(\xi-\tau_{s0})+\Delta x(\xi-\tau_s)\big)\Big),$$
$$\sigma(\xi;\theta,\varepsilon\delta\mu)=f_{0x}[\xi;\theta,\varepsilon\delta\mu]-f_{0x}[\xi],\quad \sigma_i(\xi;\theta,\varepsilon\delta\mu)=f_{0x_i}[\xi;\theta,\varepsilon\delta\mu]-f_{0x_i}[\xi].$$

It is easy to see that

$$f_0[\xi,x_0+\Delta x]-f_0[\xi]=\int\limits_0^1\frac{d}{d\theta}\,f_0[\xi;\theta,\varepsilon\delta\mu]\,d\theta$$
$$=\int\limits_0^1\bigg\{f_{0x}[\xi;\theta,\varepsilon\delta\mu]\Delta x(\xi)+\sum_{i=1}^s f_{0x_i}[\xi;\theta,\varepsilon\delta\mu]\big(x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big)\bigg\}\,d\theta$$
$$=\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)+\sum_{i=1}^s\sigma_{i1}(\xi;\varepsilon\delta\mu)\big(x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big)$$
$$+f_{0x}[\xi]\Delta x(\xi)+\sum_{i=1}^s f_{0x_i}[\xi]\big(x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big),$$

where

$$\sigma_1(\xi;\varepsilon\delta\mu)=\int\limits_0^1\sigma(\xi;\theta,\varepsilon\delta\mu)\,d\theta,\quad \sigma_{i1}(\xi;\varepsilon\delta\mu)=\int\limits_0^1\sigma_i(\xi;\theta,\varepsilon\delta\mu)\,d\theta.$$

Taking into account the last relation for $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$, we have

$$w_{01}(t;\varepsilon\delta\mu)=\sum_{p=1}^4 w_{01}^{(p)}(t;\varepsilon\delta\mu),$$

where

$$w_{01}^{(1)}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)\,d\xi,$$
$$w_{01}^{(2)}(t;\varepsilon\delta\mu)=\sum_{i=1}^s\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)\sigma_{i1}(\xi;\varepsilon\delta\mu)\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big]\,d\xi$$
$$=\sum_{i=1}^s\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)\sigma_{i1}(\xi;\varepsilon\delta\mu)\big[\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})+\varepsilon\delta\varphi(\xi-\tau_i)\big]\,d\xi,$$
$$w_{01}^{(3)}(t;\varepsilon\delta\mu)=\sum_{i=1}^s\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)f_{0x_i}[\xi]\,\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})\big]\,d\xi$$
$$=\sum_{i=1}^s\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)f_{0x_i}[\xi]\,\big[\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big]\,d\xi,$$

$$w_{01}^{(4)}(t;\varepsilon\delta\mu)=\sum_{i=1}^{s}\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)f_{0x_i}[\xi]\big[\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big]\,d\xi$$
$$=\varepsilon\sum_{i=1}^{s}\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)f_{0x_i}[\xi]\big[\delta\varphi(\xi-\tau_i)-\delta\varphi(\xi-\tau_{i0})\big]\,d\xi.$$

The function $\varphi_0(\xi)$, $\xi\in I_1$ is absolutely continuous, therefore for each fixed Lebesgue point $\xi\in(t_{00},t_{10}+\delta_2)$ of function $\dot\varphi_0(\xi-\tau_{i0})$ we get

$$\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})=\int\limits_{\xi}^{\xi-\varepsilon\delta\tau_i}\dot\varphi_0(\varsigma-\tau_{i0})\,d\varsigma=-\varepsilon\dot\varphi_0(\xi-\tau_{i0})\delta\tau_i+\gamma_i(\xi;\varepsilon\delta\mu),\tag{2.51}$$

where

$$\lim_{\varepsilon\to0}\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}=0\ \text{ uniformly for }\ \delta\mu\in\Im_-^{(1)}.\tag{2.52}$$

Thus, (2.51) is valid for almost all points of the interval $(t_{00},t_{10}+\delta_2)$. From (2.51), taking into account the boundedness of the function $\dot\varphi_0(\xi)$, we have

$$\big|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\le O(\varepsilon\delta\mu)\ \text{ and }\ \Big|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const.\tag{2.53}$$

According to (2.42) and (2.51), for the expression $w_{01}^{(p)}(t;\varepsilon\delta\mu)$, $p=\overline{1,4}$, we have

$$|w_{01}^{(1)}(t;\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon\delta\mu)\sigma_1(\varepsilon\delta\mu),\quad |w_{01}^{(2)}(t;\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon\delta\mu)\sum_{i=1}^{s}\sigma_{i1}(\varepsilon\delta\mu),$$
$$w_{01}^{(3)}(t;\varepsilon\delta\mu)=\sum_{i=1}^{s}\bigg[\gamma_{i1}(t;\varepsilon\delta\mu)-\varepsilon\bigg(\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)f_{0x_i}[\xi]\dot\varphi_0(\xi-\tau_{i0})\,d\xi\bigg)\delta\tau_i\bigg],$$
$$|w_{01}^{(4)}(t;\varepsilon\delta\mu)|\le o(\varepsilon\delta\mu)\|Y\|\sum_{i=1}^{s}\int\limits_{t_{00}}^{t_0+\tau_1}|f_{0x_i}[\xi]|\,d\xi.$$

Here

$$\sigma_1(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{00}+\tau_{10}}|\sigma_1(\xi;\varepsilon\delta\mu)|\,d\xi,\quad \sigma_{i1}(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{00}+\tau_{10}}|\sigma_{i1}(\xi;\varepsilon\delta\mu)|\,d\xi,$$
$$\|Y\|=\sup\big\{|Y\xi;t|:\ \ (\xi,t)\in\Pi\big\},\quad \gamma_{i1}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\gamma_i(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|\le\|Y\|\int\limits_{t_{00}}^{t_{00}+\tau_{10}}|f_{x_i}[\xi]|\,\Big|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{\varepsilon\to0}\sigma_1(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\sigma_{i1}(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},t_{00}+\tau_{10}]\times\Im_-^{(1)}$ (see (2.52) and (2.53)). Thus,

$$w_{01}^{(1)}(t;\varepsilon\delta\mu)=w_{01}^{(2)}(t;\varepsilon\delta\mu)=w_{01}^{(4)}(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu),$$
$$w_{01}^{(3)}(t;\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t_0+\tau_1}Y(\xi;t)f_{0x_i}[\xi]\dot\varphi_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+o(t;\varepsilon\delta\mu).\tag{2.54}$$

Further,

$$\varepsilon\int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}}Y(\xi;t)f_{0x_i}[\xi]\dot\varphi_0(\xi-\tau_{i0})d\xi=o(t;\varepsilon\delta\mu),\ \ \dot x_0(\xi-\tau_{i0})=\dot\varphi_0(\xi-\tau_{i0}),\ \ \xi\in[t_{00},t_{00}+\tau_{i0}),$$

therefore,

$$w_{01}^{(3)}(t;\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\Bigg[\int\limits_{t_{00}}^{t_{00}+\tau_{10}}Y(\xi;t)f_{0x_i}[\xi]\dot x_0(\xi-\tau_{i0})\,d\xi\Bigg]\delta\tau_i+o(t;\varepsilon\delta\mu). \tag{2.55}$$

On the basis of (2.54) and (2.55), we obtain

$$w_{01}(t;\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\Bigg[\int\limits_{t_{00}}^{t_{00}+\tau_{10}}Y(\xi;t)f_{0x_i}[\xi]\dot x_0(\xi-\tau_{i0})\,d\xi\Bigg]\delta\tau_i+o(t;\varepsilon\delta\mu).$$

Let us now transform $w_{02}(t;\varepsilon\delta\mu)$. We have

$$w_{02}(t;\varepsilon\delta\mu)=\int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}}Y(\xi;t)\big\{f_0[\xi,x_0+\Delta x]-f_0[\xi]\big\}\,d\xi-\int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}}Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi-\tau_{10})\,d\xi+o(t;\varepsilon\delta\mu).$$

Since for $\xi\in[t_0+\tau_1,t_{00}+\tau_{10}]$,

$$|\Delta x(\xi)|\le O(\varepsilon\delta\mu),\ \ |\Delta x(\xi-\tau_i)|=\varepsilon|\delta\varphi(\xi-\tau_i)|,\ \ x_0(\xi-\tau_i)=\varphi_0(\xi-\tau_i),\ \ i=\overline{2,s},$$

and

$$x_0(\xi-\tau_1)+\Delta x(\xi-\tau_1)=x(\xi-\tau_1;\mu_0+\varepsilon\delta\mu)=y(\xi-\tau_1;\mu_0+\varepsilon\delta\mu)=y_0(\xi-\tau_1)+\Delta y(\xi-\tau_1;\varepsilon\delta\mu),$$

we get

$$\begin{aligned}\lim_{\varepsilon\to0}\big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau_1)+\Delta x(\xi-\tau_1),\dots,x_0(\xi-\tau_s)+\Delta x(\xi-\tau_s)\big)\\
=\lim_{\xi\to t_{00}+\tau_{10}-}\big(\xi,x_0(\xi),y_0(\xi-\tau_{10}),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big)=w_{1i}^0,\\
\lim_{\xi\to t_{00}+\tau_{10}-}\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big)=w_0,\end{aligned}$$

i.e.,

$$\lim_{\varepsilon\to0}\sup_{\xi\in[t_0+\tau_1,t_{00}+\tau_{10}]}\big[f_0[\xi,x_0+\Delta x]-f_0[\xi]\big]=f_{01}.$$

Moreover, the function $Y(\xi;t)$ is continuous at the set $[t_{00},t_{00}+\tau_{10}]\times[t_{00}-\tau_2,t_{10}+\delta_2]\subset\Pi$. Thus,

$$\int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}}Y(\xi;t)\big\{f_0[\xi,x_0+\Delta x]-f_0[\xi]\big\}\,d\xi=-\varepsilon Y(t_{00}+\tau_{10};t)f_{10}(\delta t_0+\delta\tau_1)+o(t;\varepsilon\delta\mu).$$

The expression $-\varepsilon Y(t_{00}+\tau_{10};t)f_{10}(\delta t_0+\delta\tau_1)$ is the effect of discontinuity. Consequently,

$$\begin{aligned}w_0(t;\varepsilon\delta\mu)=-\,\varepsilon\sum_{i=1}^{s}\Bigg[\int\limits_{t_{00}}^{t_{00}+\tau_{10}}Y(\xi;t)f_{0x_i}[\xi]\dot x_0(\xi-\tau_{i0})\,d\xi\Bigg]\delta\tau_i-\varepsilon Y(t_{00}+\tau_{10};t)f_{10}(\delta t_0+\delta\tau_1)\\
-\int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}}Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi-\tau_{10})d\xi+o(t;\varepsilon\delta\mu).\end{aligned} \tag{2.56}$$

Let $\varrho_{1,1} = t_0 + \tau_1$ again and $t_0 + \tau_{10} < t_0 + \tau_1$, then we have

$$w_0(t;\varepsilon\delta\mu) = \sum_{k=1}^{2} \widehat{w}_{0k}(t;\varepsilon\delta\mu),$$

where

$$\begin{aligned}
\widehat{w}_{01} &= \int\limits_{t_{00}}^{t_0+\tau_{10}} Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\,d\xi,\\
\widehat{w}_{02} &= \int\limits_{t_0+\tau_{10}}^{t_0+\tau_1} Y(\xi;t)\big\{f_0[\xi;x_0+\Delta x]-f_0[\xi]\big\}\,d\xi + \int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi;t)\big\{f_0[\xi;x_0+\Delta x]-f_0[\xi]\big\}\,d\xi\\
&\quad - \int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t)\Big\{f_{0x}[\xi]\Delta x(\xi) + \sum_{i=2}^{s} f_{0x_i}[\xi]\Delta x(\xi-\tau_{i0})\Big\}\,d\xi\\
&\quad - \int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t)f_{x_1}[\xi]\Delta x(\xi-\tau_{10})\,d\xi.
\end{aligned}$$

For this case the formula (2.56) is valid and can be proved by the scheme described above.

Let $\varrho_{1,1} = t_{00} + \tau_{10}$, i.e., $t_{00} + \tau_{10} < t_0 + \tau_i$. In this case, by analogous transformations can be proved the formula

$$\begin{aligned}
w_0(t;\varepsilon\delta\mu) = &-\varepsilon\sum_{i=1}^{s}\Bigg[\int\limits_{t_{00}}^{t_{00}+\tau_{10}} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\Bigg]\delta\tau_i\\
&-\int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi-\tau_{10})\,d\xi + o(t;\varepsilon\delta\mu)
\end{aligned}$$

without discontinuity effect $-\varepsilon f_{01}(\delta t_0 + \delta\tau_1)$. We notice that this effect appears under transformation of the addend $w_1(t;\varepsilon\delta\mu)$. For $R_1(t;t_{00},\varepsilon\delta\mu)$, after transformations of $w_i(t;\varepsilon\delta\mu)$, $i=\overline{1,s}$, we obtain

$$\begin{aligned}
R_1(t;t_{00},\varepsilon\delta\mu) = &-\varepsilon\sum_{i=1}^{s}\Bigg[\int\limits_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\Bigg]\delta\tau_i - \sum_{i=1}^{s}\int\limits_{t_0+\tau_{i0}}^{t_{00}+\tau_{i0}} Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi-\tau_{i0})\,d\xi\\
&-\varepsilon\sum_{i=1}^{s} Y(t_{00}+\tau_{i0};t)f_{i0}(\delta t_0+\delta\tau_i) + o(t;\varepsilon\delta\mu).
\end{aligned}\tag{2.57}$$

Finally, let us estimate $R_2(t;t_{00},\varepsilon\delta\mu)$. We have

$$|R_2(t;t_{00},\varepsilon\delta\mu)| \le \varepsilon\gamma v(\varepsilon\delta\mu),$$

where

$$\begin{aligned}
v(\varepsilon\delta\mu) &= \int\limits_{t_{00}}^{t_{10}+\delta_2} \widehat{L}(t)\Big\{|\Delta x(t)| + \sum_{i=1}^{s}\big[|x_0(t-\tau_i)-x_0(t-\tau_{i0})| + |\Delta x(t-\tau_i)|\big]\Big\}\,dt,\\
\widehat{L}(t) &= \sum_{j=1}^{s} L_{\delta f_j,K_1}(t)
\end{aligned}$$

(see (2.46)). It is clear that

$$v(\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) \int\limits_{t_{00}}^{t_{10}+\delta_2} \widehat{L}(t)\,dt + \sum_{i=1}^{s} \int\limits_{t_{00}}^{\varrho_{i,1}} \widehat{L}(t)\big[|\varphi_0(t-\tau_i)-\varphi_0(t-\tau_{i0})| + \varepsilon|\delta\varphi(t-\tau_i)|\big]\,dt$$
$$+ \sum_{i=1}^{s} \int\limits_{\varrho_{i,1}}^{\varrho_{i,2}} \widehat{L}(t)\big[|x_0(t-\tau_i)-x_0(t-\tau_{i0})| + |\Delta x(t-\tau_i)|\big]\,dt$$
$$+ \sum_{i=1}^{s} \int\limits_{\varrho_{i,2}}^{t_{00}+\tau_{i0}} \widehat{L}(t)\big[|x_0(t-\tau_i)-x_0(t-\tau_{i0})| + O(\varepsilon\delta\mu)\big]\,dt.$$

Further,

$$\varphi_0(t-\tau_i)-\varphi_0(t-\tau_{i0}) = \int\limits_{t-\tau_{i0}}^{t-\tau_i} \dot{\varphi}_0(\xi)\,d\xi,$$
$$x_0(t-\tau_i)-x_0(t-\tau_{i0}) = \int\limits_{t-\tau_{i0}}^{t-\tau_i} \dot{x}_0(\xi)\,d\xi = \int\limits_{t-\tau_{i0}}^{t-\tau_i} f_0[\xi]\,d\xi.$$

Taking into account the boundedness of the functions $\dot{\varphi}_0(\xi)$ and $f_0[\xi]$, we obtain

$$\big|\varphi_0(t-\tau_i)-\varphi_0(t-\tau_{i0})\big| = O(\varepsilon\delta\mu), \quad \big|x_0(t-\tau_i)-x_0(t-\tau_{i0})\big| = O(\varepsilon\delta\mu).$$

Moreover,

$$\big|x_0(t-\tau_i)-x_0(t-\tau_{i0})\big| + |\Delta x(t-\tau_i)|, \ \ t\in[\varrho_{i,1},\varrho_{i,2}],$$

is bounded. From these relations it follows that

$$\lim_{\varepsilon\to 0} v(\varepsilon\delta\mu) = 0$$

uniformly for $\delta\mu \in \Im_{-}^{(1)}$. Thus,

$$R_2(t;t_{00},\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu). \tag{2.58}$$

From (2.47), by virtue of (2.50), (2.57) and (2.58), we obtain (2.16), where $\delta x(t;\delta\mu)$ has the form (2.17).

2.5 Lemma on the estimation of the increment of a solution with respect to the variation set $\Im_{+}^{(1)}$

Lemma 2.10. *Let $\tau_{s0} > \cdots > \tau_{10}$ and $t_{00}+\tau_{s0} \le r_2$. Moreover, the conditions* 2.2–2.3 *of Theorem* 2.1 *and the condition* (2.20) *hold. Then there exists a number $\varepsilon_2 \in (0,\varepsilon_1)$ such that*

$$\max_{t\in[t_0,r_2+\delta_1]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{2.59}$$

for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times\Im_{+}^{(1)}$. Moreover,

$$\Delta y(t_0) = \varepsilon[\delta x_0 - f_0^{+}\delta t_0] + o(\varepsilon\delta\mu). \tag{2.60}$$

Proof. Let $\varepsilon_2 \in (0,\varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times\Im_{+}^{(1)}$ the inequalities

$$t_{00}+\tau_i > t_0, \ \ t_{00}+\tau_{i0} > t_0, \ \ i=\overline{1,s}, \tag{2.61}$$

hold. On the interval $[t_0, r_2+\delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$ satisfies the equation

$$\dot{\Delta} y(t) = a(t;\varepsilon\delta\mu) + \varepsilon b(t;\varepsilon\delta\mu) \tag{2.62}$$

(see (2.32)). We rewrite the equation (2.62) in the integral form

$$\Delta y(t) = \Delta y(t_0) + \int_{t_0}^{t} \big[a(\xi;\varepsilon\delta\mu) + \varepsilon b(\xi;\varepsilon\delta\mu)\big]\,d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_0)| + a_1(t;t_0,\varepsilon\delta\mu) + \varepsilon b_1(t_0,\varepsilon\delta\mu). \tag{2.63}$$

Let us prove the formula (2.60). We have

$$\Delta y(t_0) = y(t_0;\mu_0+\varepsilon\delta\mu) - y_0(t_0) = x_{00}+\varepsilon\delta x_0 - x_{00} - \int_{t_{00}}^{t_0} f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big)\,dt \tag{2.64}$$

(see (2.61)). It is clear that if $t \in [t_{00}, t_0]$, then

$$\lim_{\varepsilon\to 0}\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi(t-\tau_{s0})\big) = \lim_{t\to t_{00}+}\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big) = w_0.$$

Consequently,

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_{00},t_0]}\big|f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big) - f_0^+\big| = 0.$$

This relation implies

$$\int_{t_{00}}^{t_0} f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big)\,dt$$
$$= -\varepsilon f_0^+\delta t_0 + \int_{t_{00}}^{t_0}\big[f_0(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})) - f_0^+\big]\,dt = -\varepsilon f_0^+\delta t_0 + o(\varepsilon\delta\mu). \tag{2.65}$$

From (2.64), by virtue of (2.65), we obtain (2.60).

Now let us prove the inequality (2.59). Let

$$\rho_{i,1} = \min\{t_{00}+\tau_i, t_{00}+\tau_{i0}\}, \quad \rho_{i,2} = \max\{t_0+\tau_i, t_{00}+\tau_{i0}\}, \ \ i=\overline{1,s}.$$

It is easy to see that $\rho_{i,2} \ge \rho_{i,1} > t_0$ and $\rho_{i,2}-\rho_{i,1} = O(\varepsilon\delta\mu)$. Let ε_2 be insomuch small that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times\Im_+^{(1)}$ the inequalities $\rho_{i,1} < \rho_{i+1,1}$, $i=\overline{1,s-1}$, hold. We now estimate $a_1(t;t_0,\varepsilon\delta\mu)$, $t\in[t_0, r_2+\delta_1]$. Let $t\in[t_0,\rho_{1,1}]$. Obviously,

$$a_1(t;t_0,\varepsilon\delta\mu) \le \int_{t_0}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t;t_0,\varepsilon\delta\mu) \tag{2.66}$$

(see (2.37)). It is clear that if $t\in[t_0,\rho_{1,1})$, then for $\xi\in[t_0,t]$ and for any $i=\overline{1,s}$, we have $\xi-\tau_i < t_0$ and $\xi-\tau_{i0} < t_{00}$, hence,

$$a_{2i}(t;t_0,\varepsilon\delta\mu) = \int_{t_0}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi$$
$$\le O(\varepsilon\delta\mu) + \int_{t_0}^{b} L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi, \ \ i=\overline{1,s}.$$

From the boundedness of the function $\dot{\varphi}_0(t)$, $t \in I_1$, follows

$$\big|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big| = \Bigg|\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} \dot{\varphi}_0(t)\,dt\Bigg| = O(\varepsilon\delta\mu).$$

Thus, for $t \in [t_0, \rho_{1,1}]$, we have

$$a_{2i}(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \ \ i=\overline{1,s}.$$

Consequently, for $t \in [t_0, \rho_{1,1}]$, we get

$$a_1(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi. \tag{2.67}$$

Let $t \in [\rho_{1,1}, \rho_{1,2}]$, then $a_1(t; t_0, \varepsilon\delta\mu) = a_1(\rho_{1,1}; t_0, \varepsilon\delta\mu) + a_1(t; \rho_{1,1}, \varepsilon\delta\mu)$. The function $|a(\xi; \varepsilon\delta\mu)|$, $\xi \in [t_0, r_2+\delta_1]$, is bounded, i.e., $|a_1(t; \rho_{1,1}, \varepsilon\delta\mu)| \le O(\varepsilon\delta\mu)$, $t \in [\rho_{1,1}, \rho_{1,2}]$. Therefore, for $t \in [\rho_{i,1}, \rho_{i,2}]$, we have

$$\begin{aligned} a_1(t; t_0, \varepsilon\delta\mu) &\le a_1(\rho_{1,1}; t_0, \varepsilon\delta\mu) + O(\varepsilon\delta\mu) \\ &\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{\rho_{1,1}} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi. \end{aligned}$$

Thus, on the interval $[t_0, \rho_{1,2}]$, the formula (2.67) is valid. Let $t \in [\rho_{1,2}, \rho_{2,1}]$, then $t-\tau_1 > t_0$, $t-\tau_{10} > t_{00}$ and $t-\tau_i < t_0$, $t-\tau_{i0} < t_{00}$, $i=\overline{2,s}$.

