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THE DIRICHLET BOUNDARY VALUE PROBLEM OF THERMO-ELECTRO-MAGNENO ELASTICITY FOR HALF SPACE

Abstract. We prove the uniqueness theorem for the Dirichlet boundary value problem of statics of the thermo-electro-magneto-elasticity theory in the case of a half-space. The corresponding unique solution is represented explicitly by means of the inverse Fourier transform under some natural restrictions imposed on the boundary vector function.

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რეზიუმე. ნახევარსივრცის შემთვევაში დამტკიცებულია თერმო-ელექტრო-მაგნეტო დრეკადობის თეორიის დირიხლეს სასაზღვრო ამოცანისათვის ერთადერთობის თეორემა. გარკვეულ ბუნებრივ შეზღუდვებში, რომლებსაც ვადებთ სასაზღვრო ვექტორ-ფუნქციას, შესაბამისი დირიხლეს სასაზღვრო ამოცანის ერთადერთი ამონახსნი წარმოდგენილია ცხადი სახით შებრუნებული ფურიეს გარდაქმნის მეშვეობით.

1. Introduction

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. In [14] it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [1]–[6], [8]–[10], [13], [15].

The mathematical model of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

In the paper we prove the uniqueness theorem of solutions for Dirichlet boundary value problems of statics for half-space.

We show that under some natural restriction on the boundary vector functions the corresponding unique solution is represented by the inverse Fourier transform.

2. Basic Equations and Formulation of Boundary Value Problems

2.1. **Field equations.** Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, $E = (E_1, E_2, E_3)^\top = -\operatorname{grad}\varphi$ and $H = (H_1, H_2, H_3) = -\operatorname{grad}\psi$ are electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials, ϑ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and S is the entropy density. We employ the notation $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial_j$, $\partial_t = \partial/\partial_t$; the superscript $(\cdot)^\top$ denotes transposition operation; the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators [11].

Constitutive relations:

$$\begin{aligned}
\sigma_{rj} = \sigma_{jr} &= c_{rjkl}\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \quad r, j = 1,2,3,