For this case, we have

$$\begin{aligned} a_1(t; t_0, \varepsilon\delta\mu) &= a_1(\rho_{1,2}; t_0, \varepsilon\delta\mu) + a_1(t; \rho_{1,2}, \varepsilon\delta\mu) \\ &\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{\rho_{1,2}} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \int\limits_{\rho_{1,2}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \int\limits_{\rho_{1,2}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi-\tau_1)|\,d\xi \\ &\quad + \int\limits_{\rho_{1,2}}^{t} L_{f_0,K_1}(\xi)|y_0(\xi-\tau_1)-y_0(\xi-\tau_{10})|\,d\xi + \sum_{i=2}^{s} a_{2i}(t; \rho_{12}, \varepsilon\delta\mu). \end{aligned}$$

It is clear that

$$\big|y_0(\xi-\tau_1)-y_0(\xi-\tau_{10})\big| \le \Bigg|\int\limits_{\xi-\tau_1}^{\xi-\tau_{10}} \big|f_0(t_{00}, \tau_{10}, \dots, \tau_{s0}, y)(t)\big|\,dt\Bigg| \le O(\varepsilon\delta\mu)$$

and

$$\begin{aligned} a_{2i}(t; \rho_{1,2}, \varepsilon\delta\mu) &= \int\limits_{\rho_{1,2}}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi \\ &\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{b} L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi \le O(\varepsilon\delta\mu), \ \ i=\overline{2,s}. \end{aligned}$$

Thus, for $t \in [t_0, \rho_{2,1}]$,

$$a_1(t;t_0,\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int_{t_0}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \int_{\rho_{1,2}-\tau_1}^{t-\tau_1} L_{f_0,K_1}(\xi+\tau_1)|\Delta y(\xi)|\,d\xi$$
$$\le O(\varepsilon\delta\mu) + \int_{t_0}^{t} \left[L_{f_0,K_1}(\xi) + \chi_1(\xi+\tau_1)L_{f_0,K_1}(\xi+\tau_1)\right]|\Delta y(\xi)|\,d\xi, \quad \rho_{1,2}-\tau_1 \ge t_0.$$

Continuing this process, we can prove that for $t \in [t_0, \rho_{s,2}]$,

$$a_1(t;t_0,\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int_{t_0}^{t} \Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]|\Delta y(\xi)|\,d\xi. \tag{2.68}$$

Let $t \in [\rho_{s,2}, r_2+\delta_1]$, then

$$a_1(t;t_0,\varepsilon\delta\mu) = a_1(\rho_{s,2};t_0,\varepsilon\delta\mu) + a_1(t;\rho_{s,2},\varepsilon\delta\mu)$$
$$\le O(\varepsilon\delta\mu) + \int_{t_0}^{\rho_{s,2}} \Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]|\Delta y(\xi)|\,d\xi$$
$$+ \int_{\rho_{s,2}}^{t} L_{f_0,K_1}(\xi)\Big[|\Delta y(\xi)| + \sum_{i=1}^{s}\big|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})\big| + \sum_{i=1}^{s}|\Delta y(\xi-\tau_i)|\,d\xi$$
$$\le O(\varepsilon\delta\mu) + \int_{t_0}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \int_{t_0}^{t}\Big[\sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]|\Delta y(\xi)|\,d\xi$$
$$+ \sum_{i=1}^{s}\int_{\rho_{s,2}-\tau_i}^{t-\tau_i} \chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi$$
$$\le O(\varepsilon\delta\mu) + \int_{t_0}^{t}\Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]|\Delta y(\xi)|\,d\xi.$$

Consequently, for $t \in [t_0, r_2+\delta_1]$, we have

$$a_1(t;t_0,\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int_{t_0}^{t}\Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]|\Delta y(\xi)|\,d\xi \tag{2.69}$$

(see (2.66) and (2.68)). Obviously,

$$b_1(t_0,\varepsilon\delta\mu) \le \gamma\int_{t_0}^{r_2+\delta_1}\sum_{i=1}^{k}\big|\delta f_i(t_0,\tau_1,\dots,\tau_s,y_0+\Delta y)(t)\big|\,dt \le \gamma\sum_{i=1}^{k}\int_I m_{\delta f_i,K_1}(t)\,dt. \tag{2.70}$$

According to (2.60), (2.69) and (2.70), the inequality (2.63) directly implies

$$|\Delta y(t)| \le O(\varepsilon\delta\mu) + \int_{t_0}^{t}\Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)\Big]|\Delta y(\xi)|\,d\xi, \quad t \in [t_0, r_2+\delta_1].$$

By the Gronwall–Bellman inequality, from the above we obtain (2.59). □

2.6 Proof of Theorem 2.2

First of all, we note that $\delta\mu \in \Im_+^{(1)}$ i.e., $t_{00} < t_0$, therefore we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } t \in [\widehat{\tau}, t_{00}), \\ \varphi(t) - y_0(t) & \text{for } t \in [t_{00}, t_0), \\ \Delta y(t) & \text{for } t \in [t_0, t_{10} + \delta_1]. \end{cases}$$

By Lemma 2.10, we get

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall\, (t, \varepsilon, \delta\mu) \in [t_0, t_{10} + \delta_1] \times (0, \varepsilon_2) \times \Im_+^{(1)} \tag{2.71}$$

and

$$\Delta x(t_0) = \varepsilon[\delta x_0 - f_0^+ \delta t_0] + o(\varepsilon\delta\mu). \tag{2.72}$$

The function $\Delta x(t)$ satisfies the equation (2.44) on the interval $[t_0, t_{10} + \delta_1]$; therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0; t)\Delta x(t_0) + \varepsilon \int_{t_0}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{p=0}^{2} R_p(t; t_0, \varepsilon\delta\mu), \ \ t \in [t_0, t_{10} + \delta_1] \tag{2.73}$$

(see (2.48)). Let $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ be insomuch small that the inequalities

$$t_0 + \tau_i < t_{10} - \delta_2, \ \ i = \overline{1, s}, \ \ t_{00} + \tau_{s0} < t_{10} - \delta_2$$

hold. The function $Y(\xi; t)$ is continuous on the set $[t_{00}, t_{00} + \tau_{s0}] \times [t_{10} - \delta_2, t_{10} + \delta_2] \subset \Pi$. Therefore,

$$Y(t_0; t)\Delta x(t_0) = \varepsilon Y(t_{00}; t)[\delta x_0 - f_0^+ \delta t_0] + o(t; \varepsilon\delta\mu) \tag{2.74}$$

(see (2.72)).

Let us transform $R(t; t_0, \varepsilon\delta\mu)$. It is not difficult to see that

$$\begin{aligned} R_0(t; t_0, \varepsilon\delta\mu) &= \sum_{i=1}^{s} \int_{t_0-\tau_i}^{t_0} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi)\, d\xi \\ &= \sum_{i=1}^{s} \Bigg[\varepsilon \int_{t_0-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t) f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + \int_{t_{00}}^{t_0} Y(\xi+\tau_{i0}; t) f_{0x_i}[\xi+\tau_{i0}]\Delta x(\xi)\, d\xi\Bigg] \\ &= \varepsilon \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi \\ &\qquad + \sum_{i=1}^{s} \int_{t_{00}+\tau_{i0}}^{t_0+\tau_{i0}} Y(\xi; t) f_{0x_i}[\xi]\Delta x(\xi - \tau_{i0})\, d\xi + o(t; \varepsilon\delta\mu). \end{aligned} \tag{2.75}$$

In a similar way, with nonessential changes, one can prove

$$\begin{aligned} R_1(t; t_0, \varepsilon\delta\mu) &= -\varepsilon \sum_{i=1}^{s} \Bigg[\int_{t_0}^{t} Y(\xi; t) f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\, d\xi\Bigg]\delta\tau_i \\ &\qquad - \sum_{i=1}^{s} \int_{t_{00}+\tau_{i0}}^{t_0+\tau_{i0}} Y(\xi; t) f_{0x_1}[\xi]\Delta x(\xi-\tau_{i0})\, d\xi - \varepsilon \sum_{i=1}^{s} f_{i0}(\delta t_0 + \delta\tau_i) + o(t; \varepsilon\delta\mu), \end{aligned} \tag{2.76}$$

$$R_2(t; t_0, \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu). \tag{2.77}$$

Finally, note that

$$\varepsilon\int\limits_{t_0}^{t}Y(\xi;t)\delta f[\xi]\,d\xi=\varepsilon\int\limits_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]d\xi+o(t;\varepsilon\delta\mu). \tag{2.78}$$

From (2.73), by virtue of (2.74)–(2.78), we obtain (2.16), where

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f_0^+\delta t_0+\beta(t;\delta\mu).$$

3 Variation formulas of solutions for equations with the continuous initial condition

3.1 Formulation of the main results

To each element

$$\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,f)\in\Lambda^{(2)}=[a,b)\times[\theta_{11},\theta_{12}]\times\cdots\times[\theta_{s1},\theta_{s2}]\times\Phi_2\times E_f^{(1)}$$

we assign the functional differential equation

$$\dot x(t)=f\big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s)\big) \tag{3.1}$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0], \tag{3.2}$$

where $\Phi_2=\{\varphi\in C(I_1,\mathbb{R}^n):\ \varphi(t)\in O\}$.

Definition 3.1. Let $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,f)\in\Lambda^{(2)}$. A function $x(t)=x(t;\mu)\in O$, $t\in[\widehat\tau,t_1]$, $t_1\in(t_0,b]$, is called a solution of the equation (3.1) with the initial condition (3.2), or a solution corresponding to the element μ and defined on the interval $[\widehat\tau,t_1]$, if it satisfies the condition (3.2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies the equation (3.1) a.e. on $[t_0,t_1]$.

Let $x_0(t)$ be a solution corresponding to a fixed element

$$\mu_0=(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,f_0)\in\Lambda^{(2)}$$

and defined on the interval $[\widehat\tau,t_{10}]$, where $t_{00},t_{10}\in(a,b)$, $t_{00}<t_{10}$, and $\tau_{i0}\in(\theta_{1i},\theta_{2i})$, $i=\overline{1,s}$.

In the space $E_{\delta\mu}^{(2)}=E_\mu^{(2)}-\mu_0$ with the elements $\delta\mu=(\delta t_0,\delta\tau_1,\dots,\delta\tau_s,\delta\varphi,\delta f)$, where $E_\mu^{(2)}=\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\times C(I_1,\mathbb{R}^n)\times E_f^{(2)}$, we introduce the set of variations

$$\begin{aligned}\Im^{(2)}=\Big\{\delta\mu=(\delta t_0,\delta\tau_1,\dots,\delta\tau_s,\delta\varphi,\delta f):\ \ |\delta t_0|\le\gamma,\ \ |\delta\tau_i|\le\gamma,\ \ i=\overline{1,s},\\ \delta\varphi=\sum_{i=1}^k\lambda_i\delta\varphi_i,\ \ \delta f=\sum_{i=1}^k\lambda_i\delta f_i,\ \ |\lambda_i|\le\gamma,\ \ i=\overline{1,k}\Big\},\end{aligned} \tag{3.3}$$

where $\delta\varphi_i\in\Phi_2-\varphi_0$, $\delta f_i\in E_f^{(2)}-f_0$, $i=\overline{1,k}$, are fixed functions; $\gamma>0$ is a fixed number.

There exist numbers $\delta_1>0$ and $\varepsilon_1>0$ such that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times\Im^{(2)}$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda^{(2)}$, and to the element $\mu_0+\varepsilon\delta\mu$ there corresponds the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat\tau,t_{10}+\delta_1]\subset I_1$.

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat\tau,t_{10}+\delta_1]$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat\tau,t_{10}+\delta_1]$.

Let us define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t)=\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t)\ \ \forall\,(t,\varepsilon,\delta\mu)\in[\widehat\tau,t_{10}+\delta_1]\times(0,\varepsilon_1)\times\Im^{(2)}. \tag{3.4}$$

Theorem 3.1. *Let the function $\varphi_0(t)$, $t \in I_1$, be absolutely continuous. Let the functions $\dot\varphi_0(t)$ and $f_0(w)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{s+1}$ be bounded. Moreover, there exist the finite limits*

$$\dot\varphi_0^- = \dot\varphi_0(t_{00}-), \quad \lim_{w \to w_0} f_0(w) = f_0^-, \;\; w \in (a, t_{00}] \times O^{s+1},$$

where

$$w_0 = \big(t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0})\big).$$

Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_-^{(2)}$,

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3.5}$$

where $\Im_-^{(2)} = \{\delta\mu \in \Im^{(2)} : \; \delta t_0 \le 0\}$ and

$$\delta x(t; \delta\mu) = Y(t_{00}; t)(\dot\varphi_0^- - f_0^-)\delta t_0 + \beta(t; \delta\mu), \tag{3.6}$$

$$\begin{aligned} \beta(t; \delta\mu) = Y(t_{00}; t)\delta\varphi(t_{00}) + \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi \\ - \int_{t_{00}}^{t} Y(\xi; t)\Big[\sum_{i=1}^{s} f_{0x_i}[\xi]\dot x_0(\xi - \tau_{i0})\delta\tau_i\Big]\, d\xi + \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi. \end{aligned} \tag{3.7}$$

Here $Y(\xi; t)$ is the $n \times n$-matrix function satisfying the equation

$$Y_\xi(\xi; t) = -Y(\xi; t) f_{0x}[\xi] - \sum_{i=1}^{s} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}], \;\; \xi \in [t_{00}, t], \tag{3.8}$$

and the condition

$$Y(t; t) = \Upsilon, \quad Y(\xi; t) = \Theta, \tag{3.9}$$

where

$$\begin{aligned} f_{0x_i}[\xi] &= f_{0x_i}\big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \dots, x_0(\xi - \tau_{s0})\big), \\ \delta f[\xi] &= \delta f\big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \dots, x_0(\xi - \tau_{s0})\big). \end{aligned}$$

Some comments. On the basis of the Cauchy formula, we can conclude that the function

$$\delta x(t) = \begin{cases} \delta\varphi(t), & t \in [\hat\tau, t_{00}), \\ \delta x(t; \delta\mu), & t \in [t_{00}, t_{10} + \delta_2], \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\delta x(t - \tau_{i0}) - \sum_{i=1}^{s} f_{0x_i}[t]\dot x_0(t - \tau_{i0})\delta\tau_i + \delta f[t]$$

with the discontinuous initial condition

$$\delta x(t) = \delta\varphi(t), \;\; t \in [\hat\tau, t_{00}), \;\; \delta x(t_{00}) = (\dot\varphi_0^- - f_0^-)\delta t_0 + \delta\varphi(t_{00}).$$

The addend

$$-\int_{t_{00}}^{t} Y(\xi; t)\Big[\sum_{i=1}^{s} f_{0x_i}[\xi]\dot x_0(\xi - \tau_{i0})\delta\tau_i\Big]\, d\xi$$

in the formula (3.7) is the effect of perturbations of the delays τ_{i0}, $i = \overline{1, s}$.

The expression

$$Y(t_{00};t)(\dot\varphi_0^- - f_0^-)\delta t_0$$

in the formula (3.6) is the effect of the continuous initial condition (3.2) and perturbation of the initial moment t_{00}.

The expression

$$Y(t_{00};t)\delta\varphi(t_{00}) + \sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(s+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$

in the formula (3.7) is the effect of perturbation of the initial function $\varphi_0(t)$.

The addend

$$\int\limits_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi$$

in (3.7) is the effect of perturbation of the right-hand side of equation

$$\dot x(t) = f_0\big(t, x(t), x(t-\tau_{10}),\dots,x(t-\tau_{s0})\big).$$

Theorem 3.2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous. Let the functions $\dot\varphi_0(t)$ and $f_0(w)$, $w=(t,x,x_1,\dots,x_s)\in I\times O^{s+1}$, be bounded. Moreover, there exist the finite limits*

$$\dot\varphi_0^+ = \dot\varphi_0(t_{00}+),\quad \lim_{w\to w_0} f_0(w) = f_0^+,\ \ w\in[t_{00},b)\times O^{s+1}.$$

Then for each $\widehat t_0\in(t_{00},t_{10})$ there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat t_0, t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im_+^{(2)}$, where $\Im_+^{(2)}=\{\delta\mu\in\Im^{(2)}:\ \delta t_0\ge 0\}$, the formula (3.5) *holds, where*

$$\delta x(t;\delta\mu) = Y(t_{00};t)(\dot\varphi_0^+ - f_0^+)\delta t_0 + \beta(t;\delta\mu). \tag{3.10}$$

The following assertion is a corollary to Theorems 3.1 and 3.2.

Theorem 3.3. *Let the assumptions of Theorems* 3.1 *and* 3.2 *be fulfilled. Moreover, $\dot\varphi_0^- - f_0^- = \dot\varphi_0^+ - f_0^+ := \widehat f_0$. Then, for each $\widehat t_0\in(t_{00},t_{10})$, there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat t_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im^{(2)}$ the formula* (3.5) *holds, where*

$$\delta x(t;\delta\mu) = Y(t_{00};t)\widehat f_0\delta t_0 + \beta(t;\delta\mu).$$

All the assumptions of Theorem 3.3 are satisfied if the function $f_0(t,x,x_1,\dots,x_s)$ is continuous and bounded, and the function $\varphi_0(t)$ is continuously differentiable. Clearly, in this case,

$$\widehat f_0 = \dot\varphi_0(t_{00}) - f_0\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0})\big).$$

Theorems 3.1–3.3 correspond to the cases where there exist the left-sided, right-sided and two-sided variations of the initial moment t_{00}, respectively.

3.2 Lemma on estimation of the increment of a solution with respect to the variation set $\Im_-^{(2)}$

To each element $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,f)\in\Lambda^{(2)}$ we assign the functional differential equation

$$\dot y(t) = f(t_0,\tau_1,\dots,\tau_s,\varphi,y)(t) \tag{3.11}$$

with the initial condition

$$y(t_0)=\varphi(t_0) \tag{3.12}$$

(see (1.16)).

Definition 3.2. Let $\mu = (t_0, \tau_1, \dots, \tau_s, \varphi, f) \in \Lambda^{(2)}$. An absolutely continuous function $y(t) = y(t;\mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (3.11) with the initial condition (3.12), or a solution corresponding to the element μ and defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = \varphi(t_0)$ and the function $y(t)$ satisfies the equation (3.11) a.e. on $[r_1, r_2]$.

Remark 3.1. Let $y(t;\mu)$, $t \in [r_1, r_2]$, be a solution corresponding to the element $\mu = (t_0, \tau_1, \dots, \tau_s, \varphi, f) \in \Lambda^{(2)}$. Then the function

$$x(t;\mu) = h(t_0, \varphi, y(\,\cdot\,;\mu))(t), \quad t \in [\widehat{\tau}, r_2], \tag{3.13}$$

is a solution of the equation (3.11) with the initial condition (3.12) (see Definition 3.1 and (1.18)).

Lemma 3.1. *Let $y_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, f_0) \in \Lambda^{(2)}$ and defined on $[r_1, r_2] \subset (a, b)$; let $t_{00} \in [r_1, r_2)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then there exist the numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(2)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(2)}$. In addition, to this element there corresponds a solution $y(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$. Moreover,*

$$\begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t) \in K_1, & t \in I_1, \\ y(t;\mu_0 + \varepsilon\delta\mu) \in K_1, & t \in [r_1 - \delta_1, r_2 + \delta_1], \end{cases} \tag{3.14}$$

and

$$\lim_{\varepsilon \to 0} y(t;\mu_0 + \varepsilon\delta\mu) = y(t;\mu_0)$$

uniformly for $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(2)}$.

This lemma is a consequence of Theorem 1.7.

Lemma 3.2. *Let $x_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{is}, \varphi_0, f_0) \in \Lambda^{(2)}$ and defined on $[\widehat{\tau}, t_{10}]$ (see Definition 3.1), let $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exist the numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(2)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(2)}$. In addition, to this element there corresponds a solution $x(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover,*

$$x(t;\mu_0 + \varepsilon\delta\mu) \in K_1, \quad t \in [\widehat{\tau}, t_{10} + \delta_1]. \tag{3.15}$$

It is easy to see that if in Lemma 3.1 one puts $r_1 = t_{00}$, $r_2 = t_{10}$, then $x_0(t) = y_0(t)$, $t \in [t_{00}, t_{10}]$, and

$$x(t;\mu_0 + \varepsilon\delta\mu) = h\big(t_0, \varphi, y(\,\cdot\,;\mu_0 + \varepsilon\delta\mu)\big)(t) \;\; \forall\,(t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im^{(2)}$$

(see (3.13)). Thus, Lemma 3.2 is a simple corollary of Lemma 3.1 (see (3.14)).

Due to the uniqueness, the solution $y(t;\mu_0)$ on the interval $[r_1 - \delta_1, r_2 + \delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1 - \delta_1, r_2 + \delta_1]$.

Lemma 3.1 allows one to define the increment of the solution $y_0(t) = y(t;\mu_0)$:

$$\Delta y(t) = \Delta y(t;\varepsilon\delta\mu) = y(t;\mu_0 + \varepsilon\delta\mu) - y_0(t) \;\; \forall\,(t, \varepsilon, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times (0, \varepsilon_1) \times \Im^{(2)}. \tag{3.16}$$

Obviously,

$$\lim_{\varepsilon \to 0} \Delta y(t;\varepsilon\delta\mu) = 0 \tag{3.17}$$

uniformly with respect to $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(2)}$ (see Lemma 3.1).

Lemma 3.3. *Let the conditions of Theorem 3.1 hold. Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that*

$$\max_{t \in [t_{00}, r_2 + \delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{3.18}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im^{(2)}_{-}$. Moreover,

$$\Delta y(t_{00}) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^- - f_0^-)\delta t_0\big] + o(\varepsilon\delta\mu). \tag{3.19}$$

Proof. Let $\varepsilon_2' \in (0, \varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2') \times \Im_-^{(2)}$ the following inequalities

$$t_0 + \tau_i > t_{00}, \quad i = \overline{1, s}, \tag{3.20}$$

hold, where $t_0 = t_{00} + \varepsilon\delta t_0$, $\tau_i = \tau_{i0} + \varepsilon\delta\tau_i$. On the interval $[t_{00}, r_2 + \delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$, where $y(t) = y(t; \mu + \varepsilon\delta\mu)$, satisfies the equation

$$\dot{\Delta} y(t) = a(t; \varepsilon\delta\mu) + \varepsilon b(t; \varepsilon\delta\mu), \tag{3.21}$$

where

$$\begin{aligned} a(t; \varepsilon\delta\mu) &= f_0(t_0, \tau_1, \dots, \tau_s, \varphi, y_0 + \Delta y)(t) - f_0(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, y_0)(t), \\ b(t; \varepsilon\delta\mu) &= \delta f(t_0, \tau_1, \dots, \tau_s, \varphi, y_0 + \Delta y)(t). \end{aligned} \tag{3.22}$$

We rewrite the equation (3.21) in the integral form

$$\Delta y(t) = \Delta y(t_{00}) + \int_{t_{00}}^{t} \big[a(\xi; \varepsilon\delta\mu) + \varepsilon b(\xi; \varepsilon\delta\mu)\big]\, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_{00})| + a_1(t; t_{00}, \varepsilon\delta\mu) + \varepsilon b_1(t_{00}; \varepsilon\delta\mu), \tag{3.23}$$

where

$$a_1(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} |a(\xi; \varepsilon\delta\mu)|\, d\xi, \quad b_1(t_{00}; \varepsilon\delta\mu) = \int_{t_{00}}^{r_2+\delta_1} |b(\xi; \varepsilon\delta\mu)|\, d\xi.$$

Let us prove the formula (3.19). We have

$$\begin{aligned} \Delta y(t_{00}) &= y(t_{00}; \mu_0 + \varepsilon\delta\mu) - y_0(t_{00}) = \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) \\ &+ \int_{t_0}^{t_{00}} \Big[f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \dots, \varphi(t - \tau_s)\big) + \varepsilon b(t; \varepsilon\delta\mu)\Big]\, dt - \varphi_0(t_{00}) \end{aligned} \tag{3.24}$$

(see (3.20)). Since

$$\int_{t_{00}}^{t_0} \dot{\varphi}_0(t)\, dt = \varepsilon\dot{\varphi}_0^- \delta t_0 + o(\varepsilon\delta\mu)$$

and

$$\lim_{\varepsilon\to 0} \delta\varphi(t_0) = \delta\varphi(t_{00}) \ \text{ uniformly with respect to } \ \delta\mu \in \Im_-^{(2)}$$

(see (3.3)), we have

$$\begin{aligned} &\varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00}) \\ &\quad = \int_{t_{00}}^{t_0} \dot{\varphi}_0(t)\, dt + \varepsilon\delta\varphi(t_{00}) + \varepsilon\big[\delta\varphi(t_0) - \delta\varphi(t_{00})\big] = \varepsilon[\dot{\varphi}_0^- \delta t_0 + \delta\varphi(t_{00})] + o(\varepsilon\delta\mu). \end{aligned} \tag{3.25}$$

It is clear that if $t \in [t_0, t_{00}]$, then

$$\lim_{\varepsilon\to 0} \big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \dots, \varphi(t - \tau_s)\big) = \lim_{t\to t_{00}-} \big(t, y_0(t), \varphi_0(t - \tau_{10}), \dots, \varphi_0(t - \tau_{s0})\big) = w_0$$

(see (3.17)). Consequently,

$$\lim_{\varepsilon\to 0} \sup_{t\in[t_0, t_{00}]} \big|f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \dots, \varphi(t - \tau_s)\big) - f_0^-\big| = 0.$$

This relation implies that

$$\int_{t_0}^{t_{00}} f_0\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big)\,dt = -\varepsilon f_0^-\delta t_0$$

$$+\int_{t_0}^{t_{00}} \Big[f_0\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big) - f_0^-\Big]\,dt = -\varepsilon f_0^-\delta t_0 + o(\varepsilon\delta\mu). \quad (3.26)$$

Further, we have

$$\int_{t_0}^{t_{00}} |b(t;\varepsilon\delta\mu)|\,dt \le \int_{t_0}^{t_{00}} \sum_{i=1}^{k} |\lambda_i|\,\big|\delta f_i(t_0,\tau_1,\dots,\tau_s,\varphi,y_0+\Delta y)(t)\big|\,dt \le \gamma \sum_{i=1}^{k} \int_{t_0}^{t_{00}} m_{\delta f_i,K_1}(t)\,dt \quad (3.27)$$

(see (3.16), (3.3) and (3.14)).

From (3.24), by virtue of (3.25)–(3.27), we obtain (3.19).

Let us now prove the inequality (3.18). Towards this end, we estimate $a_1(t;t_{00},\varepsilon\delta\mu)$, $t\in[t_{00},r_2+\delta_1]$. Obviously,

$$a_1(t;t_{00},\varepsilon\delta\mu) \le \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t;t_{00},\varepsilon\delta\mu), \quad (3.28)$$

where

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|h(t_0,\varphi,y_0+\Delta y)(\xi-\tau_i) - h(t_{00},\varphi_0,y_0)(\xi-\tau_{i0})\big|\,d\xi$$

(see (3.22)).

Let $t_{00}+\tau_{i0}\le r_2$ and let $\varepsilon_2{}'$ be insomuch small that $t_{00}+\tau_i < r_2+\delta_1$.

Furthermore, let

$$\rho_{i1} = \min\{t_0+\tau_i, t_{00}+\tau_{i0}\}, \quad \rho_{i2} = \max\{t_{00}+\tau_i, t_{00}+\tau_{i0}\}.$$

It is easy to see that

$$\rho_{i2} \ge \rho_{i1} > t_{00} \ \text{ and } \ \rho_{i2}-\rho_{i1} = O(\varepsilon\delta\mu).$$

Let $t\in[t_{00},\rho_{i1})$, then for $\xi\in[t_{00},t]$, we have $\xi-\tau_i<t_0$ and $\xi-\tau_{i0}<t_{00}$, and hence,

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi.$$

From the boundedness of the function $\dot\varphi_0(t)$, $t\in I_1$, it follows that

$$\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big| = \big|\varphi_0(\xi-\tau_i)+\varepsilon\delta\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|$$

$$= O(\varepsilon\delta\mu) + \bigg|\int_{\xi-\tau_{i0}}^{\xi-\tau_i} \dot\varphi_0(t)\,dt\bigg| = O(\varepsilon\delta\mu). \quad (3.29)$$

Thus, for $t\in[t_{00},\rho_{i1}]$, we have

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \quad i=\overline{1,s}. \quad (3.30)$$

Let $t\in[\rho_{i1},\rho_{i2}]$, then

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) \le a_{2i}(\rho_{i1};t_{00},\varepsilon\delta\mu) + a_{2i}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + a_{2i}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu).$$

Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_i$, i.e., $t_0 + \tau_i < t_{00} + \tau_{i0} < t_{00} + \tau_i$. We have

$$\begin{aligned}
a_{2i}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu) &\le \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-y_0(\xi-\tau_{i0})\big|\,d\xi \\
&\le \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-\varphi(\xi-\tau_i)\big|\,d\xi \\
&\quad + \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-\varphi(\xi-\tau_i)\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_{i0})-y_0(\xi-\tau_{i0})\big|\,d\xi \\
&\le o(\varepsilon\delta\mu) + \int_{t_0+\tau_i}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-\varphi(\xi-\tau_i)\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_{i0})-y_0(\xi-\tau_{i0})\big|\,d\xi \\
&= o(\varepsilon\delta\mu) + \int_{t_0}^{t_{00}} L_{f_0,K_1}(\xi+\tau_i)\big|y(\xi;\mu_0+\varepsilon\delta\mu)-\varphi(\xi)\big|\,d\xi \\
&\quad + \int_{t_{00}}^{t_{00}+\tau_i-\tau_{i0}} L_{f_0,K_1}(\xi+\tau_{i0})|\varphi_0(\xi)-y_0(\xi)|\,d\xi
\end{aligned}$$

(see (3.29)). The functions $f_0(t,x,x_1,\dots,x_s), (t,x,x_1,\dots,x_s) \in I \times O^{1+s}$ and $\dot\varphi_0(t)$, $t \in I_1$, are bounded; therefore,

$$\begin{aligned}
&\big|y(\xi;\mu_0+\varepsilon\delta\mu)-\varphi(\xi)\big| \\
&= \left|\varphi(t_0) + \int_{t_0}^{\xi} \big[f_0(t_0,\tau_1,\dots,\tau_s,\varphi,y_0+\Delta y)(t) + b(t;\varepsilon\delta\mu)\big]\,dt - \varphi(\xi)\right| \le O(\varepsilon\delta\mu),\quad \xi \in [t_0,t_{00}], \qquad (3.31)
\end{aligned}$$

and

$$|\varphi_0(\xi)-y_0(\xi)| = \left|\varphi_0(\xi)-\varphi_0(t_{00}) - \int_{t_{00}}^{\xi} f_0(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,y_0)(t)\,dt\right| \le O(\varepsilon\delta\mu),$$
$$\xi \in [t_{00}, t_{00}+\tau_i-\tau_{i0}]\ \ (\tau_i > \tau_{i0}).$$

Thus,

$$a_{2i}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu)=o(\varepsilon\delta\mu).$$

Let $\rho_{i1}=t_0+\tau_i$ and $\rho_{i2}=t_{00}+\tau_{i0}$, then

$$a_{i2}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu)=\int\limits_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi=o(\varepsilon\delta\mu).$$

Let $\rho_{i1}=t_{00}+\tau_{i0}$ and $\rho_{i2}=t_{00}+\tau_i$, i.e., $t_{00}+\tau_{i0}<t_0+\tau_i<t_{00}+\tau_i$. We have

$$\begin{aligned}a_{i2}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu)\le\int\limits_{t_{00}+\tau_{i0}}^{t_0+\tau_i} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi\\+\int\limits_{t_0+\tau_i}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu)-y_0(\xi-\tau_{i0})\big|\,d\xi=o(\varepsilon\delta\mu).\end{aligned}$$

Consequently, for $t\in[t_{00},\rho_{i2}]$, the inequality (3.30) holds.

Let $t\in[\rho_{i2},r_2+\delta_1]$, then $t-\tau_i\ge t_0$ and $t-\tau_{i0}\ge t_{00}$, therefore,

$$\begin{aligned}a_{2i}(t;t_{00},\varepsilon\delta\mu)&=a_{2i}(\rho_{i2};t_{00},\varepsilon\delta\mu)+\int\limits_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\big|y_0(\xi-\tau_i)+\Delta y(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi\\&\le O(\varepsilon\delta\mu)+\int\limits_{\rho_{i2}-\tau_i}^{t-\tau_i} L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi+\int\limits_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\big|y_0(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi\\&\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\chi(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi+\int\limits_{\rho_{i2}}^{r_2+\delta_1} L_{f_0,K_1}(\xi)\big|y_0(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi.\end{aligned}$$

Further, for $\xi\in[\rho_{i2},r_2+\delta_1]$,

$$\big|y_0(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\le\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_I}\big|f_0(t_{00},\tau_{i1},\dots,\tau_{is},y_0)(t)\big|\,dt\le O(\varepsilon\delta\mu).$$

Thus, for $t\in[t_{00},r_2+\delta_1]$, we get

$$a_{2i}(t;t_{00},\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi. \tag{3.32}$$

We now consider the case where $t_{00}+\tau_{i0}>r_2$. Let $\delta_2\in(0,\delta_1)$ and $\varepsilon_2''\in(0,\varepsilon_1)$ be insomuch small numbers that $t_{00}+\tau_{i0}>r_2+\delta_2$ and $t_0+\tau_i>r_2+\delta_2$ for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2'')\times\Im_-^{(2)}$.

It is easy to see that

$$a_{2i}(t;t_{00},\varepsilon\delta\mu)\le\int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,dt\le O(\varepsilon\delta\mu).$$

Thus, for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},r_2+\delta_2]\times(0,\varepsilon_2)\times\Im_-^{(2)}$ and $i=\overline{1,s}$, where $\varepsilon_2=\min(\varepsilon_2',\varepsilon_2'')$, the inequality (3.32) holds.

Consequently, we have

$$a_1(t;t_{00},\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int_{t_{00}}^{t}\Big[L_{f,K_1}(\xi)+\sum_{i=1}^{s}\chi_1(\xi+\tau_i)L_{f,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi, \qquad t\in[t_{00},r_2+\delta_1], \tag{3.33}$$

(see (3.28)). Obviously,

$$b_1(t_{00},\varepsilon\delta\mu)\le\gamma\int_{t_{00}}^{r_2+\delta_2}\sum_{i=1}^{k}\big|\delta f_i(t_0,\tau_1,\dots,\tau_s,y_0+\Delta y)(t)\big|\,dt\le\gamma\sum_{i=1}^{k}\int_I m_{\delta f_i,K_1}(t)\,dt. \tag{3.34}$$

According to (3.19), (3.33) and (3.34), the inequality (3.23) directly implies

$$|\Delta y(t)|\le O(\varepsilon\delta\mu)+\int_{t_{00}}^{t}\Big[L_{f,K_1}(\xi)+\sum_{i=1}^{s}\chi_1(\xi+\tau_i)L_{f,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi,\ \ t\in[t_{00},r_2+\delta_2].$$

By the Gronwall–Bellman inequality lemma, from the above we obtain (3.18). □

3.3 Proof of Theorem 3.1

Let $r_1=t_{00}$ and $r_2=t_{10}$ as in Lemma 3.1, then

$$x_0(t)=\begin{cases}\varphi_0(t), & t\in[\widehat{\tau},t_{00}),\\ y_0(t), & t\in[t_{00},t_{10}],\end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times\Im_-^{(2)}$,

$$x(t;\mu_0+\varepsilon\delta\mu)=\begin{cases}\varphi(t), & t\in[\widehat{\tau},t_0),\\ y(t;\mu_0+\varepsilon\delta\mu), & t\in[t_0,t_{10}+\delta_1]\end{cases}$$

(see (3.13)).

We note that $\delta\mu\in\Im_-^{(2)}$, i.e., $t_0<t_{00}$, therefore

$$\Delta x(t)=\begin{cases}\varepsilon\delta\varphi(t) & \text{for } t\in[\widehat{\tau},t_0),\\ y(t;\mu_0+\varepsilon\delta\mu)-\varphi_0(t) & \text{for } t\in[t_0,t_{00}),\\ \Delta y(t) & \text{for } t\in[t_{00},t_{10}+\delta_1]\end{cases}$$

(see (3.4) and (3.16)).

By Lemma 3.3 and the relation

$$\big|y(t;\mu_0+\varepsilon\delta\mu)-\varphi_0(t)\big|\le O(\varepsilon\delta\mu),\ \ t\in[t_0,t_{00}],$$

we have

$$|\Delta x(t)|\le O(\varepsilon\delta\mu)\ \ \forall\,(t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im_-^{(2)}, \tag{3.35}$$

$$\Delta x(t_{00})=\varepsilon\big[\delta\varphi(t_{00})+(\dot{\varphi}_0^- - f_0^-)\delta t_0\big]+o(\varepsilon\delta\mu). \tag{3.36}$$

The function $\Delta x(t)$ satisfies the equation

$$\begin{aligned}\dot{\Delta}x(t)&=f_0[t,x_0+\Delta x]+\varepsilon\delta f[t,x_0+\Delta x]-f_0[t]\\ &=f_{0x}[t]\Delta x(t)+\sum_{i=1}^{s}f_{0x_i}[t]\Delta x(t-\tau_{i0})+\varepsilon\delta f[t]+\sum_{i=1}^{2}\vartheta_i(t;\varepsilon\delta\mu)\end{aligned} \tag{3.37}$$

on the interval $[t_{00},t_{10}+\delta_2]$, where

$$\begin{aligned}
f_0[t,x_0+\Delta x] &= f_0\Big(t,x_0(t)+\Delta x(t),x_0(t-\tau_1)+\Delta x(t-\tau_1),\dots,x_0(t-\tau_s)+\Delta x(t-\tau_s)\Big),\\
f_0[t] &= f_0\big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0})\big),\\
\delta f[t,x_0+\Delta x] &= \delta f\Big(t,x_0(t)+\Delta x(t),x_0(t-\tau_1)+\Delta x(t-\tau_1),\dots,x_0(t-\tau_s)+\Delta x(t-\tau_s)\Big),\\
\delta f[t] &= \delta f\big(t,x_0(t),x_0(t-\tau_1),\dots,x_0(t-\tau_s)\big),\\
\vartheta_1(t;\varepsilon\delta\mu) &= f_0[t,x_0+\Delta x]-f_0[t]-f_{0x}[t]\Delta x(t)-\sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t-\tau_{i0}), \qquad (3.38)\\
\vartheta_2(t;\varepsilon\delta\mu) &= \varepsilon\big[\delta f[t,x_0+\Delta x]-\delta f[t]\big]. \qquad (3.39)
\end{aligned}$$

By using the Cauchy formula, one can represent the solution of the equation (3.37) in the form

$$\Delta x(t)=Y(t_{00};t)\Delta x(t_{00})+\varepsilon\int_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi+\sum_{p=0}^{2} R_p(t;t_{00},\varepsilon\delta\mu),\ \ t\in[t_{00},t_{10}+\delta_2], \tag{3.40}$$

where

$$\begin{cases}
R_0(t;t_{00},\varepsilon\delta\mu)=\displaystyle\sum_{i=1}^{s} R_{i0}(t;t_{00},\varepsilon\delta\mu),\\
R_{i0}(t;t_{00},\varepsilon\delta\mu)=\displaystyle\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi,\\
R_p(t;t_{00},\varepsilon\delta\mu)=\displaystyle\int_{t_{00}}^{t} Y(\xi;t)\vartheta_p(\xi;\varepsilon\delta\mu)\,d\xi, \quad p=1,2,
\end{cases} \tag{3.41}$$

and $Y(\xi;t)$ is the matrix function satisfying the equation (3.8) and the condition (3.9). The function $Y(\xi;t)$ is continuous on the set

$$[t_{00}-\delta_2,t_{00}]\times[t_{00},t_{10}+\delta_2]\subset\Pi.$$

Therefore,

$$Y(t_{00};t)\Delta x(t_{00})=\varepsilon Y(t_{00};t)\big[\delta\varphi(t_{00})+(\dot\varphi_0^- - f_0^-)\delta t_0\big]+o(t;\varepsilon\delta\mu) \tag{3.42}$$

(see (3.36)). One can readily see that

$$\begin{aligned}
R_{i0}(t;t_{00},\varepsilon\delta\mu) &= \varepsilon\int_{t_{00}-\tau_{i0}}^{t_0} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int_{t_0}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\\
&= \varepsilon\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)d\xi+o(t;\varepsilon\delta\mu)
\end{aligned}$$

(see (3.35)). Thus

$$R_0(t;t_{00},\varepsilon\delta\mu)=\varepsilon\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{3.43}$$

We introduce the notations:

$$\begin{aligned}
f_0[t;\theta,\varepsilon\delta\mu]=f_0\Big(t,x_0(t)+\theta\Delta x(t),x_0(t-\tau_{10})+\theta(x_0(t-\tau_1)-x_0(t-\tau_{10})+\Delta x(t-\tau_1)),\dots,\\
x_0(t-\tau_{s0})+\theta\big(x_0(t-\tau_s)-x_0(t-\tau_{s0})+\Delta x(t-\tau_s)\big)\Big),
\end{aligned}$$

$$\sigma(t;\theta,\varepsilon\delta\mu)=f_{0x}[t;\theta,\varepsilon\delta\mu]-f_{0x}[t], \quad \varrho_i(t;\theta,\varepsilon\delta\mu)=f_{0x_i}[t;\theta,\varepsilon\delta\mu]-f_{0x_i}[t].$$

It is easy to see that

$$f_0([t,x_0+\Delta x]-f_0[t]=\int\limits_0^1\frac{d}{d\theta}\,f_0[t;\theta,\varepsilon\delta\mu]\,d\theta$$

$$=\int\limits_0^1\Big\{f_{0x}[t;\theta,\varepsilon\delta\mu]\Delta x(t)+\sum_{i=1}^s f_{0x_i}[t;\theta,\varepsilon\delta\mu]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big)\Big\}\,d\theta$$

$$=\Big[\int\limits_0^1\sigma(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\Delta x(t)+\sum_{i=1}^s\Big[\int\limits_0^1\varrho_i(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big)$$

$$+f_{0x}[t]\Delta x(t)+\sum_{i=1}^s f_{0x_i}[t]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big).$$

Taking into account the last relation for $t\in[t_{00},t_{10}+\delta_2]$, we have

$$R_1(t;t_{00},\varepsilon\delta\mu)=\sum_{p=3}^6 R_p(t;t_{00},\varepsilon\delta\mu),$$

where

$$R_3(t;t_{00},\varepsilon\delta\mu)=\int\limits_{t_{00}}^t Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)\,d\xi,\quad \sigma_1(\xi;\varepsilon\delta\mu)=\int\limits_0^1\sigma(\xi;s,\varepsilon\delta\mu)\,ds,$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^s\int\limits_{t_{00}}^t Y(\xi;t)\varrho_{i2}(\xi;\varepsilon\delta\mu)\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big]\,d\xi,$$

$$\varrho_{i2}(\xi;\varepsilon\delta\mu)=\int\limits_0^1\varrho_{i1}(\xi;\theta,\varepsilon\delta\mu)\,d\theta,$$

$$R_5(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^s\int\limits_{t_{00}}^t Y(\xi;t)f_{0x_i}[\xi]\,\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})\big]\,d\xi,$$

$$R_6(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^s\int\limits_{t_{00}}^t Y(\xi;t)f_{0x_i}[\xi]\,\big[\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big]\,d\xi$$

(see (3.38)). The function $x_0(t)$, $t\in[\widehat{\tau},t_{10}+\delta_2]$, is absolutely continuous, and for each fixed Lebesgue point $\xi_i\in(t_{00},t_{10}+\delta_2)$ of the function $\dot{x}_0(\xi-\tau_{i0})$ we get

$$x_0(\xi_i-\tau_i)-x_0(\xi_i-\tau_{i0})=\int\limits_{\xi_i}^{\xi_i-\varepsilon\delta\tau_i}\dot{x}_0(\varsigma-\tau_{i0})\,d\varsigma=-\varepsilon\dot{x}_0(\xi_i-\tau_{i0})\delta\tau_i+\gamma_i(\xi_i;\varepsilon\delta\mu),\tag{3.44}$$

where

$$\lim_{\varepsilon\to0}\frac{\gamma_i(\xi_i;\varepsilon\delta\mu)}{\varepsilon}=0\ \text{ uniformly for }\ \delta\mu\in\Im_-^{(2)}.\tag{3.45}$$

Thus (3.44) is valid for almost all points of the interval $(t_{00},t_{10}+\delta_2)$. From (3.44), taking into account the boundedness of the function

$$\dot{x}_0(t)=\begin{cases}\dot{\varphi}_0(t), & t\in[\widehat{\tau},t_{00}],\\ f_0\big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0})\big), & t\in(t_{00},t_{10}+\delta_2],\end{cases}$$

it follows that
$$\big|x_0(\xi_i-\tau_i)-x_0(\xi_i-\tau_{i0})\big|\le O(\varepsilon\delta\mu)\ \text{ and }\ \Big|\frac{\gamma_i(\xi_i;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const. \tag{3.46}$$
It is clear that
$$\big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big|=\begin{cases}o(\xi;\varepsilon\delta\mu) & \text{for } \xi\in[t_{00},\rho_{i1}],\\ O(\xi;\varepsilon\delta\mu) & \text{for } \xi\in[\rho_{i1},\rho_{i2}]\end{cases} \tag{3.47}$$
(see (3.35)).

Let $\xi\in[\rho_{i2},t_{10}+\delta_1]$, then $\xi-\tau_i\ge t_{00}$, $\xi-\tau_{i0}\ge t_{00}$. Therefore,
$$\begin{aligned}\big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big| &\le \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}|\dot{\Delta}x(s)|\,ds\\ &\le \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} L_{f_0,K_1}(\varsigma)\Big[|\Delta x(\varsigma)|+\sum_{i=1}^{s}\big|x_0(\varsigma-\tau_i)-x_0(\varsigma-\tau_{i0})\big|+|\Delta x(\varsigma-\tau_i)|\Big]\,d\varsigma\\ &\qquad+\varepsilon\alpha\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}\sum_{j=1}^{k} m_{\delta f_j,K_1}(\varsigma)\,d\varsigma=o(\xi;\varepsilon\delta\mu)\end{aligned} \tag{3.48}$$
(see (3.37), (3.15), (3.46) and (3.35)).

According to (3.35), (3.44) and (3.46)–(3.48), for the expressions $R_p(t;t_{00},\varepsilon\delta\mu)$, $p=3,4,5$, we have
$$|R_3(t;t_{00},\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu),\ \ \sigma_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\sigma_1(\xi;\varepsilon\delta\mu)|\,d\xi,$$
$$|R_4(t;t_{00},\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon\delta\mu)\sum_{i=1}^{s}\rho_{i2}(\varepsilon\delta\mu),\ \ \rho_{i2}(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\rho_{i1}(\xi;\varepsilon\delta\mu)|\,d\xi,$$
$$R_5(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+\sum_{i=1}^{s}\gamma_{i1}(t;\varepsilon\delta\mu),$$
where
$$\|Y\|=\sup\big\{|Y(\xi;t)|:\ (\xi,t)\in\Pi\big\},\ \ \gamma_{i1}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\gamma_i(\xi;\varepsilon\delta\mu)\,d\xi.$$
Obviously,
$$\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|\le\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_1}|f_{0x_i}[\xi]|\,\Big|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$
By the Lebesgue theorem on the passage to the limit under the integral sign, we have
$$\lim_{\varepsilon\to0}\sigma_2(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\rho_{i2}(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$
uniformly for $(t,\delta\mu)\in[t_{00},t_{10}+\delta_1]\times\Im_-^{(2)}$ (see (3.45)). Thus,
$$R_p(t;t_{00},\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu),\ \ p=3,4, \tag{3.49}$$
$$R_5(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+o(t;\varepsilon\delta\mu). \tag{3.50}$$

Further,

$$|R_6(t;t_{00},\varepsilon\delta\mu)| \le \|Y\| \int\limits_{t_{00}}^{t_{10}+\delta_1} \sum_{i=1}^{s} |f_{0x_i}[\xi]| \, \big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big| \, d\xi = o(\varepsilon\delta\mu). \tag{3.51}$$

On the basis of (3.49)–(3.51), we obtain

$$R_1(t;t_{00},\varepsilon\delta\mu) = -\varepsilon \sum_{i}^{s} \bigg[\int\limits_{t_{00}}^{t} Y(\xi;t) f_{0x_i}[\xi]\dot x_0(\xi-\tau_{i0})\, d\xi \bigg] \delta\tau_i + o(t;\varepsilon\delta\mu). \tag{3.52}$$

Next,

$$|R_2(t;t_{00},\varepsilon\delta\mu)|$$
$$\le \varepsilon\gamma \int\limits_{t_{00}}^{t_{10}+\delta_1} \sum_{j=1}^{k} L_{\delta f_j,K_1}(\xi) \Big[|\Delta x(\xi)| + \sum_{i=1}^{s} \big(|x_0(\xi-\tau)-x_0(\xi-\tau_0)| + |\Delta x(\xi-\tau)| \big) \Big] \, d\xi \le o(\varepsilon\delta\mu) \tag{3.53}$$

(see (3.39)).

From (3.40), by virtue of (3.42), (3.43), (3.52) and (3.53), we obtain (3.5), where $\delta x(t;\delta\mu)$ has the form (3.6).

3.4 Lemma on estimation of the increment of a solution with respect to the variation set $\Im_{+}^{(2)}$

Lemma 3.4. *Let the conditions of Theorem* 3.2 *hold. Then there exist the numbers* $\varepsilon_2 \in (0,\varepsilon_1)$ *and* $\delta_2 \in (0,\delta_1)$ *such that*

$$\max_{t\in[t_0,r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{3.54}$$

for arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_2] \times \Im_{+}^{(2)}$. *Moreover,*

$$\Delta y(t_0) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot\varphi_0^{+} - f_0^{+})\delta t_0\big] + o(\varepsilon\delta\mu). \tag{3.55}$$

Proof. Let a number $\varepsilon_2' \in (0,\varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2') \times \Im_{+}^{(2)}$, the inequalities

$$t_{00}+\tau_i > t_0, \ \ t_{00}+\tau_{i0} > t_0, \ \ i=\overline{1,s}, \tag{3.56}$$

hold, where $t_0 = t_{00} + \varepsilon\delta t_0$. On the interval $[t_0, r_2+\delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$ satisfies the equation

$$\dot\Delta y(t) = a(t;\varepsilon\delta\mu) + \varepsilon b(t;\varepsilon\delta\mu) \tag{3.57}$$

(see (3.21)). We rewrite the equation (3.57) in the integral form

$$\Delta y(t) = \Delta y(t_0) + \int\limits_{t_0}^{t} \big[a(\xi;\varepsilon\delta\mu) + \varepsilon b(\xi;\varepsilon\delta\mu)\big]\, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_0)| + a_1(t;t_0,\varepsilon\delta\mu) + \varepsilon b(t_0,\varepsilon\delta\mu). \tag{3.58}$$

Let us prove the formula (3.55). We have

$$\Delta y(t_0) = y(t_0;\mu_0+\varepsilon\delta\mu) - y_0(t_0)$$
$$= \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00}) - \int\limits_{t_{00}}^{t_0} \big[f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big)\big]\, dt \tag{3.59}$$

(see (3.56)). Since

$$\int_{t_{00}}^{t_0} \dot\varphi_0(t)\,dt = \varepsilon\dot\varphi_0^+\delta t_0 + o(\varepsilon\delta\mu)$$

and

$$\lim_{\varepsilon\to 0}\delta\varphi(t_0) = \delta\varphi(t_{00}) \ \text{ uniformly with respect to } \ \delta\mu\in\Im_+^{(2)}$$

(see (3.3)), we get

$$\varphi_0(t_0)+\varepsilon\delta\varphi(t_0)-\varphi_0(t_{00}) \\ = \int_{t_{00}}^{t_0}\dot\varphi_0(t)\,dt + \varepsilon\delta\varphi(t_{00}) + \varepsilon[\delta\varphi(t_0)-\delta\varphi(t_{00})] = \varepsilon[\dot\varphi_0^+\delta t_0+\delta\varphi(t_{00})]+o(\varepsilon\delta\mu). \quad (3.60)$$

It is clear that if $t\in[t_{00},t_0]$, then

$$\lim_{\varepsilon\to 0}\big(t,y_0(t)+\Delta y(t),\varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big) = \lim_{t\to t_{00}+}\big(t,y_0(t),\varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big) = w_0$$

(see (3.17)). Consequently,

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_{00},t_0]}\Big|f_0(t,y_0(t)+\Delta y(t),\varphi(t-\tau_1),\dots,\varphi(t-\tau_s)) - f_0^+\Big| = 0.$$

This relation implies that

$$\int_{t_0}^{t_{00}} f_0\big(t,y_0(t)+\Delta y(t),\varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big)\,dt \\ = -\varepsilon f_0^+\delta t_0 + \int_{t_0}^{t_{00}}\big[f_0\big(t,y_0(t)+\Delta y(t),\varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big)-f_0^+\big]\,dt = -\varepsilon f_0^+\delta t_0 + o(\varepsilon\delta\mu). \quad (3.61)$$

From (3.59), by virtue of (3.60) and (3.61), we obtain (3.55).

In order to prove the inequality (3.54) we estimate $a_1(t;t_0,\varepsilon\delta\mu)$, $t\in[t_0,r_2+\delta_1]$. Obviously,

$$a_1(t;t_0,\varepsilon\delta\mu) \le \int_{t_0}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t;t_0,\varepsilon\delta\mu) \quad (3.62)$$

(see (3.28)).

Let there exist $t_{00}+\tau_{i0}\le r_2$ and let $\varepsilon_2{}'\in(0,\varepsilon_1)$ be insomuch small that $t_0+\tau_i<r_2+\delta_1$. Furthermore, let

$$\rho_{i1}=\min\{t_{00}+\tau_i,t_{00}+\tau_{i0}\},\ \ \rho_{i2}=\max\{t_0+\tau_i,t_{00}+\tau_{i0}\}.$$

It is easy to see that

$$\rho_{i2}\ge\rho_{i1}>t_0,\ \ \rho_{i2}-\rho_{i1}=O(\varepsilon\delta\mu).$$

Let $t\in[t_0,\rho_{i1})$, then for $\xi\in[t_0,t]$ we have $\xi-\tau_i<t_0$ and $\xi-\tau_{i0}<t_{00}$. Therefore,

$$a_{2i}(t;t_0,\varepsilon\delta\mu) = \int_{t_0}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0}\big|\,d\xi.$$

From the boundedness of the function $\dot\varphi_0(t)$, $t\in I_1$, follows

$$\begin{aligned}\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big| &= \big|\varphi_0(\xi-\tau_i)+\varepsilon\delta\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\\ &= O(\varepsilon\delta\mu)+\Bigg|\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}\dot\varphi_0(t)\,dt\Bigg| = O(\varepsilon\delta\mu).\end{aligned}\tag{3.63}$$

Thus, for $t\in[t_0,\rho_{i1}]$, we have

$$a_{2i}(t;t_0,\varepsilon\delta\mu)\le O(\varepsilon\delta\mu),\ \ i=\overline{1,s}.\tag{3.64}$$

Let $t\in[\rho_{i1},\rho_{i2}]$, then

$$a_{2i}(t;t_0,\varepsilon\delta\mu)\le a_{2i}(\rho_{i1};t_0,\varepsilon\delta\mu)+a_{2i}(\rho_{21};\rho_{i1},\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+a_{2i}(\rho_{21};\rho_{i1},\varepsilon\delta\mu)$$

(see (3.64)).

Let $\rho_{i1}=t_{00}+\tau_i$ and $\rho_{i2}=t_0+\tau_i$, i.e., $t_{00}+\tau_i<t_{00}+\tau_{i0}<t_0+\tau_i$. We have

$$\begin{aligned}a_{2i}(\rho_{i2};\rho_{i1},\varepsilon\delta\mu)\le \int\limits_{t_{00}+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,d\xi\\ +\int\limits_{t_{00}+\tau_{i0}}^{t_0+\tau_i} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi\le o(\varepsilon\delta\mu)\end{aligned}$$

(see (3.63)). Consequently, for $t\in[t_0,\rho_{i2}]$, the inequality (3.64) holds.

Let $t\in[\rho_{i2},r_2+\delta_1]$, then $t-\tau_i\ge t_0$ and $t-\tau_{i0}\ge t_{00}$. Therefore

$$\begin{aligned}a_{2i}(t;t_0,\varepsilon\delta\mu) &= a_{2i}(\rho_{i2};t_0,\varepsilon\delta\mu)+\int\limits_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\big|y_0(\xi-\tau_i)+\Delta y(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi\\ &\le O(\varepsilon\delta\mu)+\int\limits_{\rho_{i2}-\tau_i}^{t-\tau_i} L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi+\int\limits_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\big|y_0(\xi-\tau_i)-y_0(\xi-\tau_{i0})\big|\,d\xi\\ &\le O(\varepsilon\delta\mu)+\int\limits_{t_0}^{t}\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi.\end{aligned}$$

Consequently, in this case we have

$$a_1(t;t_0,\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int\limits_{t_0}^{t}\Big[L_{f,K_1}(\xi)+\sum_{i=1}^{s}\chi_1(\xi+\tau_i)L_{f,K_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi,\ \ t\in[t_0,r_2+\delta_1],\tag{3.65}$$

(see (3.62)).

We now consider the case where $t_{00}+\tau_{i0}>r_2$. Let the numbers $\delta_2\in(0,\delta_1)$ and $\varepsilon_2''\in(0,\varepsilon_1)$ be insomuch small that $t_0+\tau_i>r_2+\delta_2$ for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2'')\times\Im_+^{(2)}$. It is easy to see that

$$a_{2i}(t;t_0,\varepsilon\delta\mu)\le\int\limits_{t_0}^{t}L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\big|\,dt\le O(\varepsilon\delta\mu).$$

Thus, for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},r_2+\delta_2]\times(0,\varepsilon_2)\times\Im_+^{(2)}$, where $\varepsilon_2=\min(\varepsilon_2',\varepsilon_2'')$, the inequality (3.65) holds.

Obviously,

$$b(t_0,\varepsilon\delta\mu)\le\gamma\int\limits_{t_0}^{r_2+\delta_2}\sum_{i=1}^{k}\big|\delta f_i(t_0,\tau_1,\dots,\tau_s,y_0+\Delta y)(t)\big|\,dt\le\gamma\sum_{i=1}^{k}\int\limits_{I}m_{\delta f_i,K_1}(t)\,dt.\tag{3.66}$$

According to (3.55), (3.65) and (3.66), the inequality (3.58) directly implies

$$|\Delta y(t)| \le O(\varepsilon\delta\mu) + \int_{t_0}^{t} \Big[L_{f,K_1}(\xi) + \sum_{i=1}^{s} \chi_1(\xi+\tau_i) L_{f,K_1}(\xi+\tau_i)\Big] |\Delta y(\xi)|\, d\xi, \ \ t \in [t_0, r_2+\delta_2].$$

By the Gronwall–Bellman inequality, from the above we obtain (3.54). □

3.5 Proof of Theorem 3.2

First of all, we note that $\delta\mu \in \Im_+^{(2)}$ i.e., $t_{00} < t_0$, therefore we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } \ t \in [\widehat{\tau}, t_{00}), \\ \varphi(t) - y_0(t) & \text{for } \ t \in [t_{00}, t_0), \\ \Delta y(t) & \text{for } \ t \in [t_0, t_{10}+\delta_1]. \end{cases}$$

In a similar way (see (3.31)), one can prove

$$|\varphi(t) - y_0(t)| = O(t; \varepsilon\delta\mu), \ \ t \in [t_{00}, t_0].$$

According to the last relation and Lemma 3.4, we have

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10}+\delta_2] \times [0, \varepsilon_2] \times \Im_+^{(2)}$$

and

$$\Delta x(t_0) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f_0^+)\big]\delta t_0 + o(\varepsilon\delta\mu).$$

Let $\widehat{t} \in (t_{00}, t_{10})$ be a fixed point, and let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that $t_0 < \widehat{t}$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_+^{(2)}$. The function $\Delta x(t)$ satisfies the equation (3.37) on the interval $[t_0, t_{10}+\delta_2]$. Therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0; t)\Delta x(t_0) + \varepsilon \int_{t_0}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{i=0}^{2} R_i(t; t_0, \varepsilon\delta\mu), \tag{3.67}$$

where $Y(\xi; t)$ is the matrix function satisfying the equation (3.8) and the condition (3.9). The matrix function $Y(\xi; t)$ is continuous on $[t_{00}, \widehat{t}) \times [\widehat{t}, t_{10}+\delta_2]$, therefore

$$Y(t_0; t)\Delta x(t_0) = \varepsilon Y(t_{00}; t)\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f_0^+)\big]\delta t_0 + o(\varepsilon\delta\mu). \tag{3.68}$$

Let us now transform

$$R_0(t; t_0, \varepsilon\delta\mu) = \sum_{i=1}^{s} R_{i0}(t; t_0, \varepsilon\delta\mu).$$

It is not difficult to see that

$$\begin{aligned} R_{i0}(t; t_0, \varepsilon\delta\mu) &= \varepsilon \int_{t_0-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t) f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + \int_{t_{00}}^{t_0} Y(\xi+\tau_0; t) f_{0x_i}[\xi+\tau_{i0}]\Delta x(\xi)\, d\xi \\ &= \varepsilon \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t) f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + o(t; \varepsilon\delta\mu). \end{aligned}$$

Thus,

$$R_0(t; t_0, \varepsilon\delta\mu) = \varepsilon \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t) f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + o(t; \varepsilon\delta\mu). \tag{3.69}$$

In a similar way, with nonessential changes, for $t \in [\widehat{t}, t_{10} + \delta_2]$, one can prove

$$R_1(t; t_0, \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \int_{t_{00}}^{t} Y(\xi; t)\big[f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\delta\tau_i\big]\, d\xi + o(t; \varepsilon\delta\mu), \tag{3.70}$$

$$R_2(t; t_0, \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu). \tag{3.71}$$

Finally, we note that for $t \in [\widehat{t}, t_{10} + \delta_2]$,

$$\varepsilon \int_{t_0}^{t} Y(\xi; t)\delta f[\xi]\, d\xi = \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + o(t; \varepsilon\delta\mu). \tag{3.72}$$

Taking into account (3.68)–(3.72), from (3.67) we obtain (3.5), where $\delta x(t; \varepsilon\delta\mu)$ has the form (3.10).

4 Optimal control problems and necessary conditions of optimality

4.1 Preliminaries and necessary criticality condition

In this subsection by E_z we will denote a vector space. The k-dimensional vector space E_z^k will be identified with the space $\mathbb{R}^k$. By the module of an element $z \in E_z^k$ we will mean the Euclidean module

$$|z|^2 = z^\top z = \sum_{i=1}^{k} (z^i)^2.$$

In what follows, finite-dimensional vector spaces will be endowed with the Euclidean topology. Let $z_i \in E_z$, $i = \overline{1, k}$. The set

$$L = \Big\{ z = \sum_{i=1}^{k} \lambda_i z_i : \ \ \lambda_i \in \mathbb{R}, \ \ i = \overline{1, k} \Big\}$$

is called the finite-dimensional linear manifold generated by the points z_i, $i = \overline{1, k}$. If $z_0 \in L$, then we say that the manifold L passes through the point z_0 and it will be denoted by L_{z_0}. In what follows, we will write the manifold L_{z_0} in the equivalent form

$$L_{z_0} = \Big\{ z = z_0 + \sum_{i=1}^{k} \lambda_i z_i : \ \ \lambda_i \in \mathbb{R}, \ \ i = \overline{1, k} \Big\}. \tag{4.1}$$

For each fixed $\alpha > 0$, the set

$$\Big\{ \sum_{i=1}^{k} \lambda_i z_i : \ \ |\lambda_i| \le \alpha, \ \ i = \overline{1, k} \Big\} \tag{4.2}$$

is a convex bounded neighborhood of zero in the space $L_{z_0} - z_0$.

Definition 4.1. We say that points $z_i \in E_z$, $i = \overline{0, k}$, are in a general position if the vectors $z_i - z_0$, $i = \overline{1, k}$, are linearly independent.

From this definition it follows that for any z_i, $i = \overline{1, k}$, the system of vectors $z_0 - z_i, \dots, z_{i-1} - z_i, z_{i+1} - z_i, \dots, z_k - z_i$ is linearly independent, as well.

Definition 4.2. Let the points z_i, $i = \overline{0, k}$, be in a general position. The convex hull of points z_i, $i = \overline{0, k}$, i.e., $\operatorname{co}(\{z_0, \dots, z_k\})$, is called a k-dimensional simplex.

Clearly, a k-dimensional simplex is a convex compact set in the linear finite dimensional manifold generated by the points z_i, $i=\overline{0,k}$. It is easy to note that

$$\begin{aligned}\operatorname{co}(\{z_0,\dots,z_k\}) &= z_0+\operatorname{co}(\{0,z_1-z_0,\dots,z_k-z_0\})\\ &= z_0+\Big\{\sum_{i=1}^{k}\lambda_i(z_i-z_0):\ \ \lambda_i\geq 0,\ \ \sum_{i=1}^{k}\lambda_i\leq 1\Big\}.\end{aligned}\tag{4.3}$$

From the relations (4.2), (4.3) and Definition 4.2 follow Lemmas 4.1 and 4.2.

Lemma 4.1. *The simplex* $\operatorname{co}(\{z_0,\dots,z_k\})$ *has a nonempty interior.*

Lemma 4.2. *Each point z of a simplex* $\operatorname{co}(\{z_0,\dots,z_k\})$ *can be uniquely represented in the form*

$$z=\sum_{i=0}^{k}\lambda_i z_i,\ \ \text{where}\ \ \lambda_i\geq 0,\ \ i=\overline{0,k},\ \ \text{and}\ \ \sum_{i=0}^{k}\lambda_i=1.$$

Lemma 4.3. *Let $M\subset E_z^k$ and, moreover, let $0\in\operatorname{int}M$. Then in E_z^k there exists a k-dimensional simplex which is contained in M and contains $0\in E_z^k$ as an interior point.*

Proof. Let $\operatorname{co}(\{z_0,\dots,z_k\})\subset E_z^k$ be a k-dimensional simplex. By Lemma 4.1, there exists $\widehat{z}\in\operatorname{int}\operatorname{co}(\{z_0,\dots,z_k\})$ such that the k-dimensional simplex

$$-\widehat{z}+\operatorname{co}(\{z_0,\dots,z_k\})=\operatorname{co}(\{z_0-\widehat{z},\dots,z_k-\widehat{z}\})$$

contains $0\in E_z^k$ as an interior point. By assumption, there exists a convex neighborhood

$$V=\big\{z\in E_z^k:\ |z|<\varepsilon_0\big\},\ \ \varepsilon_0>0,$$

of zero contained in M.

Let $\varepsilon>0$ be a number such that $\varepsilon(z_i-\widehat{z})\in V$, $i=\overline{0,k}$. Hence the k-dimensional simplex $\varepsilon\operatorname{co}(\{z_0-\widehat{z},\dots,z_k-\widehat{z}\})$ is contained in M and contains $0\in E_z^k$ as an interior point. □

Lemma 4.4. *Let a linear mapping*

$$g:E_z\to E_g^k \tag{4.4}$$

and a k-dimensional simplex $\operatorname{co}(\{g_0,\dots,g_k\})\subset E_g^k$ be given. Let z_i, $i=\overline{0,k}$, be certain inverse images of the points g_i, $i=\overline{0,k}$, under the mapping (4.4)*, respectively. Then $\operatorname{co}(\{z_0,\dots,z_k\})\subset E_z$ is a k-dimensional simplex, and the restriction of the mapping*

$$g:\operatorname{co}(\{z_0,\dots,z_k\})\longrightarrow\operatorname{co}(\{g_0,\dots,g_k\}) \tag{4.5}$$

is a homeomorphism.

Proof. Let there exist numbers λ_i, $i=\overline{1,k}$, such that

$$\sum_{i=1}^{k}\lambda_i(z_i-z_0)=0,\quad \sum_{i=1}^{k}|\lambda_i|\neq 0.$$

Obviously,

$$g\Big(\sum_{i=1}^{k}\lambda_i(z_i-z_0)\Big)=\sum_{i=1}^{k}\lambda_i(g_i-g_0)=0;$$

in turn, this contradicts the linear independence of the elements g_i-g_0, $i=\overline{1,k}$.

Therefore, $\operatorname{co}(\{z_0,\dots,z_k\})$ is a k-dimensional simplex. By Lemma 4.2, the mapping (4.5) is a homeomorphism. □

Let the Hausdorff vector topology be given in E_z, which transforms E_z into a topological vector space.

Lemma 4.5. *Let $W \subset E_z$ and let a mapping*

$$P : W \to E_p^k, \tag{4.6}$$

continuous in the topology induced from E_z, be given. Further, let $K \subset W$ be a compact set. Then for any $\varepsilon > 0$, there exists a neighborhood $V_\varepsilon \subset E_z$ of zero such that

$$|P(z') - P(z'')| \le \varepsilon \ \ \forall (z', z'') \in K \times W, \ \ z' - z'' \in V_\varepsilon.$$

Proof. For each point $z' \in K$, there exists a convex neighborhood $V(z') \subset E_z$ of zero such that

$$|P(z') - P(z)| \le \frac{\varepsilon}{3} \ \ \forall z \in (z' + V(z')) \cap W.$$

The system of sets $\{z' + V(z') : \ z' \in K\}$ composes an open covering of the compact set K. Hence there exists a finite subcovering $\{z_i' + V(z_i') : \ i = \overline{1,m}\}$ of the set K.

Clearly, for $z \in (z_i' + V(z_i')) \cap W$,

$$|P(z_i') - P(z)| \le \frac{\varepsilon}{3}\,. \tag{4.7}$$

By the continuity of the mapping (4.6), for $2\varepsilon/3$, there exist convex neighborhoods $V_i \supset V(z_i')$, $i = \overline{1,m}$, of zero such that

$$|P(z_i') - P(z)| \le 2\,\frac{\varepsilon}{3} \ \ \forall z \in (z_i' + V_i) \cap W. \tag{4.8}$$

Obviously, the sets

$$\widehat{V}_i = V_i - V(z_i') = V_i + (-1)V(z_i'), \ \ V_\varepsilon = \bigcap_{i=1}^{m} \widehat{V}_i$$

are the neighborhoods of zero in E_z, and for an arbitrary point $z \in (z_i' + V(z_i') + \widehat{V}_i) \cap W$, the inequality (4.8) holds.

Let $(z', z'') \in K \times W$, $z' - z'' \in V_\varepsilon$ and the point z' belong to some of the sets $z_k' + V(z_k')$, $1 \le k \le m$. Further,

$$z'' - z_k' = z'' - z' + z' - z_k' \in V_\varepsilon + V(z_k') \subset \widehat{V}_k + V(z_k') = V_k.$$

Taking into account the inequalities (4.7) and (4.8), we have

$$|P(z') - P(z'')| \le |P(z') - P(z_k')| + |P(z_k') - P(z'')| \le \frac{\varepsilon}{3} + \frac{2}{3}\,\varepsilon = \varepsilon.$$ □

Definition 4.3. The set Ψ of a subsets from E_z is called a filter if it satisfies the following conditions:

(a) if $A \in \Psi$ and $B \in \Psi$, then $A \cap B \in \Psi$;

(b) if $A \in \Psi$ and $B \supset A$, then $B \in \Psi$;

(c) $\varnothing \notin \Psi$.

The set of all neighborhoods of a fixed point of the space E_z serves as an example of a filter.

Definition 4.4. A set $\widehat{\Re}$ of a subset of E_z is called a basis of a filter if it has the following properties:

(a) for any $A \in \widehat{\Re}$ and $B \in \widehat{\Re}$, there exists $C \in \widehat{\Re}$ such that $C \subset A \cap B$;

(b) $\varnothing \notin \Re$.

The set Ψ of all subsets each of which contains a certain set from $\widehat{\Re}$ is the filter generated by the basis $\widehat{\Re}$.

Theorem 4.1 (Carathéodory)**.** *Let $M \subset E_z^k$. Then any point $z \in \operatorname{co}(M)$ can be represented in the form*

$$z = \sum_{i=0}^{k} \lambda_i z_i,$$

where $z_i \in M$, $\lambda_i \geq 0$, $i = \overline{0,k}$, and $\sum\limits_{i=0}^{k} \lambda_i = 1$.

Theorem 4.2 (Brouwer)**.** *Let $\operatorname{co}(\{z_0,\dots,z_k\}) \subset E_z$ be a k-dimensional simplex. Then each continuous mapping*

$$g : \operatorname{co}(\{z_0,\dots,z_k\}) \longrightarrow \operatorname{co}(\{z_0,\dots,z_k\})$$

has a fixed point, i.e., there exists a point $z \in \operatorname{co}(\{z_0,\dots,z_k\})$ such that $g(z) = z$.

Theorem 4.3. *Let $M \subset E_z^k$ be a convex set and $0 \in \partial M$. Then there exists a nonzero k-dimensional vector $\pi = (\pi_1,\dots,\pi_k)$ such that*

$$\pi z = \sum_{i=1}^{k} \pi_i z^i \leq 0 \ \ \forall\, z \in M.$$

Let $E_z = E_x^k \times E_\varsigma$ be a vector space of points $z = (x,\varsigma)$. Assume that $D \subset E_z$ is a certain set and a mapping

$$P : D \to E_p^m \tag{4.9}$$

is given. Let Ψ be an arbitrary filter in E_z.

Definition 4.5. We say that the mapping (4.9) is defined on the filter Ψ if there exists an element $W \in \Psi$ such that $W \subset D$.

Definition 4.6. Let the mapping (4.9) be defined on the filter Ψ. The mapping (4.9) is said to be critical on the filter Ψ if for any point z_0 belonging to all elements of the filter Ψ, there exists an element $W \subset \Psi$ such that $W \subset D$ and $P(z_0) \in \partial P(W)$.

Definition 4.7. We say that the mapping (4.9) defined on the filter Ψ is continuous on Ψ if there exists an element $W \in \Psi$ such that $W \subset D$ and the restriction

$$P : W \to E_p^m$$

of the mapping (4.9) is continuous in the topology induced from E_z.

Let $X \subset E_z^k$ be a locally convex topological subspace, i.e., for an arbitrary neighborhood $V_x \subset X$ of a point $x \in X$, there exists a convex neighborhood $\widehat{V}_x \subset X$ contained in V_x. The following lemma is easily proved.

Lemma 4.6. *Let $\widehat{x} \in X$ be a fixed point. Further, let $V_0 \subset X - \widehat{x}$ be a convex bounded neighborhood of zero, and let $V_1 \subset X - \widehat{x}$ be a certain neighborhood of zero. Then there exists a number $\varepsilon_0 > 0$ such that*

$$\varepsilon V_0 \subset V_1 \ \ \forall\, \varepsilon \in (0,\varepsilon_0).$$

Definition 4.8. A set $D \subset X \times E_\varsigma$ is said to be finitely locally convex if for an arbitrary point $z = (x,\varsigma) \in D$ and for arbitrary manifold $L_\varsigma \subset E_\varsigma$, there exist convex neighborhoods $V_x \subset X$ and $V_\varsigma \subset E_\varsigma$ of the points x and ς, respectively, such that

$$V_x \times V_\varsigma \subset D.$$

Lemma 4.6 and Definition 4.8 directly imply the following lemma.

Lemma 4.7. *Let D be a finitely locally convex set, and let $z_0 = (x_0,\varsigma_0) \in D$. Further, let $V_0 \subset X - x_0$ and $V \subset L_{\varsigma_0} - \varsigma_0$ be bounded convex neighborhoods of zero (see (4.2)). Then there exists a number $\varepsilon_0 > 0$ such that*

$$z_0 + \varepsilon\delta z \in D \ \ \forall\, (\varepsilon,\delta z) \in (0,\varepsilon_0) \times V_0 \times V, \ \ \delta z = (\delta x, \delta\varsigma). \tag{4.10}$$

Definition 4.9. We say that the mapping (4.9) has a differential at a point $z_0 = (x_0, \varsigma_0) \in D$ if there exists a linear mapping

$$dP_{z_0} : E_{\delta z} = E_z - z_0 \to E^m_{\delta p} \tag{4.11}$$

such that for any manifold

$$L_{\varsigma_0} = \Big\{ \varsigma_0 + \sum_{i=1}^{k} \lambda_i \delta\varsigma_i : \ \ \lambda_i \in \mathbb{R}, \ \ i = \overline{1,k} \Big\} \subset E_\varsigma$$

(see (4.1)) the representation

$$P(z_0 + \varepsilon\delta z) - P(z_0) = \varepsilon dP_{z_0}(\delta z) + o(\varepsilon\delta z) \ \ \forall\, (\varepsilon, \delta z) \in (0, \varepsilon_0) \times V_0 \times V$$

holds, where $V_0 \subset X - x_0$ and $V \subset L_{\varsigma_0} - \varsigma_0$ are bounded neighborhoods of zero; $\varepsilon_0 > 0$ is the number for which (4.10) holds;

$$\lim_{\varepsilon\to 0} \frac{o(\varepsilon\delta z)}{\varepsilon} = 0 \ \text{ uniformly in } \ \delta z \in V_0 \times V.$$

The mapping (4.11) is called the differential of the mapping (4.9) at the point z_0.

Definition 4.10 (Gamkrelidze)**.** A filter Ψ in E_z is said to be quasiconvex if for any element $W \in \Psi$ and any natural number k, there exists an element $W_1 = W_1(W, k) \in \Psi$ such that for arbitrary points $z_i \in W_1$, $i = \overline{0,k}$, and an arbitrary neighborhood of zero $V \subset E_z$, there exists a continuous mapping

$$\phi : \text{co}(\{z_0, \dots, z_k\}) \longrightarrow W. \tag{4.12}$$

satisfying the condition

$$(z - \phi(z)) \in V \ \ \forall\, z \in \text{co}(\{z_0, \dots, z_k\}).$$

Obviously, in Definition 4.10, we can assume that $W_1 \subset W$, since any element $W_2 \subset W \cap W_1$ of the filter has the indicated property of the element W_1. Therefore, in what follows, we will assume that $W_1 \subset W$.

Definition 4.11. A filter Ψ is said to be convex if there exists a basis of the filter consisting of convex sets.

Lemma 4.8. *Every convex filter Ψ in E_z is quasiconvex.*

Proof. For any element $W \in \Psi$, there exists a convex element $W_2 \subset W$, which can be taken as W_1; as the mapping (4.12), it is necessary to take the identity mapping. □

Let E_z be a topological vector space, $X \subset E^k_x$ be a locally convex topological space, and $D \subset X \times E_\varsigma$ be a finitely locally convex set.

Let a mapping

$$P : D \to E^m_p \tag{4.13}$$

be given, and let Ψ be a filter in E_z.

By co$[\Psi]$ we denote the convex filter whose elements are the sets co(W), where W is an arbitrary element of the filter Ψ.

Theorem formulated below is an analogue of R. V. Gamkrelidze and G. L. Kharatishvili's Theorem on the necessary criticality condition to mappings defined on a finitely locally convex set. The proof of the following theorem is performed according to the scheme presented in [7–9] with only nonessential changes.

Theorem 4.4. *Let the mapping* (4.13) *be continuous on* co$[\Psi]$ *and critical on* Ψ*. Further, let the filter* Ψ *be quasiconvex. Then for any point* $z_0 = (x_0, \varsigma_0)$ *belonging to all sets of the filter* Ψ *at which the mapping* (4.13) *has the differential* (4.11)*, there exists an element* $\widehat{W} \in \Psi$ *such that zero of the space* E^m_{dp} *is a boundary point of the set*

$$dP_{z_0}(\text{co}(\widehat{W}) - z_0) \subset E^m_{dp}. \tag{4.14}$$

Proof. By the assumption, there exist elements $W_i \in \Psi$, $i = 1, 2$, such that $\operatorname{co}(W_1) \subset D$, $W_2 \subset D$, and, moreover, the mapping
$$P : \operatorname{co}(W_1) \to E_p^m \tag{4.15}$$
is continuous and $P(z_0) \in \partial P(W_2)$. Clearly, $W_3 = W_1 \cap W_2 \in \Psi$ and $P(z_0) \in \partial P(W_3)$.

Let the conditions of the theorem hold, but for any $W \in \Psi$ lying in D, the point $0 \in E_{dp}^m$ is an interior point of the set
$$dP_{z_0}(\operatorname{co}(W) - z_0) \subset E_{dp}^m.$$

Let us show that this contradicts the choice of the element W_3. Precisely, we prove the solvability of the following equation
$$P(z) = P(z_0) + p, \quad z \in W_3, \tag{4.16}$$
with respect to z and for any vector $p \in E_p^m$ whose module is sufficiently small, and, therefore, we prove that $P(z_0)$ is an interior point of the set $P(W_3) \subset E_p^m$, which contradicts the choice of W_3. By $W_4 = W_4(W_3; (m+1)^2) \subset W_3$ we denote the element of the quasiconvex filter Ψ (see Definition 4.10) satisfying the following condition: for any neighborhood of zero $V \subset E_z$ and any $1 + (m+1)^2$ points $z_0, \dots, z_{(m+1)^2}$ from W_4, there exists a continuous mapping
$$\phi : \operatorname{co}(\{z_0, \dots, z_{(m+1)^2}\}) \longrightarrow W_3 \tag{4.17}$$
satisfying the condition
$$(z - \phi(z)) \in V \quad \forall\, z \in \operatorname{co}(\{z_0, \dots, z_{(m+1)^2}\}). \tag{4.18}$$
According to the assumption made, $0 \in E_{dp}^m$ is an interior point of the convex set
$$dP_{z_0}(\operatorname{co}(W_4) - z_0) \subset E_{dp}^m. \tag{4.19}$$
Hence there exist $m+1$ points $dp_i \in dP_{z_0}(\operatorname{co}(W_4) - z_0)$ that are in general position, and, moreover, the m-dimensional simplex $\operatorname{co}(\{dp_0, \dots, dp_m\})$ containing $0 \in E_{dp}^m$ as an interior point (see Lemma 4.3).

By the linearity of the mapping (4.11),
$$dP_{z_0}(\operatorname{co}(W_4) - z_0) = \operatorname{co}(dP_{z_0}(W_4 - z_0)).$$
Each of the points
$$dp_i \in \operatorname{co}(dP_{z_0}(W_4 - z_0)), \quad i = \overline{0, m},$$
is represented in the form
$$dp_i = \sum_{j=0}^m \mu_{ij} dp_{ij}, \quad dp_{ij} \in dP_{z_0}(W_4 - z_0), \quad \mu_{ij} \geq 0, \quad \sum_{j=0}^m \mu_{ij} = 1$$
(see Theorem 4.1). Let $\delta z_{ij} \in W_4 - z_0$ be some inverse images of the points dp_{ij} under the mapping
$$dP_{z_0} : W_4 - z_0 \longrightarrow E_{dp}^m,$$
and let
$$\delta z_i = \sum_{j=0}^m \mu_{ij} \delta z_{ij}, \quad i = \overline{0, m}. \tag{4.20}$$
Obviously,
$$dP_{z_0}(\delta z_i) = dp_i, \quad i = \overline{0, m}.$$
By Lemma 4.4, the points $\delta z_i = (\delta x_i, \delta \varsigma_i)$, $i = \overline{0, m}$, are in general position and the mapping
$$dP_{z_0} : \operatorname{co}(\{\delta z_0, \dots, \delta z_m\}) \longrightarrow \operatorname{co}(\{dp_0, \dots, dp_m\}) \tag{4.21}$$
is a homeomorphism.

Let $z \in \operatorname{co}(\{z_0, z_0+\delta z_0, \dots, z_0+\delta z_m\})$. Then

$$z = z_0 + \sum_{i=0}^{m} \lambda_i \delta z_i = z_0 + \sum_{i=0}^{m}\sum_{j=0}^{k} \lambda_i \mu_{ij} \delta z_{ij} = \Big(1 - \sum_{i=0}^{m}\sum_{j=0}^{m} \lambda_i \mu_{ij}\Big) z_0 + \sum_{i=0}^{m}\sum_{j=0}^{m} \lambda_i \mu_{ij}(z_0 + \delta z_{ij}),$$

$$\lambda_i \geq 0, \ \ \sum_{i=0}^{m} \lambda_i \leq 1$$

(see (4.3) and (4.20)). Hence

$$\operatorname{co}\big(\{z_0, z_0+\delta z_0, \dots, z_0+\delta z_m\}\big) \subset \operatorname{co}\big(\{z_0, z_0+\delta z_{00}, \dots, z_0+\delta z_{ij}, \dots, z_0+\delta z_{mm}\}\big). \tag{4.22}$$

Further, let us show that for $\varepsilon \in [0,1]$, the inclusion

$$z_0 + \varepsilon \operatorname{co}(\{\delta z_0, \dots, \delta z_m\}) \subset \operatorname{co}\big(\{z_0, z_0+\delta z_0, \dots, z_0+\delta z_m\}\big) \tag{4.23}$$

holds. Indeed, it is clear that every point $z_0 + \varepsilon \operatorname{co}(\{\delta z_0, \dots, \delta z_m\})$ is represented in the form

$$z = z_0 + \varepsilon \sum_{i=0}^{m} \lambda_i \delta z_i = (1-\varepsilon) z_0 + \varepsilon \sum_{i=0}^{m} \lambda_i (z_0 + \delta z_i) \in \operatorname{co}\big(\{z_0, z_0+\delta z_0, \dots, z_0+\delta z_m\}\big).$$

The inclusions (4.22) and (4.23) imply

$$\begin{aligned} z_0 + \varepsilon \operatorname{co}(\{\delta z_0, \dots, \delta z_m\}) &\subset \operatorname{co}\big(\{z_0, z_0+\delta z_0, \dots, z_0+\delta z_m\}\big) \\ &\subset \operatorname{co}(W_4) \subset \operatorname{co}(W_3) \subset D \ \forall \varepsilon \in [0,1]. \end{aligned} \tag{4.24}$$

Taking into account

$$D - z_0 \subset (X - x_0) \times (E_\varsigma - \varsigma_0),$$

we see that the latter relation directly implies the inclusion

$$\varepsilon \operatorname{co}(\{\delta x_0, \dots, \delta x_m\}) \subset (X - x_0), \ \ \varepsilon \in [0,1]. \tag{4.25}$$

Let $L_{\varsigma_0} \subset E_\varsigma$ be the manifold generated by the points $\varsigma_0, \delta\varsigma_0, \dots, \delta\varsigma_m$:

$$L_{\varsigma_0} = \Big\{\varsigma_0 + \sum_{i=0}^{m+1} \lambda_i \delta\varsigma_i : \ \ \lambda_i \in \mathbb{R}, \ \ i = \overline{0, m+1}\Big\}, \ \ \delta\varsigma_{m+1} = \varsigma_0.$$

Obviously,

$$\varepsilon \operatorname{co}(\{\delta\varsigma_0, \dots, \delta\varsigma_m\}) \subset L_{\varsigma_0} - \varsigma_0. \tag{4.26}$$

Let $V_0 \subset X - x_0$ and $V \subset L_{\varsigma_0} - \varsigma_0$ be convex bounded neighborhoods of zero. There exists a number $\varepsilon_1 \in (0,1)$ such that

$$\varepsilon_1 \operatorname{co}(\{\delta x_0, \dots, \delta x_m\}) \subset V_0, \ \ \varepsilon_1 \operatorname{co}(\{\delta\varsigma_0, \dots, \delta\varsigma_m\}) \subset V$$

(see (4.25) and (4.26)). Hence

$$\varepsilon_1 \operatorname{co}(\{\delta z_0, \dots, \delta z_m\}) \subset V_0 \times V. \tag{4.27}$$

Let $w(\varepsilon) = \varepsilon\varepsilon_1$, $\varepsilon \in (0,1)$. Lemma 4.7 implies the existence of a number $\varepsilon_2 \in (0,1)$ such that

$$z_0 + w(\varepsilon)\delta z \in D \ \ \forall (\varepsilon, \delta z) \in (0, \varepsilon_2) \times V_0 \times V.$$

Denote by $d > 0$ the distance from the point $0 \in E^m_{dp}$ to the boundary of the simplex $\operatorname{co}(\{dp_0, \dots, dp_m\})$.

The differentiability of the mapping (4.13) at the point z_0 implies the existence of a number $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$P(z_0 + w(\varepsilon)\delta z) = P(z_0) + w(\varepsilon) dP_{z_0}(\delta z) + o(w(\varepsilon)\delta z) \ \ \forall (\varepsilon, \delta z) \in (0, \varepsilon_3) \times V_0 \times V; \tag{4.28}$$

moreover,

$$\frac{|o(w(\varepsilon)\delta z)|}{w(\varepsilon)} \le \frac{d}{3} \ \ \forall\,(\varepsilon,\delta z)\in(0,\varepsilon_3)\times V_0\times V. \tag{4.29}$$

Obviously, on account of (4.27), the relations (4.28) and (4.29) hold for $(\varepsilon,\delta z)\in(0,\varepsilon_3)\times \operatorname{co}(\{\delta z_0,\dots,\delta z_m\})$.

The mapping (4.15) is continuous on $\operatorname{co}(W_3)$ in the topology of the space $X\times L_{\varsigma_0}$. Therefore, $P(z_0+w(\varepsilon)\delta z)$ is continuous in $\delta z\in \operatorname{co}(\{\delta z_0,\dots,\delta z_m\})$ (see (4.24)). Using (4.28), we conclude from the above-said that for each $\varepsilon\in(0,\varepsilon_3)$, the function $o(w(\varepsilon)\delta z)$ is continuous on $\operatorname{co}(\{\delta x_0,\dots,\delta x_m\})$.

Further, the continuity of the mapping P on $\operatorname{co}(W_3)$ and the compactness of the set $z_0+w(\varepsilon)\operatorname{co}(\{\delta z_0,\dots,\delta z_k\})\subset \operatorname{co}(W_3)$ imply that for each $\varepsilon\in(0,\varepsilon_3)$, there exists a neighborhood of zero $V_\varepsilon\subset E_z$ such that for

$$z'\in z_0+w(\varepsilon)\operatorname{co}(\{\delta x_0,\dots,\delta x_m\}),\ \ z''\in \operatorname{co}(W_3),\ \ z'-z''\in V_\varepsilon, \tag{4.30}$$

we have

$$|P(z')-P(z'')|\le w(\varepsilon)\,\frac{d}{3} \tag{4.31}$$

(see Lemma 4.5).

The conditions (4.17), (4.18) and the relation (4.22) directly imply the existence of a family of continuous mappings

$$\phi_\varepsilon:\operatorname{co}\big(\{z_0,z_0+\delta z_0,\dots,z_0+\delta z_m\}\big)\longrightarrow W_4,$$

depending on $\varepsilon\in(0,\varepsilon_3)$ and satisfying the condition

$$z-\phi_\varepsilon(z)\in V_\varepsilon\ \ \forall\,z\in \operatorname{co}\big(\{z_0,z_0+\delta z_0,\dots,z_0+\delta z_m\}\big).$$

For $\varepsilon\in(0,\varepsilon_3)$, the simplex $z_0+w(\varepsilon)\operatorname{co}(\{\delta z_0,\dots,\delta z_m\})$ is contained in $\operatorname{co}(\{z_0,z_0+\delta z_0,\dots,z_0+\delta z_m\})$ (see (4.23), and, therefore,

$$z-\phi_\varepsilon(z)\in V_\varepsilon\ \ \forall\,z\in z_0+w(\varepsilon)\operatorname{co}(\{\delta z_0,\dots,\delta z_m\}). \tag{4.32}$$

Let us now show that the equation

$$P(\phi_\varepsilon(z))=P(z_0)+w(\varepsilon)p,\ \ z\in z_0+w(\varepsilon)\operatorname{co}(\{\delta z_0,\dots,\delta z_m\}) \tag{4.33}$$

is solvable in z for a sufficiently small ε and an arbitrary $p\in E_p^m$ satisfying the condition

$$|p|\le\frac{d}{3}\,. \tag{4.34}$$

Indeed, we rewrite this equation in the form

$$P(z)=P(z_0)+w(\varepsilon)p+P(z)-P(\phi_\varepsilon(z)),\ \ z\in z_0+w(\varepsilon)\operatorname{co}(\{\delta z_0,\dots,\delta z_m\}),$$

or, using (4.28), in the form of the following equation in δz:

$$dP_{z_0}(\delta z)=p-\frac{o(w(\varepsilon)\delta z)}{w(\varepsilon)}+\frac{P(z_0+w(\varepsilon)\delta z)-P(\phi_\varepsilon(z_0+w(\varepsilon)\delta z)}{w(\varepsilon))}\,,\ \ \delta z\in \operatorname{co}(\{\delta z_0,\dots,\delta z_m\}). \tag{4.35}$$

The relations (4.29)–(4.32) and (4.34) imply

$$\Big(p-\frac{o(w(\varepsilon)\delta z)}{w(\varepsilon)}+\frac{P(z_0+w(\varepsilon)\delta z)-P(\phi_\varepsilon(z_0+w(\varepsilon)\delta z)}{w(\varepsilon))}\Big)\in \operatorname{co}(\{\delta p_0,\dots,\delta p_m\})$$

and hence the equation (4.35) is equivalent to the equation

$$\delta z=dP_{z_0}^{-1}\Big(p-\frac{o(w(\varepsilon)\delta z)}{w(\varepsilon)}+\frac{P(z_0+w(\varepsilon)\delta z)-P(\phi_\varepsilon(z_0+w(\varepsilon)\delta z)}{w(\varepsilon)}\Big), \tag{4.36}$$

where

$$dP_{z_0}^{-1} : \mathrm{co}(\{dp_0, \dots, dp_m\}) \longrightarrow \mathrm{co}(\{\delta z_0, \dots, \delta z_m\})$$

is a continuous mapping, inverse to the mapping (4.21).

We can consider the right-hand side of the equation (4.36) as a continuous self-mapping of the simplex $\mathrm{co}(\{\delta z_0, \dots, \delta z_m\})$, and hence each fixed point of this mapping is a solution of the equation (4.36) (see Theorem 4.2). Thus, we have proved the solvability of the equation (4.33) for an arbitrary p satisfying (4.34) and, therefore, the solvability of the equation (4.16) for p whose modules are sufficiently small. □

Theorem 4.5. *Let the conditions of Theorem 4.4 hold. Then for any point z_0 belonging to all sets Ψ at which the differential (4.11) exists, there exist an element $\widehat{W} \in \Psi$ and a vector $\pi = (\pi_1, \dots, \pi_m) \neq 0$ such that*

$$\pi dP_{z_0}(\delta z) = \sum_{i=1}^{m} \pi_i dP_{z_0}^i(\delta z) \leq 0 \ \ \forall \in \mathrm{cone}(\mathrm{co}(\widehat{W}) - z_0), \tag{4.37}$$

where $\mathrm{cone}(\widehat{W})$ is the cone generated by the set $\widehat{W}$.

Proof. Set (4.14), being the image of a convex set under a linear mapping, is also convex. Since $0 \in E_{dp}^m$ is a boundary point of the convex set (4.14), by Theorem 4.3, there exists a nonzero m-dimensional vector for which

$$\pi dP_{z_0}(\delta z) \leq 0 \ \ \forall \, \delta z \in \mathrm{co}(\widehat{W} - z_0).$$

This implies (4.37). □

4.2 Gamkrelidze's approximation lemma

Let $U_0 \subset \mathbb{R}^r$ be an open set. Now let us consider the function $f(t, x, x_1, \dots, x_s, u)$, $(t, x, x_1, \dots, x_s, u) \in I \times O^{s+1} \times U_0$, satisfying the following conditions: for almost all $t \in I$, the function $f : I \times O^{s+1} \times U_0 \to \mathbb{R}^n$ is continuous and continuously differentiable in $(x, x_1, \dots, x_s) \in O^{s+1}$; for each $(x, x_1, \dots, x_s, u) \in O^{s+1} \times U_0$, the function $f(t, x, x_1, \dots, x_s, u)$ and the matrices $f_x(t, x, \cdot), f_{x_i}(t, x, \cdot)$, $i = \overline{1, s}$, are measurable on I; for any compact sets $K \subset O$ and $M \subset U_0$, there exists a function $m_{K,M}(t) \in L_1(I, \mathbb{R}_+)$ such that for any $(x, x_1, \dots, x_s, u) \in K^{s+1} \times M$ and almost all $t \in I$,

$$|f(t, x, x_1, \dots, x_s, u)| + |f_x(t, x, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, x, \cdot)| \leq m_{K,M}(t).$$

Introduce the set

$$F = \big\{ f(t, x, x_1, \dots, x_s) = f(t, x, x_1, \dots, x_s, u(t)) : \ u \in \Omega(I, U) \big\},$$

where $U \subset U_0$ is a given set. The set F can be identified with a subset of the space $E_f^{(1)}$. A family of subintervals

$$\sigma = \big\{ I_\beta = [t_\beta, t_{\beta+1}] : \ \beta = \overline{1, m} \big\},$$

where $a = t_1 < t_2 < \dots < t_{m-1} < t_m = b$, is called a σ-partition of the interval I.

Let the points $f_i(t, x, x_1, \dots, x_s) = f(t, x, x_1, \dots, x_s, u_i(t)) \in F$, $i = \overline{1, k+1}$, and the σ-partition of the interval be given. Using these data, to each point λ of the k-dimensional simplex

$$\Sigma = \Big\{ \lambda = (\lambda_1, \dots, \lambda_{k+1}) : \ \lambda_i \geq 0, \ \sum_{i=1}^{k+1} \lambda_i = 1 \Big\}$$

we can uniquely put in correspondence the subdivision of each intervals I_β into $k+1$ subintervals $I_{\beta_i}(\lambda)$, $i = \overline{1, k+1}$, defined by the condition

$$\mathrm{mes}\, I_{\beta_i}(\lambda) = \lambda_i \,\mathrm{mes}\, I_\beta, \ \ i = \overline{1, k+1}, \tag{4.38}$$

if $\lambda_i = 0$, then the corresponding interval degenerates into a point. Define the mapping

$$\phi_\sigma : \Sigma \to F \tag{4.39}$$

by the formula

$$\phi_\sigma(\lambda) = f_\lambda(t, x, x_1, \dots, x_s) = f\big(t, x, x_1, \dots, x_s, u_\lambda(t)\big),$$

where

$$u_\lambda(t) = u_i(t), \ \ t \in I_{\beta_i}(\lambda), \ \ \beta = \overline{1,m}, \ \ i = \overline{1,k+1}.$$

It is clear that

$$f_\lambda(t, x, x_1, \dots, x_s) = f_i(t, x, x_1, \dots, x_s), \ \ t \in I_{\beta_i}(\lambda), \ \ (x, x_1, \dots, x_s) \in O^{s+1}, \quad \beta = \overline{1,m}, \ \ i = \overline{1,k+1}. \tag{4.40}$$

The relations (4.38) and (4.40) play principal role in proving the following

Lemma 4.9 (Gamkrelidze's approximation lemma [6,7,10])**.** *For an arbitrary σ-partition, the mapping (4.39) is continuous, i.e., for an arbitrary point $\widehat{\lambda} \in \Sigma$ and an arbitrary neighborhood $V_{K,\varepsilon} \in \Re$, there exists a number $\delta > 0$ such that*

$$(f_\lambda - f_{\widehat{\lambda}}) \in V_{K,\varepsilon} \ \ \forall \lambda \in \big\{\lambda \in \Sigma : \ |\lambda - \widehat{\lambda}| < \delta\big\}.$$

Moreover, for an arbitrary neighborhood $V_{K,\varepsilon} \in \Re$, there exists a σ-partition such that for $\forall\,\lambda \in \Sigma$, we have

$$\Big(\sum_{i=1}^{k+1} \lambda_i f_i - f_\lambda\Big) \in V_{K,\varepsilon},$$

i.e.,

$$\bigg|\int_{t'}^{t''} \Big[\sum_{i=0}^{s} \lambda_i f_i(t, x, x_1, \dots, x_s) - f_\lambda(t, x, x_1, \dots, x_s)\Big]\,dt\bigg| \le \varepsilon \quad \forall\,(t', t'', x, x_1, \dots, x_s, \lambda) \in I^2 \times K^{s+1} \times \Sigma.$$

Let $\theta_\nu > \cdots > \theta_1 > 0$ be the given numbers with $\theta_i = m_i h$, where m_i, $i = \overline{1,\nu}$, are natural numbers and $h > 0$ is a real number. Let the function $f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_\nu)$, $(t,x,x_1,\dots,x_s,u_1,\dots,u_\nu) \in I \times O^{s+1} \times U_0^{\nu+1}$, satisfy the following conditions: for almost all $t \in I$, the function $f : I \times O^{s+1} \times U_0^{\nu+1} \to \mathbb{R}^n$ is continuous and continuously differentiable in $(x, x_1, \dots, x_s) \in O^{s+1}$; for each $(x, x_1, \dots, x_s, u, u_1, \dots, u_\nu) \in O^{1+s} \times U_0^{\nu+1}$, the function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_\nu)$ and the matrices $f_x(t, x, \cdot)$, $f_{x_i}(t, x, \cdot)$, $i = \overline{1,s}$, are measurable on I; for any compact sets $K \subset O$ and $M \subset U_0$, there exists a function $m_{K,M}(t) \in L_1(I, \mathbb{R}_+)$ such that for any $(x, x_1, \dots, x_s, u, u_1, \dots, u_\nu) \in K^{s+1} \times M^{\nu+1}$ and almost all $t \in I$,

$$\big|f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_\nu)\big| + |f_x(t, x, \,\cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, x, \,\cdot)| \le m_{K,M}(t).$$

Introduce the set

$$F_1 = \Big\{f(t, x, x_1, \dots, x_s) = f\big(t, x, x_1, \dots, x_s, u(t), u(t-\theta_1), \dots, u(t-\theta_\nu)\big) : \ u \in \Omega(I_2, U)\Big\},$$

where $I_2 = [a - \theta_\nu, b], \Omega(I_2, U) \subset E_u(I_2)$.

Consider the functions $f_i(t, x, x_1, \dots, x_s) = f(t, x, x_1, \dots, x_s, u_i(t), u_i(t-\theta_1), \dots, u_i(t-\theta_\nu)) \in F_1$, $i = \overline{0,s}$. In this case, we consider the $\widehat{\sigma}$-partition which means that we partition the interval $[a-\theta_\nu, b]$ in the following way. Let $\gamma > 0$ be the minimum number satisfying the condition $b + \gamma - a + \theta_{nu} = lh$, where l is a natural number and let $I^{(\alpha)}$, $\alpha = \overline{1,l}$, be a system of intervals of length h adjacent to each other such that the left endpoint of the interval $I^{(1)}$ coincides with the point $a - \theta_\nu$, the right endpoint of it coincides with the left endpoint of the subsequent interval $I^{(2)}$, etc., and the right endpoint of $I^{(l)}$ coincides with the endpoint $b+\gamma$. Next, we divide each of the intervals $I^{(\alpha)}$ by a partial interval $I_\beta^{(\alpha)}$, $\beta=\overline{1,m}$, in a unified way so that the right endpoint of one of the partial intervals $I_\beta^{(l)}$ coincides with the point b. To an arbitrary point $\lambda\in\Sigma$, we put in correspondence a subdivision into partial intervals $I_{\beta_i}^{(\alpha)}(\lambda)$ common for all $I_\beta^{(\alpha)}$ and defined by the condition (4.38).

Let the points $f_i(t,x,x_1,\dots,x_s)=f(t,x,x_1,\dots,x_s,u_i(t),u_i(t-\theta_1),\dots,u_i(t-\theta_\nu))\in F_1$, $i=\overline{0,s}$, and $\widehat{\sigma}$-partition of the interval be given.

Let us define the mapping

$$\phi_{\widehat{\sigma}}:\Sigma\to F_1 \tag{4.41}$$

by the formula

$$\phi_{\widehat{\sigma}}(\lambda)=f_\lambda(t,x,x_1,\dots,x_s)=f\big(t,x,x_1,\dots,x_s,u_\lambda(t),u_\lambda(t-\theta_1),\dots,u_\lambda(t-\theta_\nu)\big),$$

where

$$u_\lambda(t)=u_i(t),\ \ t\in I_{\beta_i}^{(\alpha)}(\lambda),\ \ \alpha=\overline{1,l},\ \ \beta=\overline{1,m},\ \ i=\overline{1,k+1}.$$

It is clear that

$$f_\lambda(t,x,x_1,\dots,x_s)=f_i(t,x,x_1,\dots,x_s),\ \ t\in I_{\beta_i}^{(\alpha)}(\lambda).$$

The latter relation allows one to prove generalization of Lemma 4.9.

Lemma 4.10. *For an arbitrary $\widehat{\sigma}$-partition, the mapping* (4.41) *is continuous. Moreover, for an arbitrary neighborhood $V_{K,\varepsilon}\in\Re$, there exists a $\widehat{\sigma}$-partition such that for $\forall\,\lambda\in\Sigma$, we have*

$$\Big(\sum_{i=1}^{k+1}\lambda_i f_i-f_\lambda\Big)\in V_{K,\varepsilon}. \tag{4.42}$$

Lemma 4.11 ([20, p. 66])**.** *Let $z_i\in E_z$, $i=\overline{1,k+1}$. There exist a subset $\Sigma_0\subset\Sigma$ and a function $\phi(z)$, $z\in\operatorname{co}(\{z_1,\dots,z_{k+1}\})$, such that the mapping*

$$\phi:\operatorname{co}(\{z_1,\dots,z_{k+1}\})\longrightarrow\Sigma_0\ \ (z\longmapsto\lambda\in\Sigma_0) \tag{4.43}$$

is a homeomorphism.

Lemma 4.12. *Let $f_i(t,x,x_1,\dots,x_s)\in F_1$, $i=\overline{1,k+1}$. Then for an arbitrary $V_{K,\varepsilon}\in\Re$, there exists a continuous mapping*

$$\phi_0:\operatorname{co}(\{f_1,\dots,f_{k+1}\})\longrightarrow F_1 \tag{4.44}$$

satisfying the condition

$$(z-\phi_0(z)\in V_{K,\varepsilon}\ \ \forall\,z\in\operatorname{co}(\{f_1,\dots,f_{k+1}\}). \tag{4.45}$$

Proof. On the set Σ_0 we define the mapping (4.41), i.e., $\phi_{\widehat{\sigma}}(\lambda)=f_\lambda\in F_1\ \forall\,\lambda\in\Sigma_0$. By Lemma 4.10, the mapping (4.41) is continuous and (4.42) is valid. Define now the continuous mapping (4.44) by the formula $\phi_0(z)=\phi_{\widehat{\sigma}}(\phi(z))$, $z\in\operatorname{co}(\{f_1,\dots,f_{k+1}\})$, where

$$z\longmapsto\phi(z)=\lambda\in\Sigma_0\ \text{ and }\ \phi_{\widehat{\sigma}}(\phi(z))=f_{\phi(z)}=f_\lambda\in F_1$$

(see (4.43)). The relation (4.42) implies (4.45). □

4.3 Example of a quasiconvex filter

Let $f_0(t,x,x_1,\dots,x_s)=f(t,x,x_1,\dots,x_s,u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_\nu))\in F_1$ be a fixed point. In F_1, let us define the filter Ψ using the basis

$$\Re_1=\big\{W_{K,\delta}:\ \ K\subset O\ \text{ is a compact set},\ \ \delta>0\ \text{ is an arbitrary number}\big\},$$

where

$$W_{K,\delta}=\Big\{f(t,x,x_1,\dots,x_s)=f\big(t,x,x_1,\dots,x_s,u(t),u(t-\theta_1),\dots,u(t-\theta_\nu)\big)\in F_1:H_1(f-f_0:K)\le\delta\Big\},$$

$$H_1(f:K)=\int\limits_I\Big[\sup_{(x,x_1,\dots,x_s)\in K^{s+1}}\Big(|f(t,x,x_1,\dots,x_s)+|f_x(t,x,\,\cdot\,)|+\sum_{i=1}^{s}|f_{x_i}(t,x,\,\cdot\,)|\Big)\Big]\,dt,\quad f\in E_f^{(1)},$$

(see Lemma 2.1).

Lemma 4.13. *The filter Ψ is quasiconvex.*

Proof. Let an arbitrary element $W\in\Psi$ and an arbitrary natural number k be given. There exists an element $W_{K,\delta}\in\Re_1$ such that $W_{K,\delta}\subset W$. Let us show that as W_1 in the definition of a quasiconvex filter, we can take $W_{K,\frac{\delta}{k+1}}$.

Assume that the points

$$f_i(t,x,x_1,\dots,x_s)=f\big(t,x,x_1,\dots,x_s,u_i(t),u_i(t-\theta_1),\dots,u_i(t-\theta_\nu)\big)\in W_{K,\frac{\delta}{k+1}},\quad i=\overline{1,k+1},$$

are such that

$$H_1(f_i-f_0);K)\leq\frac{\delta}{k+1},\quad i=\overline{1,k+1}.$$

By Lemma 4.12, there exists a continuous mapping

$$\phi_0:\operatorname{co}(\{f_1,\dots,f_{k+1}\})\longrightarrow F_1$$

defined by the formula

$$\phi_0=\phi_{\widehat{\sigma}}(\phi(z))=f_\lambda,\quad\lambda\in\Sigma_0,$$

and satisfying the condition

$$(z-\phi_0(z))\in V_{K,\varepsilon}\ \ \forall\,z\in\operatorname{co}(\{f_1,\dots,f_{k+1}\}).$$

It remains to prove that $f_\lambda\in W_{K,\delta}\ \forall\,\lambda\in\Sigma_0$. For this purpose, let us estimate the quantity $H_1(f_\lambda-f_0;K)$. Owing to the specific character of the $\widehat{\sigma}$-partition, we have

$$f_\lambda(t,x,x_1,\dots,x_s)=f_i(t,x,x_1,\dots,x_s),\quad t\in I_{\beta_i}^{(\alpha)}(\lambda)\cap I.$$

Taking into account the latter assertion, we have

$$\begin{aligned}
H_1(f_\lambda-f_0;K)&=\sum_{\alpha=1}^{l}\int\limits_{I^{(\alpha)}\cap I}\Big[\sup_{(x,x_1,\dots,x_s)\in K^k}\Big(\big|f_\lambda(t,x,\,\cdot\,)-f_0(t,x,\,\cdot\,)\big|\\
&\qquad+\Big|\frac{\partial}{\partial x}\,f_\lambda(t,x,\,\cdot\,)-\frac{\partial}{\partial x}\,f_0(t,x\,\cdot\,)\Big|+\sum_{j=1}^{s}\Big|\frac{\partial}{\partial x_j}\,f_\lambda(t,x,\,\cdot\,)-\frac{\partial}{\partial x_j}\,f_0(t,x,\,\cdot\,)\Big|\Big)\Big]\,dt\\
&\leq\sum_{\alpha=1}^{l}\sum_{\beta=1}^{m}\sum_{i=1}^{k+1}\int\limits_{I_{\beta_i}^{(\alpha)}(\lambda)\cap I}\Big[\sup_{(x,x_1,\dots,x_s)\in K^k}\Big(\big|f_i(t,x,\,\cdot\,)-f_0(t,x,\,\cdot\,)\big|\\
&\qquad+\Big|\frac{\partial}{\partial x}\,f_i(t,x,\,\cdot\,)-\frac{\partial}{\partial x}\,f_0(t,x,\,\cdot\,)\Big|+\sum_{j=1}^{s}\Big|\frac{\partial}{\partial x_j}\,f_\lambda(t,x,\,\cdot\,)\,\frac{\partial}{\partial x_j}f_0(t,x,\,\cdot\,)\Big|\Big)\Big]\,dt\\
&\leq\sum_{i=1}^{k+1}H_1(f_i-f_0;K)\leq\delta.
\end{aligned}$$

Hence $\phi_0(z)\in W_{K,\delta}$. □

Lemma 4.14. *In the space $E_f^{(1)}$, let the set*

$$W^{(1)}=\big\{f\in E_f^{(1)}:\ \ H_1(f-f_0;K_0)\leq\delta_0\big\},$$

where $\delta_0 > 0$ is a fixed number and $K_0 \subset O$ is a compact set, be given. Then for an arbitrary $W \in \Psi$, the inclusion

$$\operatorname{cone}\big([W^{(1)}]_W - f_0\big) \supset F_1 - f_0 \tag{4.46}$$

holds. Here $[W]_{w^{(1)}}$ denotes the closure (with respect to $W^{(1)}$) of the set $W^{(1)} \cap W$ in the topology on $W^{(1)}$ induced by the topology on E_f.

Proof. Clearly, $W_{K_0,\delta_0} \subset W^{(1)}$ and there exists W_{K_1,δ_1} contained in W. Therefore,

$$W^{(1)} \cap W \supset W_{K_1,\delta_1} \cap W_{K_0,\delta_0} \supset W_{K_2,\delta_2}, \tag{4.47}$$

where $K_2 = K_0 \cup K_1$, $\delta_2 = \min\{\delta_1, \delta_0\}$. To prove the inclusion (4.46), it suffices to show that

$$\operatorname{cone}\big([W^{(1)}]_{w_{K_2,\delta_2}} - f_0\big) \supset F_1 - f_0$$

(see (4.47)). Let $f - f_0 \in F_1 - f_0$ and $z_\lambda = (1-\lambda)f_0 + \lambda f$, $\lambda \in [0,1]$; let $\{\varepsilon_i\}$ be a sequence converging to zero. By Lemma 4.12, we can construct a sequence of continuous mappings

$$\phi_0^{(i)} : \operatorname{co}(\{f_0, f\}) \longrightarrow F_1, \ \ i = 1, 2, \dots,$$

such that

$$z_\lambda - \phi_0^{(i)}(z_\lambda) \in V_{K_2,\varepsilon_i}, \ \ \lambda \in [0,1], \ \ i = 1, 2, \dots, \tag{4.48}$$

where

$$\phi_0^{(i)}(z_\lambda) = f_\lambda(t, x, x_1, \dots, x_s) = f\big(t, x, x_1, \dots, x_s, u_\lambda(t), u_\lambda(t-\theta_1), \dots, u_\lambda(t-\theta_\nu)\big),$$
$$u_\lambda(t) = \begin{cases} u_0(t), & t \in I_{\beta_1}^{(\alpha)} \cap I_2, \\ u(t), & t \in I_{\beta_2}^{(\alpha)} \cap I_2, \end{cases} \qquad \alpha = \overline{1,l}, \ \ \beta = \overline{1,m_i}, \ \ m_i = m(\varepsilon_i).$$

Let us now prove the existence of $\lambda_0 \in (0,1)$ such that

$$\phi_0^{(i)}(z_\lambda) \in W_{K_2,\delta_2}, \ \ i = 1, 2, \dots, \ \ \forall\, \lambda \in [0, \lambda_0]. \tag{4.49}$$

For the expression $H_1(f - f_0; K_2)$, taking into account the relation $f_\lambda(t, x, x_1, \dots, x_s) = f_0(t, x, x_1, \dots, x_s)$, $t \in t \in I_{\beta_1}^{(\alpha)}(\lambda) \cap I$, we have

$$H_1(f_\lambda - f_0; K_2) = \sum_{\alpha=1}^{l} \int\limits_{I_{2i}^{(\alpha)}(\lambda)} \bigg[\sup_{(x,x_1,\dots,x_s) \in K_2^{1+s}} \Big(\big| f(t,x,\,\cdot\,) - f_0(t,x,\,\cdot\,) \big|$$
$$+ \Big| \frac{\partial}{\partial x} f(t,x,\,\cdot\,) - \frac{\partial}{\partial x} f_0(t,x,\,\cdot\,) \Big| + \sum_{j=1}^{s} \Big| \frac{\partial}{\partial x_j} f(t,x,\,\cdot\,) - \frac{\partial}{\partial x_j} f_0(t,x,\,\cdot\,) \Big| \Big) \bigg]\, dt,$$

where

$$I_{2i}^{(\alpha)}(\lambda) = \bigcup_{\beta=1}^{m_i} \big(I_{\beta_2}^{(\alpha)}(\lambda) \cap I \big).$$

The specific character of the $\widehat{\sigma}$-partition implies

$$\operatorname{mes}\Big(\sum_{\alpha=1}^{l} I_{2i}^{(\alpha)}(\lambda) \Big) \longrightarrow \sum_{\alpha=1}^{l} \sum_{\beta=1}^{m_1} \operatorname{mes} I_{\beta_2}^{\alpha}(\lambda) = \lambda \sum_{\alpha=1}^{l} \sum_{\beta=1}^{m_1} \operatorname{mes} I_{\beta}^{\alpha} \le \lambda \operatorname{mes} I_2.$$

Therefore,

$$\operatorname{mes}\Big(\sum_{\alpha=1}^{l} I_{2i}^{(\alpha)}(\lambda) \Big) \longrightarrow 0 \ \text{ as } \ \lambda \to 0$$

uniformly in $i = 1, 2, \dots$. Hence there exists $\lambda_0 \in (0,1)$ for which

$$H_1(f_\lambda - f_0; K_2) \le \delta_2.$$

The inclusion (4.48) is proved. The condition (4.49) implies $\phi_0^{(i)}(z_\lambda) \to z_\lambda$ as $i \to \infty$. Therefore,

$$z_\lambda \in [W^{(1)}]_{W_{K_2,\delta_2}} \ \text{ for } \ \lambda \in [0, \lambda_0],$$

and hence

$$z_\lambda - f_0 \in \operatorname{cone}\big([W^{(1)}]_{W_{K_2,\delta_2}}\big), \ \ \lambda \in [0, \lambda_0],$$

but $z_\lambda - f_0 = \lambda(f - f_0)$. Thus $f - f_0 \in \operatorname{cone}([W^{(1)}]_{W_{K_2,\delta_2}})$. □

4.4 The optimal control problem with the discontinuous initial condition

Consider the optimal control problem

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_\nu)\big), \tag{4.50}$$
$$t \in [t_0, t_1] \subset I, \ \ u \in \Omega(I_2, U),$$

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_0, \ \ \varphi \in \Phi_1, \ \ x_0 \in X_0, \tag{4.51}$$

$$q^i\big(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x(t_1)\big) = 0, \ \ i = \overline{1, l}, \tag{4.52}$$

$$q^0\big(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x(t_1)\big) \longrightarrow \min, \tag{4.53}$$

where $\theta_\nu > \dots > \theta_1 > 0$, $\Phi_2 = \{\varphi \in \mathrm{PC}(I_2, \mathbb{R}^n) : \ \varphi(t) \in N\}$, $N \subset O$ is a convex set; $X_0 \subset O$ is a convex compact set; the scalar-valued functions $q^i(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x_1)$, $i = \overline{0, l}$, are continuously differentiable on $I^2 \times [\theta_{11}, \theta_{12}] \times \dots \times [\theta_{s1}, \theta_{s2}] \times O^2$.

The problem (4.50)–(4.53) is called an optimal control problem with the discontinuous initial condition.

Definition 4.12. Let $v = (t_0, t_1, \tau_1, \dots, \tau_s, x_0, \varphi, u) \in A = (a,b) \times (a,b) \times (\theta_{11}, \theta_{12}) \times \dots \times (\theta_{s1}, \theta_{s2}) \times X_0 \times \Phi_2 \times \Omega(I, U)$. A function $x(t) = x(t; v) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution of the equation (4.50) with the discontinuous initial condition (4.51), or a solution corresponding to the element v and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (4.51) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (4.50) a.e. on $[t_0, t_1]$.

Definition 4.13. An element $v = (t_0, t_1, \tau_1, \dots, \tau_s, x_0, \varphi, u) \in A$ is said to be admissible if the corresponding solution $x(t) = x(t; v)$ satisfies the boundary conditions (4.52).

Denote by A_0 the set of admissible elements.

Definition 4.14. An element $v_0 = (t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in A_0$ is said to be optimal if there exist a number $\delta_0 > 0$ and a compact set $K_0 \subset O$ such that for an arbitrary element $v \in A_0$ satisfying the condition

$$|t_{00} - t_0| + |t_{10} - t_1| + \sum_{i=1}^{s} |\tau_{i0} - \tau_i| + |x_{00} - x_0| + \|\varphi_0 - \varphi\|_{I_1} + H_1(f_0 - f; K_0) \le \delta_0,$$

the inequality

$$q^0\big(t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, x_0(t_{10})\big) \le q^0\big(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x(t_1)\big)$$

holds. Here

$$f_0 = f_0(t, x, x_1, \dots, x_s) = f\big(t, x, x_1, \dots, x_s, u_0(t), u_0(t-\theta_1), \dots, u_0(t-\theta_k)\big)$$

and

$$f = f(t, x, x_1, \dots, x_s) = f\big(t, x, x_1, \dots, x_s, u(t), u(t-\theta_1), \dots, u(t-\theta_k)\big).$$

Theorem 4.6. *Let v_0 be an optimal element and let the following conditions hold:*

4.1. $\tau_{s0} > \cdots > \tau_{10}$ *and* $t_{00} + \tau_{s0} < t_{10}$, *with* $\tau_{i0} \in (\theta_{i0}, \theta_{i+10})$, $i = \overline{1, s-1}$;

4.2. $\theta_i = m_i h$, $i = \overline{1, \nu}$, *where* m_i, $i = \overline{1, \nu}$, *are natural numbers,* $h > 0$ *is a real number;*

4.3. *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot\varphi_0(t)$ *is bounded;*

4.4. *the function* $f_0(w), w = (t, x, x_1, \dots, x_s) \in I \times O^{s+1}$, *is bounded;*

4.5. *there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^-, \quad w \in (a, t_{00}] \times O^{s+1},$$

where $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$;

4.6. *there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} [f_0(w_{1i}) - f_0(w_{2i})] = f_{0i},$$

where $w_{1i}, w_{2i} \in (a, b) \times O^{s+1}$, $i = \overline{1, s}$,

$$\begin{aligned} w_{1i}^0 = \Big(&t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),\\ &x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big),\\ w_{2i}^0 = \Big(&t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),\\ &\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big); \end{aligned}$$

4.7. *there exists the finite limit*

$$\lim_{w \to w_{s+1}} f_0(w) = f_{s+1}^-, \quad w \in (t_{00}, t_{10}] \times O^{s+1}, \quad w_{s+1} = \big(t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_{10}), \dots, x_s(t_{10} - \tau_{s0})\big).$$

Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, *with* $\pi_0 \le 0$, *and a solution* $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ *of the equation*

$$\dot\psi(t) = -\psi(t) f_{0x}[t] - \sum_{i=1}^{s} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}], \quad t \in [t_{00}, t_{10}], \quad \psi(t) = 0, \quad t > t_{10}, \tag{4.54}$$

such that the following conditions hold:

4.8. *the conditions for the moments* t_{00} *and* t_{10}:

$$\pi Q_{0t_0} \ge \psi(t_{00}) f_0^- + \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0}) f_{0i}, \quad \pi Q_{0t_1} \ge -\psi(t_{10}) f_{s+1}^-,$$

where

$$Q_0 = (q^0, \dots, q^l)^\top, \quad Q_{0t_0} = \frac{\partial}{\partial t_0} Q_0;$$

4.9. *the conditions for the delays* τ_{i0}, $i = \overline{1, s}$,

$$\pi Q_{0\tau_i} = \psi(t_{00} + \tau_{i0}) f_{0i} + \int_{t_{00}}^{t_{10}} \psi(t) f_{0x_i}[t] \dot x_0(t - \tau_{i0})\, dt = 0, \quad i = \overline{1, s};$$

4.10. *the condition of the vector x_{00},*

$$(\pi Q_{0x_0}+\psi(t_{00}))x_{00}=\max_{x_0\in X_0}(\pi Q_{0x_0}+\psi(t_{00}))x_0;$$

4.11. *the integral maximum principle for the initial function $\varphi_0(t)$,*

$$\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi_0(t)\,dt=\max_{\varphi(t)\in\Phi_1}\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi(t)\,dt;$$

4.12. *the integral maximum principle for the control function $u_0(t)$,*

$$\int\limits_{t_{00}}^{t_{10}}\psi(t)f_0[t]\,dt$$
$$=\max_{u(t)\in\Omega(I_2,U)}\int\limits_{t_{00}}^{t_{10}}\psi(t)f\Big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0\big(t-\tau_{i0},u(t),u(t-\theta_1),\dots,u(t-\theta_\nu)\big)\Big)\,dt;$$

4.13. *the condition for the function $\psi(t)$,*

$$\psi(t_{10})=\pi Q_{0x_1}.$$

Theorem 4.7. *Let v_0 be an optimal element and let the conditions* 4.1–4.4 *and* 4.6 *hold. Moreover, there exist the finite limits*

$$\lim_{w\to w_0}f_0(w)=f_0^+,\ \ w\in[t_{00},b)\times O^{s+1},\qquad \lim_{w\to w_{s+1}}f_0(w)=f_{s+1}^+,\ \ w\in[t_{10},b)\times O^{s+1}. \tag{4.55}$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq0$, with $\pi_0\le0$, and a solution of the equation (4.54) *such that the conditions* 4.9–4.13 *hold. Moreover,*

$$\pi Q_{0t_0}\le\psi(t_{00})f_0^-+\sum_{i=1}^{s}\psi(t_{00}+\tau_{i0})f_{0i},\quad \pi Q_{0t_1}\le-\psi(t_{10})f_{s+1}^-.$$

Theorem 4.8. *Let v_0 be an optimal element and let the conditions of Theorem* 4.6 *hold. Moreover, there exist the finite limits f_0^+, f_{s+1}^+, with $f_0^-=f_0^+:=\widehat{f}_0$, $f_{s+1}^-=f_{s+1}^+:=\widehat{f}_{s+1}$. Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq0$, with $\pi_0\le0$, and a solution of the equation* (4.54) *such that the conditions* 4.9–4.13 *hold. Moreover,*

$$\pi Q_{0t_0}=\psi(t_{00})\widehat{f}_0+\sum_{i=1}^{s}\psi(t_{00}+\tau_{i0})f_{0i},\quad \pi Q_{0t_1}=-\psi(t_{10})\widehat{f}_{s+1}.$$

Theorem 4.9. *Let v_0 be an optimal element and let the conditions* 4.1–4.5 *and* 4.7 *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}[f_0(w_{1i})-f_0(w_{2i})]=f_{0i}^-,$$

where $w_{1i},w_{2i}\in(a,t_{00}+\tau_{i0})\times O^{s+1}$, $i=\overline{1,s}$. Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq0$, with $\pi_0\le0$, and a solution of the equation (4.54) *such that the conditions* 4.8–4.13 *hold. Moreover,*

$$\pi Q_{0\tau_i}\ge\psi(t_{00}+\tau_{i0})f_{0i}^-+\int\limits_{t_{00}}^{t_{10}}\psi(t)f_{0x_i}[t]\dot{x}_0(t-\tau_{i0})\,dt,\ \ i=\overline{1,s}.$$

Theorem 4.10. *Let v_0 be an optimal element and let the conditions* 4.1–4.5 *and* (4.55) *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}\big[f_0(w_{1i})-f_0(w_{2i})\big]=f_{0i}^+,$$

where $w_{1i},w_{2i}\in[t_{00}+\tau_{i0},b)\times O^{s+1}$, $i=\overline{1,s}$. Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, with $\pi_0\le 0$, and a solution of the equation (4.54) *such that the conditions* 4.8–4.13 *hold. Moreover,*

$$\pi Q_{0\tau_i}\le \psi(t_{00}+\tau_{i0})f_{0i}^+ +\int_{t_{00}}^{t_{10}}\psi(t)f_{0x_i}[t]\dot x_0(t-\tau_{i0})\,dt,\ \ i=\overline{1,s}.$$

4.5 Proof of Theorem 4.6

Auxiliary assertions. Let $K\subset O$ be a compact set and let $\alpha>0$ be a certain given number. In the spaces $E_f^{(1)}$ and E_f, we define, respectively, the sets

$$W_{K,\alpha}=\big\{\delta f\in E_f^{(1)}:\ H_1(\delta f;K)\le\alpha\big\},$$
$$W(K;\alpha)=\Big\{\delta f\in E_f:\ \exists\, m_{\delta f,K}(t),L_{\delta f,K}(t)\in L_1(I,R_+),\ \int_I\big[m_{\delta f,K}(t)+L_{\delta f,K}(t)\big]\,dt\le\alpha\Big\}.$$

Lemma 4.15. *Let $K_i\subset O$, $i=1,2$, be compact sets, and, moreover, let $K_1\subset \operatorname{int}K_2$ and $\alpha_1>0$ be a certain number. Then there exists a number $\alpha_2>0$ such that*

$$W_{K_2,\alpha_1}\subset W(K_1;\alpha_2). \tag{4.56}$$

Proof. Let $\delta f\in W_{K_2,\alpha_1}$. Hence

$$\int_I\sup\Big\{|\delta f(t,x,x_1,\dots,x_s)|+|\delta f_x(t,x,\cdot\,)|+\sum_{i=1}^s|\delta f_{x_i}(t,x,\cdot\,)|:\ (x,x_1,\dots,x_s)\in K_2^{s+1}\Big\}\,dt\le\alpha_1.$$

For a.e. $t\in I$ and every $(x',x_1',\dots,x_s')\in K_1^{s+1},(x',x_1',\dots,x_s')\in K_1^{s+1}$ the inequality

$$\big|\delta f(t,x',x_1',\dots,x_s')-\delta f(t,x'',x_1'',\dots,x_s'')\big|\le L_{\delta f,K_1}(t)\Big[|x'-x''|+\sum_{i=1}^s|x_i'-x_i''|\Big]$$

holds, where

$$L_{\delta f,K_1}(t)=n^2s(\alpha_0+1)$$
$$\times\sup\Big\{|\delta f(t,x,x_1,\dots,x_s)|+|\delta f_x(t,x,\cdot\,)|+\sum_{i=1}^s|\delta f_{x_i}(t,x,\cdot\,)|:\ (x,x_1,\dots,x_s)\in K_2\Big\}$$

(see Lemma 2.2).

On the other hand, it is obvious that for $(t,x,x_1,\dots,x_s)\in I\times K_1^{s+1}$, we have

$$|\delta f(t,x,x_1,\dots,x_s)|\le m_{\delta f,K_1}(t)=\sup\Big\{|\delta f(t,x,x_1,\dots,x_s)|:\ (x,x_1,\dots,x_s)\in K_1^{s+1}\Big\}.$$

Using the relations obtained above, we get

$$\int_I\big[m_{\delta f,K_1}(t)+L_{\delta f,K_1}(t)\big]\,dt\le\alpha_1\big[1+n^2s(\alpha_0+1)\big]:=\alpha_2.$$

The inclusion (4.56) is proved. □

To each element

$$\kappa = (t_0, t_1, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in (a,b) \times (a,b) \times (\theta_{11}, \theta_{12}) \times \cdots \times (\theta_{s1}, \theta_{s2}) \times X_0 \times \Phi_1 \times E_f^{(1)}$$

we put in correspondence the functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s)\big), \;\; t \in [t_0, t_1],$$

with the initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0), \;\; x(t_0) = x_0.$$

Definition 4.15. The solution corresponding to an element $\kappa = (t_0, t_1, \tau_1, \dots, \tau_s, x_0, \varphi, f)$ is called a solution $x(t;\mu)$, $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f)$, defined on $[\widehat{\tau}, t_1]$, and denoted by $x(t;\kappa)$.

Therefore,

$$x_0(t) = x(t; v_0) = x(t; \kappa_0) = x(t; \mu_0), \;\; t \in [\widehat{\tau}, t_{10}], \tag{4.57}$$

where

$$\kappa_0 = (t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, f_0), \quad \mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0, f_0).$$

The following lemma is a direct consequence of Theorem 1.2.

Lemma 4.16. *Let $\alpha_1 > 0$ be a certain given number, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl} \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exists a number $\delta_1 > 0$ such that to each element*

$$\begin{aligned} \kappa \in V(\kappa_0; K_1, \delta_1, \alpha_1) = {} & (B(t_{00}; \delta_1) \cap I) \times (B(t_{10}; \delta_1) \cap I) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots \\ & \times (B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}, \delta_1) \cap O) \times (B(\varphi_0; \delta_1) \cap \Phi_2) \times \big[f_0 + (W_{K_1,\alpha_1} \cap V_{K_1,\delta_1})\big] \end{aligned}$$

there corresponds the solution $x(t;\kappa) \in K_1$, $t \in [\widehat{\tau}, t_1]$. Moreover, for each $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_1)$ such that for an arbitrary $\kappa \in V(\kappa_0; K_1, \delta_1, \alpha_1)$, the inequality

$$|x(t_{10}; \kappa_0) - x(t_1; \kappa)| \le \varepsilon$$

holds.

Remark 4.1. Lemma 4.16 remains valid if we replace the set $V(\kappa_0; K_1, \delta_1, \alpha_1)$ by the set

$$\begin{aligned} V(\kappa_0; K_1, \delta_1) = {} & (B(t_{00}; \delta_1) \cap I) \times (B(t_{10}; \delta_1) \cap I) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots \\ & \times (B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}, \delta_1) \cap O) \times (B(\varphi_0; \delta_1) \cap \Phi_2) \times [f_0 + W_{K_1,\delta_1}]. \end{aligned}$$

Let us now consider the topological vector space

$$E_\kappa = \mathbb{R}^{2+s+n} \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E_f^{(1)}$$

with the points $\kappa = (y, \varsigma)$, where $y = (t_0, t_1, \tau_1, \dots, \tau_s, x_0)^\top$, $\varsigma = (\varphi, f)$.

The set

$$X = [a, t_{00}] \times [t_{00}, t_{10}] \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \subset \mathbb{R}^{2+s+n}$$

is a locally convex subspace in the topology induced from $\mathbb{R}^{2+s+n}$.

By $D_0 \subset E_\kappa$ we denote the set of elements $\kappa \in X \times \Phi_2 \times E_f^{(1)}$ such that the solution $x(t;\kappa)$ corresponds to each of them. The set D_0 is nonempty, since $\kappa_0 \in D_0$.

Lemma 4.17. *The set D_0 is finitely convex.*

Proof. Let $\widehat{\kappa} = (\widehat{y}, \widehat{\varsigma}) \in D_0$ be an arbitrary fixed point, and $L_{\widehat{\varsigma}} \subset E_\varsigma$ be a linear manifold, i.e.,

$$L_{\widehat{\varsigma}} = \Big\{ \widehat{\varsigma} + \delta\varsigma : \;\; \delta\varsigma = \sum_{i=1}^{k} \lambda_i \delta\varsigma_i, \;\; \lambda_i \in \mathbb{R}, \;\; i = \overline{1,k} \Big\},$$

where $\delta\varsigma_i \in E_\varsigma$, $i = \overline{1,k}$, are fixed points. There exists a number $\delta_1 > 0$ such that with each element $\kappa \in V(\widehat{\kappa}; K_1, \delta_1)$ we associate the solution $x(t; \widehat{\varsigma}) \in K_1$ (see Remark 4.1).

Let a number $\delta \in (0, \delta_1)$ be insomuch small that the neighborhood of the point $\widehat{\varsigma}$

$$V_{\widehat{\varsigma}} = \Big\{\widehat{\varsigma} + \sum_{i=1}^{k} \lambda_i \delta\varsigma_i : \ \ |\lambda_i| \le \delta, \ \ i = \overline{1,k}\Big\}$$

is contained in the set

$$(B(\widehat{\varphi}; \delta_1) \cap \Phi_2) \times [\widehat{f} + W_{K_1,\delta_1}].$$

Therefore, there exist convex neighborhoods

$$V_{\widehat{y}} = (B(\widehat{t}_0; \delta) \cap (a, \widehat{t}_0)) \times (B(\widehat{t}_1; \delta) \cap (\widehat{t}_0, \widehat{t}_1)) \times (B(\widehat{x}_0; \delta_1) \cap O) \subset X, \ \ V_{\widehat{\varsigma}} \subset L_{\widehat{\varsigma}}$$

such that

$$V_{\widehat{y}} \times V_{\widehat{\varsigma}} \subset D_0.$$

Hence the set D_0 is finitely locally convex with respect to the space $X \times E_\varsigma$. □

On the set D_0, let us define the mapping

$$S : D_0 \to \mathbb{R}^n$$

by the formula

$$S(\kappa) = x(t_1; \kappa).$$

Lemma 4.18. *The mapping S is differentiable at the point κ_0 and*

$$dS_{\kappa_0}(\delta\kappa) = \delta x(t_{10}; \delta\kappa) + f_{s+1}^{-}\delta t_1 \ \ \forall\, \delta\kappa = (\delta t_0, \delta t_1, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_\kappa - \kappa_0, \tag{4.58}$$

where

$$\begin{aligned}\delta x(t_{10}; \delta\kappa) = \delta x(t_{10}; \delta\mu) = &-\Big[Y(t_{00}; t) f_0^{-} + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}\Big]\delta t_0 \\ &- \sum_{i=1}^{s}\Big[Y(t_{00} + \tau_{i0}; t) f_{0i} + \int_{t_{00}}^{t} Y(\xi; t) f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\, d\xi\Big]\delta\tau_i + Y(t_{00}; t)\delta x_0 \\ &+ \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi + \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi \end{aligned} \tag{4.59}$$

and $\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_\mu^{(1)} - \mu_0$.

Proof. Let $L_{\varsigma_0} \subset E_\varsigma$ be a linear manifold, and let

$$V_0 \subset X - y_0, \ \ V \subset L_{\varsigma_0} - \varsigma_0$$

be bounded convex neighborhoods of zero, where $y_0 = (t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00})^\top$ and $\varsigma_0 = (\varphi_0, f_0)$.

The finite local convexity of the set D_0 implies the existence of a number $\varepsilon_0 > 0$ such that for an arbitrary $(\varepsilon, \delta\varsigma) \in (0, \varepsilon_0) \times V_0 \times V$, $\varsigma_0 + \varepsilon\delta\varsigma \in D_0$, and

$$\begin{aligned} x(t_{10} + \varepsilon\delta t_1; \kappa_0 + \varepsilon\delta\kappa) - x(t_{10} + \varepsilon\delta t_1; \kappa_0) &= x(t_{10} + \varepsilon\delta t_1; \mu_0 + \varepsilon\delta\mu) - x(t_{10} + \varepsilon\delta t_1; \mu_0) \\ &= \Delta x(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu) = \varepsilon\delta x(t_{10} + \varepsilon\delta t_1; \delta\mu) + o(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu), \end{aligned}$$

where the variation $\delta x(t_{10} + \varepsilon\delta t_1; \delta\mu)$ is calculated by the formula (2.7).

We have

$$\begin{aligned} S(\kappa_0+\varepsilon\delta\kappa)-S(\kappa_0) &= x(t_{10}+\varepsilon\delta t_1;\kappa_0+\varepsilon\delta\kappa)-x_0(t_{10}) \\ &= x(t_{10}+\varepsilon\delta t_1;\kappa_0+\varepsilon\delta\kappa)-x_0(t_{10}+\varepsilon\delta t_1)+x_0(t_{10}+\varepsilon\delta t_1)-x_0(t_{10}) \\ &= \varepsilon\delta x(t_{10}+\varepsilon\delta t_1;\delta\mu)+o(t_{10}+\varepsilon\delta t_1;\varepsilon\delta\mu)+\int\limits_{t_{10}}^{t_{10}+\varepsilon\delta t_1} f_0[t]\,dt. \end{aligned} \tag{4.60}$$

It is easy to note that

$$\lim_{i\to 0}\delta x(t_{10}+\varepsilon\delta t_1;\delta\mu)=\delta x(t_{10};\delta\mu)$$

uniformly in $\delta\kappa\in V_0\times V$, (i.e., uniformly for the corresponding $\delta\mu$) and

$$\int\limits_{t_{10}}^{t_{10}+\varepsilon\delta t_1} f_0[t]\,dt=\varepsilon f_{s+1}^{-}\delta t_1+o(\varepsilon\delta\kappa).$$

Taking into account these relations and the variation formula (2.7), from (4.60) we obtain

$$S(\kappa_0+\varepsilon\delta\kappa)-S(\kappa_0)=\varepsilon\big[\delta x(t_{10};\delta\kappa)+f_{s+1}^{-}\delta t_1\big]+o(\varepsilon\delta\kappa)=\varepsilon dS_{\kappa_0}(\delta\kappa)+o(\varepsilon\delta\kappa), \tag{4.61}$$

where $\delta x(t_{10};\delta\kappa)$ has the form (2.59). □

Differentiability of the mapping at the point z_0. Consider the vector space

$$E_z=\mathbb{R}\times E_\kappa$$

of points $z=(\xi,\kappa)$.

Introduce the sets

$$X=\mathbb{R}_+\times X_0,\quad D=\mathbb{R}_+\times D_0.$$

The set is finitely locally convex in the subspace $X\times E_\varsigma\subset E_z$ (see Lemma 4.17).

On the set D, let us define the mapping

$$P:D\to\mathbb{R}^{l+1}$$

by the formula

$$P(z)=Q(t_0,t_1,\tau_1,\dots,\tau_s,x_0,S(\kappa))+(\xi,0,\dots,0)^\top,$$

where $Q=q^0,\dots,q^l$ and $S(\kappa)=x(t_1;\kappa)$.

Lemma 4.19. *The mapping P is differentiable at the point $z_0=(0,\kappa_0)$ and*

$$\begin{aligned} dP_{z_0}(\delta z) &= \Big\{Q_{0t_0}-Q_{0x_1}Y(t_{00};t_{10})f_0^{-}-\sum_{i=1}^{s}Q_{0x_1}Y(t_{00}+\tau_{i0};t_{10})f_{0i}\Big\}\delta t_0+\big\{Q_{0t_1}+Q_{0x_1}f_{s+1}^{-}\big\}\delta t_1 \\ &\quad+\sum_{i=1}^{s}\Big\{Q_{0\tau_i}-Q_{0x_1}Y(t_{00}+\tau_{i0};t_{10})f_{0i}-\int\limits_{t_{00}}^{t_{10}}Q_{0x_1}Y(t;t_{10})f_{0x_i}[t]\dot x_0(t-\tau_{i0})\,dt\Big\}\delta\tau_i \\ &\quad+\big\{Q_{0x_0}+Q_{0x_1}Y(t_{00};t_{10})\big\}\delta x_0+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Q_{0x_1}Y(t+\tau_{i0};t)f_{0x_i}[t+\tau_{i0}]\delta\varphi(t)\,dt \\ &\quad+\int\limits_{t_{00}}^{t_{10}}Q_{0x_1}Y(t;t_{10})\delta f[t]\,dt+(\delta\xi,0,\dots,0)^\top,\quad \delta z=(\delta\xi,\delta\kappa)\in E_z-z_0. \end{aligned} \tag{4.62}$$

Proof. Let $L_{\varsigma_0} \subset E_\varsigma$ be an arbitrary linear manifold and let

$$V_0 \subset X - (0, y_0)^\top, \ \ V \subset L_{\varsigma_0} - \varsigma_0$$

be arbitrary bounded convex neighborhoods of zero. There exists a number $\varepsilon_0 > 0$ such that for arbitrary $\varepsilon \in (0, \varepsilon_0)$ and $\delta z \in V_0 \times V$,

$$z_0 + \varepsilon\delta z \in D,$$

and the formula (4.61) holds.

We have

$$\begin{aligned} P(z_0 + \varepsilon\delta z) - P(z_0) = {} & Q\big(t_{00}+\varepsilon\delta t_0, t_{10}+\varepsilon\delta t_1, \tau_{10}+\varepsilon\delta\tau_1, \dots, \tau_{s0}+\varepsilon\delta\tau_s, x_{00}+\varepsilon\delta x_0, S(\kappa_0+\varepsilon\delta\kappa)\big) \\ & - Q\big(t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, S(\kappa_0)\big) + \varepsilon(\delta\xi, 0, \dots, 0)^\top. \end{aligned}$$

Let a number $\varepsilon_0 > 0$ be insomuch small that

$$S(\kappa_0) + t(S(\kappa_0 + \varepsilon\delta\kappa) - S(\kappa_0)) \in O \ \ \forall\, (t, \varepsilon) \in (0,1) \times (0, \varepsilon_0), \ \ \forall\, \delta z \in V_0 \times V,$$

where $\delta z = (\delta\xi, \delta\kappa)$ (see Lemma 4.16).

Let us now transform the difference

$$\begin{aligned} & Q\big(t_{00} + \varepsilon\delta t_0, t_{10} + \varepsilon\delta t_1, \tau_{10} + \varepsilon\delta\tau_1, \dots, \tau_{s0} + \varepsilon\delta\tau_s, x_{00} + \varepsilon\delta x_0, S(\kappa_0 + \varepsilon\delta\kappa)\big) \\ & \qquad\qquad - Q\big(t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, S(\kappa_0)\big) \\ & = \int_0^1 \frac{d}{dt}\, Q\Big(t_{00} + \varepsilon t\delta t_0, t_{10} + \varepsilon t\delta t_1, \tau_{10} + \varepsilon t\delta\tau_1, \dots, \\ & \qquad \tau_{s0} + \varepsilon t\delta\tau_s, x_{00} + \varepsilon t\delta x_0, S(\kappa_0) + t\big(S(\kappa_0 + \varepsilon\delta\kappa) - S(\kappa_0)\big)\Big)\, dt \\ & = \varepsilon\Big[Q_{0t_0}\delta t_0 + Q_{0t_1}\delta t_1 + \sum_{i=1}^s Q_{0\tau_i}\delta\tau_i + Q_{0x_0}\delta x_0 + Q_{0x_1} dS_{\kappa_0}(\delta\kappa)\Big] + \alpha(\varepsilon\delta z), \end{aligned}$$

where

$$\begin{aligned} \alpha(\varepsilon\delta z) = \varepsilon \int_0^1 \Big\{ & \big[Q_{0t_0}[\varepsilon;t] - Q_{0t_0}\big]\delta t_0 + \big[Q_{0t_1}[\varepsilon;t] - Q_{0t_1}\big]\delta t_1 + \sum_{i=1}^s \big[Q_{0\tau_i}[\varepsilon;t] - Q_{0\tau_i}\big]\delta\tau_i \\ & + \big[Q_{0x_0}[\varepsilon;t] - Q_{0x_0}\big]\delta x_0 + \big[Q_{0x_1}[\varepsilon;t] - Q_{0x_1}\big] S_{\kappa_0}(\delta\kappa) + Q_{0x_1}[\varepsilon;t] o(\varepsilon\delta\kappa)\Big\}\, dt, \end{aligned}$$

$$\begin{aligned} Q_{0t_0}[\varepsilon;t] = Q_{t_0}\Big(& t_{00} + \varepsilon t\delta t_0, t_{10} + \varepsilon t\delta t_1, \tau_{10} + \varepsilon t\delta\tau_1, \dots, \\ & \tau_{s0} + \varepsilon t\delta\tau_s, x_{00} + \varepsilon t\delta x_0, S(\kappa_0) + t\big(S(\kappa_0 + \varepsilon\delta\kappa) - S(\kappa_0)\big)\Big). \end{aligned}$$

It is easy to note that

$$\lim_{\varepsilon\to 0} \big[Q_{0t_i}[\varepsilon;t] - Q_{0t_i}\big] = 0, \ \ i = 1,2, \quad \lim_{\varepsilon\to 0} \big[Q_{0\tau_i}[\varepsilon;t] - Q_{0\tau_i}\big] = 0, \ \ i = \overline{1,s},$$
$$\lim_{\varepsilon\to 0} \big[Q_{0x_i}[\varepsilon;t] - Q_{0x_i}\big] = 0, \ \ i = 0,1.$$

Therefore, $\alpha(\varepsilon\delta z) = o(\varepsilon\delta z)$. Thus,

$$P(z_0+\varepsilon\delta z)-P(z_0)=\varepsilon\Big[Q_{0t_0}\delta t_0+Q_{0t_1}\delta t_1+\sum_{i=1}^s Q_{0\tau_i}\delta\tau_i+Q_{0x_1}dS_{\kappa_0}(\delta\kappa)+(\delta\xi,0,\dots,0)^\top\Big]+o(\varepsilon\delta z).$$

Due to the relations (4.58) and (4.59) from the above equality we get (4.62). □

Quasiconvexity of the filter Ψ_{z_0}. Continuity of the mapping P on the filter $\mathrm{co}[\Psi_{z_0}]$. In the topological vector space E_z, let us define the filter Ψ_{z_0} as the direct product

$$\Psi_{z_0} = \Psi_{\widehat{y}_0} \times \Psi_{\varphi_0} \times \Psi$$

of two filters $\Psi_{\widehat{y}_0}$, $\widehat{y}_0 = (0, y_0)^\top$, and Ψ_{φ_0} which are defined, respectively, by the convex bases

$$\Big\{(B_0 \cap \mathbb{R}_+) \times (B_{t_{00}} \cap (a, t_{00}]) \times (B_{t_{01}} \cap (a, t_{10}]) \times (B_{\tau_{10}} \cap (\theta_{11}, \theta_{12})) \times \cdots \\ \times (B_{\tau_{s0}} \cap (\theta_{s1}, \theta_{s2})) \times (B_{x_{00}} \cap O) : \ \ B_0, \dots, B_{x_{00}} \ \text{ are convex neighborhoods}\Big\},$$

$$\Big\{B_{\varphi_0} \cap \Phi_1 : \ \ B_{\varphi_0} \subset \mathrm{PC}(I_1, \mathbb{R}^n) \ \text{ is a convex neighborhood}\Big\}.$$

The filter Ψ has been introduced in Subsection 4.3.

There exists a number $\delta_1 > 0$ such that the set

$$W = \mathbb{R}_+ \times (B(t_{00}; \delta_1) \cap (a, t_{00}]) \times (B(t_{01}; \delta_1) \cap (a, t_{10}]) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots \\ \times B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}; \delta_1) \cap O)) \times (B(\varphi_0; \delta_1) \cap \Phi_1)) \times W_{f_0}^{(1)}(K_1, \delta_1) \subset D$$

and, moreover, the mapping

$$P : W \to \mathbb{R}_p^{l+1}$$

is continuous in the topology induced from E_z. Here

$$W_{f_0}^{(1)}(K_1, \delta_1) = \big\{f \in E_f^{(1)} : \ \ H_1(f - f_0 : K_1) \le \delta_1\big\}.$$

The element W_{K_1,δ_1} of the filter Ψ is contained in the convex set $W_{f_0}^{(1)}(K_1, \delta_1)$. Therefore,

$$\mathrm{co}(W_{z_0}) \subset W \subset D,$$

where

$$W_{z_0} = \mathbb{R}_+ \times (B(t_{00}; \delta_1) \cap (a, t_{00}]) \times (B(t_{01}; \delta_1) \cap (a, t_{10}]) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots \\ \times B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}; \delta_1) \cap O)) \times (B(\varphi_0; \delta_1) \cap \Phi_1)) \times W_{K_1,\delta_1} \in \Psi_{z_0}.$$

Hence there exists an element $W_{z_0} \in \Psi$ such that

$$P : \mathrm{co}(W_{z_0}) \to \mathbb{R}^{l+1}$$

is continuous. Therefore, the mapping P is defined and continuous on the filter $\mathrm{co}([\Psi_{z_0}])$.

Criticality of the mapping P on the filter Ψ_{z_0}. The point $z_0 = (0, \kappa_0)$ belongs to all elements of the filter Ψ_{z_0}, and, moreover,

$$P(z_0) = \big(q^0\big(t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, x_0(t_{10})\big), 0, \dots, 0\big)^\top.$$

Introduce the set

$$\mho = \Big\{\kappa = (t_0, t_1, \tau_1, \dots, \tau_s, x_0, \varphi, f) : \ \ f = f\big(t, x, x_1, \dots, x_s, u(t), u(t - \theta_1), \dots, u(t - \theta_\nu)\big), \\ w = (t_0, t_1, \tau_1, \dots, \tau_s, x_0, \varphi, u) \in W_0\Big\}.$$

For an arbitrary element

$$z = (\xi, \kappa) \in W_{z_0} \cap (\mathbb{R}_+ \times \mho),$$

where $W_{z_0} \in \Psi_{z_0}$, we have

$$P(z) = \big(q^0(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x(t_1); \kappa), 0, \dots, 0\big)^\top.$$

The element $w_0 \in W_0$ is optimal; therefore, there exists an element $W_{z_0}(K_2;\delta_2) \in \Psi_{z_0}$, where $\delta_2 \in (0,\widehat{\delta})$ and $K_2 \subset O$ is a compact set containing $\widehat{K}$ such that for an arbitrary element

$$z \in W_{z_0}(K_2;\delta_2) \cap (\mathbb{R}_+ \times \mho)$$

the inequality

$$q^0\big(t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, x_0(t_{10})\big) \le q^0\big(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x(t_1;\kappa)\big) + \xi$$

holds. It is easy to see that

$$P(W_{z_0} \cap (\mathbb{R}_+ \times \mho)) \subset \mathbb{R}_0 = \big\{(p^1, 0 \dots, 0)^\top \in \mathbb{R}^{p+1}\big\}$$

and the point $P(z_0)$ is a boundary point of the set $P(W_{z_0}(K_2;\delta_2) \cap (\mathbb{R}_+ \times \mho))$ with respect to the space $\mathbb{R}_0$.

Therefore, $P(z_0) \in \partial(P(W_{z_0}(K_2;\delta_2) \cap \mathbb{R}_0))$, and, the more so, $P(z_0) \in \partial(P(W_{z_0}(K_2;\delta_2))$.

Deduction of the necessary optimality conditions. All the conditions of Theorem 4.5 hold. Therefore, there exist a nonzero vector $\pi = (\pi_0, \dots, \pi_l)$ and an element $\widehat{W}_{z_0} \in \Psi_{z_0}$ such that the inequality

$$\pi dP_{z_0}(\delta z) \le 0 \ \ \forall\, \delta z \in \operatorname{cone}(\widehat{W}_{z_0} - z_0) \tag{4.63}$$

holds, where $dP_{z_0}(\delta z)$ has the form (4.62).

Introduce the function

$$\psi(t) = \pi Q_{0x_1} Y(t : t_{10}); \tag{4.64}$$

as is easily seen, it satisfies the equation (4.54) and the conditions

$$\psi(t_{10}) = \pi Q_{0x_1}, \ \ \psi(t) = 0, t > t_{10}. \tag{4.65}$$

Taking into account (4.62), (4.64) and (4.65), from the inequality (4.63) we obtain

$$\begin{aligned}
&\Big\{\pi Q_{0t_0} - \psi(t_{00}) f_0^- - \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0}) f_{0i}\Big\}\delta t_0 + \big\{\pi Q_{0t_1} + \psi(t_{10}) f_{s+1}^-\big\}\delta t_1 \\
&\quad + \sum_{i=1}^{s}\Big\{\pi Q_{0\tau_i} - \psi(t_{00} + \tau_{i0}) f_{0i} - \int_{t_{00}}^{t_{10}} \psi(t) f_{0x_i}[t]\dot{x}_0(t - \tau_{i0})\, dt\Big\}\delta\tau_i + \big\{\pi Q_{0x_0} + \psi(t_{00})\big\}\delta x_0 \\
&\quad + \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}]\delta\varphi(t)\, dt + \int_{t_{00}}^{t_{10}} \psi(t)\delta f[t]\, dt + \pi_0 \delta\xi, \ \ \delta z \in \operatorname{cone}(\widehat{W}_{z_0} - z_0).
\end{aligned} \tag{4.66}$$

The condition $\delta z \in \operatorname{cone}(\widehat{W}_{z_0} - z_0)$ is equivalent to the conditions

$$\begin{gathered}
\delta\xi \in \mathbb{R}_+, \ \ \delta t_0 \in (-\infty, 0], \ \ \delta t_1 \in (-\infty, 0], \quad \delta\tau_i \in \mathbb{R}, \ \ i = \overline{1, s}, \\
\delta x_0 \in \operatorname{cone}(\widehat{W}_{x_{00}} - x_{00}), \ \ \delta\varphi \in \operatorname{cone}(\widehat{W}_{\varphi_0} - \varphi_0), \ \ \delta f \in \operatorname{cone}(\widehat{W}_{f_0} - f_0),
\end{gathered}$$

where

$$\widehat{W}_{x_{00}} = B_{x_{00}} \cap X_0, \ \ \widehat{W}_{\varphi_0} = B_{\varphi_0} \cap \Phi_1 \in \Psi_{\varphi_0}, \ \ \widehat{W}_{f_0} \in \Psi_{f_0}.$$

Let $\delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta\varphi = \delta f = 0$ in (4.66), we obtain

$$\pi_0 \delta\xi \le 0 \ \ \forall\, \delta\xi \in \mathbb{R}_+.$$

This implies

$$\pi_0 \le 0.$$

Setting $\delta\xi = \delta t_0 = \delta\tau_1 = \cdots = \delta\tau_i = 0$, and $\delta x_0 = \delta\varphi = \delta f = 0$; then, taking into account the fact that $\delta t_0 \in (-\infty, 0]$, from (4.66) for the initial moment t_{00} we obtain the following condition:

$$\pi Q_{0t_0} \geq \psi(t_{00}) f_0^- + \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0}) f_{0i}.$$

If $\delta\xi = \delta t_0 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta\varphi = \delta f = 0$ in the inequality (4.66), then for the final moment t_{10} we obtain the following condition:

$$\pi Q_{0t_1} \geq -\psi(t_{10}) f_{s+1}^-.$$

If $\delta\xi = \delta t_0 = \delta t_1 = 0$ and $\delta x_0 = \delta\varphi = \delta f = 0$, we get

$$\sum_{i=1}^{s} \left\{ \pi Q_{0\tau_i} - \psi(t_{00} + \tau_{i0}) f_{0i} - \int_{t_{00}}^{t_{10}} \psi(t) f_{0x_i}[t] \dot{x}_0(t - \tau_{i0})\, dt \right\} \delta\tau_i \leq 0 \ \ \forall\, \delta\tau_i \in \mathbb{R}, \ \ i = \overline{1, s}.$$

From the above follow the conditions for the delays τ_{i0}, $i = \overline{1, s}$:

$$\pi Q_{0\tau_i} = \psi(t_{00} + \tau_{i0}) f_{0i} + \int_{t_{00}}^{t_{10}} \psi(t) f_{0x_i}[t] \dot{x}_0(t - \tau_{i0})\, dt, \ \ i = \overline{1, s}.$$

Let $\delta\xi = \delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta\varphi = \delta f = 0$ in (4.66). Then

$$\{\pi Q_{0x_0} + \psi(t_{00})\} \delta x_0 \leq 0, \ \ \delta x_0 \in \operatorname{cone}((B_{x_{00}} \cap X_0) - x_{00}).$$

Let us prove the inclusion

$$\operatorname{cone}((B_{x_{00}} \cap X_0) - x_{00}) \supset X_0 - x_{00}.$$

Indeed, let $x_0 \in X_0$ be arbitrary point. The set x_0 is convex, therefore, for an arbitrary $\varepsilon \in [0, 1]$, the point $x_\varepsilon = x_{00} + \varepsilon(x_0 - x_{00}) \in X_0$. On the other hand, for a sufficiently small $\varepsilon > 0$, $x_\varepsilon \in B_{x_{00}}$. Hence $x_\varepsilon - x_{00} = \varepsilon(x_0 - x_{00}) \in (B_{x_{00}} \cap X_0) - x_{00}$. This implies $x_0 - x_{00} \in \operatorname{cone}((B_{x_{00}} \cap X_0) - x_{00})$. Thus,

$$\{\pi Q_{0x_0} + \psi(t_{00})\} x_{00} = \max_{x_0 \in X_0} \{\pi Q_{0x_0} + \psi(t_{00})\} x_0.$$

Let $\delta\xi = \delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta f = 0$. We have

$$\sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}] \delta\varphi(t)\, dt \leq 0 \ \ \forall\, \delta\varphi \in \operatorname{cone}(\widehat{W}_{\varphi_0} - \varphi_0).$$

Analogously, we can prove

$$\operatorname{cone}(\widehat{W}_{\varphi_0} - \varphi_0) \supset \Phi_1 - \varphi_0.$$

Thus,

$$\sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}] \varphi_0(t)\, dt = \max_{\varphi(t) \in \Phi_1} \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}] \varphi(t)\, dt.$$

We now consider the case where $\delta\xi = \delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta\varphi = 0$. From (4.66) we obtain

$$\int_{t_{00}}^{t_{10}} \psi(t) \delta f[t]\, dt \leq 0, \ \ \delta f \in \operatorname{cone}(\widehat{W}_{f_0} - f_0).$$

Now, using the last inequality, let us prove the integral maximum principle. For this purpose, we have to prove the continuity of the mapping

$$\delta f \longrightarrow \int_{t_{00}}^{t_{10}} \delta f[t]dt, \quad \delta f[t] = \delta f\big(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0})\big) \tag{4.67}$$

on the set $W^{(1)}(K_1;\alpha)$ in the topology induced from $E_f^{(1)}$. Here $K_1 \subset O$ is a compact set containing a certain neighborhood of the set $\varphi_0(I_2) \cup x_0([t_{00}, t_{10}])$ and $\alpha > 0$ is a certain number.

Let $\delta f_i \in W^{(1)}(K_1;\alpha)$, $i = 1, 2, \dots$, and $\lim\limits_{i\to\infty} H_0(\delta f_i; K_1) = 0$. The mapping (4.67) is continuous if

$$\lim_{i\to\infty} \int_{t_{00}}^{t_{10}} \psi(t)\delta f_i[t]\, dt = 0. \tag{4.68}$$

Integration by parts yields

$$\int_{t_{00}}^{t_{10}} \delta\psi(t) f_i[t]dt = \psi(t_{00}) \int_{t_{00}}^{t_{10}} \delta f_i[t]\, dt - \int_{t_{00}}^{t_{10}} \psi(t)\bigg(\int_{t_{00}}^{t} \delta f_i[\xi]\, d\xi\bigg)\, dt.$$

By Lemma 1.5, we have

$$\lim_{i\to\infty} \int_{t_{00}}^{t} \delta f_i[\xi]\, d\xi = 0$$

uniformly in $t \in [t_{00}, t_{10}]$.

Therefore, the relation (4.68) holds. The continuity of the mapping (4.67) allows us to strengthen inequality given above:

$$\int_{t_{00}}^{t_{10}} \psi(t)\delta f[t]dt \le 0, \quad \delta f \in \operatorname{cone}([W^{(1)}(K_1;\alpha)]_{\widehat{W}_{f_0}} - f_0).$$

According to Lemma 4.14,

$$\operatorname{cone}([W^{(1)}(K_1;\alpha)]_{\widehat{W}_{f_0}} - f_0) \supset F_1 - f_0. \tag{4.69}$$

From (4.69) it follows the integral maximum principle

$$\int_{t_{00}}^{t_{10}} \psi(t) f_0[t]\, dt = \max_{u(t)\in\Omega(I,U)} \int_{t_{00}}^{t_{10}} \psi(t) f\Big(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0}), u(t), u\big(t-\theta_1, \dots, u(t-\theta_\nu)\big)\Big)\, dt.$$

Theorem 4.6 is proved.

To conclude this subsection, it should be noted that Theorems 4.7–4.10 are proved in a similar way by using the corresponding variation formulas of a solution.

4.6 The optimal control problem with the continuous initial condition

Consider the optimal control problem

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_\nu)\big), \tag{4.70}$$

$$t \in [t_0, t_1] \subset I, \quad u \in \Omega(I_2, U),$$

$$x(t) = \varphi(t), \quad t \in [\widehat{\tau}, t_0], \quad \varphi \in \Phi_3, \tag{4.71}$$

$$q^i\big(t_0, t_1, \tau_1, \dots, \tau_s, \varphi(t_0), x(t_1)\big) = 0, \quad i = \overline{1, l}, \tag{4.72}$$

$$q^0\big(t_0, t_1, \tau_1, \dots, \tau_s, \varphi(t_0), x(t_1)\big) \longrightarrow \min, \tag{4.73}$$

where $\Phi_3 = \{\varphi \in C(I_1, \mathbb{R}^n) : \ \varphi(t) \in N\}$, $N \subset O$ is a convex set.

The problem (4.70)–(4.73) is called an optimal control problem with the continuous initial condition.

Definition 4.16. Let $v = (t_0, t_1, \tau_1, \dots, \tau_s, \varphi, u) \in A_1 = (a,b) \times (a,b) \times (\theta_{11}, \theta_{12}) \times \cdots \times (\theta_{s1}, \theta_{s2}) \times \Phi_3 \times \Omega(I,U))$. A function $x(t) = x(t; v) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution of the equation (4.70) with the discontinuous initial condition (4.71), or a solution corresponding to the element v and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (4.71) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (4.70) a.e. on $[t_0.t_1]$.

Definition 4.17. An element $v = (t_0, t_1, \tau_1, \dots, \tau_s, \varphi, u) \in A_1$ is said to be admissible if the corresponding solution $x(t) = x(t; v)$ satisfies the boundary conditions (4.72).

Denote by A_{10} the set of admissible elements.

Definition 4.18. An element $v_0 = (t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, u_0) \in V_{10}$ is said to be optimal if there exist a number $\delta_0 > 0$ and a compact set $K_0 \subset O$ such that for an arbitrary element $v \in A_{10}$ satisfying the condition

$$|t_{00} - t_0| + |t_{10} - t_1| + \sum_{i=1}^{s} |\tau_{i0} - \tau_i| + \|\varphi_0 - \varphi\|_{I_1} + H_1(f_0 - f; K_0) \le \delta_0$$

the inequality

$$q^0\big(t_{00}, t_{10}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t_{00}), x_0(t_{10})\big) \le q^0\big(t_0, t_1, \tau_1, \dots, \tau_s, \varphi(t_0), x(t_1)\big)$$

holds.

Theorem 4.11. *Let v_0 be an optimal element and let the following conditions hold:*

4.14. $\tau_{s0} > \cdots > \tau_{10}$ *and with* $\tau_{i0} \in (\theta_{i0}, \theta_{i+10})$, $i = \overline{1, s-1}$*;*

4.15. $\theta_i = m_i h$, $i = \overline{1, \nu}$, *where* m_i, $i = \overline{1, \nu}$, *are natural numbers,* $h > 0$ *is a real number;*

4.16. *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot{\varphi}_0(t)$ *is bounded;*

4.17. *the function* $f_0(w)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{1+s}$, *is bounded;*

4.18. *there exist the finite limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \quad \lim_{w \to w_0} f_0(w) = f_0^-, \ \ w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$*;*

4.19. *there exists the finite limit*

$$\lim_{w \to w_{s+1}} f_0(w) = f_{s+1}^-, \ \ w \in (t_{00}, t_{10}] \times O^{s+1}, \ \ w_{s+1} = \big(t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_{10}), \dots, x_s(t_{10} - \tau_{s0})\big).$$

Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$ *with* $\pi_0 \le 0$, *and a solution* $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ *of the equation*

$$\begin{aligned} \dot{\psi}(t) = -\psi(t) f_{0x}[t] - \sum_{i=1}^{s} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}], \ \ t \in [t_{00}, t_{10}], \\ \psi(t) = 0, \ \ t > t_{10}, \end{aligned} \tag{4.74}$$

such that the following conditions hold:

4.20. *the conditions for the moments* t_{00} *and* t_{10}*:*

$$\pi Q_{0t_0} \ge \psi(t_{00})[\dot{\varphi}_0^- - f_0^-], \ \ \pi Q_{0t_1} \ge -\psi(t_{10}) f_{s+1}^-;$$

4.21. *the conditions for the delays* τ_{i0}, $i=\overline{1,s}$*:*

$$\pi Q_{0\tau_i}=\int\limits_{t_{00}}^{t_{10}}\psi(t)f_{0x_i}[t]\dot{x}_0(t-\tau_{i0})\,dt,\ \ i=\overline{1,s};$$

4.22. *the maximum principle for the initial function* $\varphi_0(t)$*:*

$$[Q_{0x_0}+\psi(t_{00})]\varphi_0(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi_0(t)\,dt,$$

$$=\max_{\varphi(t)\in\Phi_2}[Q_{0x_0}+\psi(t_{00})]\varphi(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi(t)\,dt;$$

4.23. *the integral maximum principle for the control function* $u_0(t)$*:*

$$\int\limits_{t_{00}}^{t_{10}}\psi(t)f_0[t]\,dt$$

$$=\max_{u(t)\in\Omega(I_2,U)}\int\limits_{t_{00}}^{t_{10}}\psi(t)f\Big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0\big(t-\tau_{i0},u(t),u(t-\theta_1),\dots,u(t-\theta_\nu)\big)\Big)\,dt;$$

4.24. *the condition for the function* $\psi(t)$

$$\psi(t_{10})=\pi Q_{0x_1}.$$

Theorem 4.12. *Let* v_0 *be an optimal element and let the conditions* 4.14–4.17 *hold. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0}f_0(w)=f_0^+,\ \ w\in[t_{00},b)\times O^{1+s},$$
$$\lim_{w\to w_{s+1}}f_0(w)=f_{s+1}^+,\ \ w\in[t_{10},b)\times O^{s+1}.$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$ *with* $\pi_0\le 0$, *and a solution of the equation* (4.74) *such that the conditions* 4.21–4.24 *hold. Moreover,*

$$\pi Q_{0t_0}\le\psi(t_{00})[\dot{\varphi}_0^+-f_0^+],\ \ \pi Q_{0t_1}\le-\psi(t_{10})f_{s+1}^+.$$

Theorem 4.13. *Let* v_0 *be an optimal element and let the conditions of Theorems* 4.11 *and* 4.12 *hold. Moreover,* $\dot{\varphi}_0^- - f_0^- = \dot{\varphi}_0^+ - f_0^+ := \widehat{f}_0$, $f_{s+1}^- = f_{s+1}^+ := \widehat{f}_{s+1}$. *Then there exist a vector* $\pi=(\pi_0,\dots,\pi_l)\neq 0$, *with* $\pi_0\le 0$, *and a solution of the equation* (4.74) *such that the conditions* 4.21–4.24 *hold. Moreover,*

$$\pi Q_{0t_0}=\psi(t_{00})\widehat{f}_0,\ \ \pi Q_{0t_1}=-\psi(t_{10})\widehat{f}_{s+1}.$$

By variation formulas of a solution (see Section 3), Theorems 4.11–4.13 are proved by the analogous scheme.

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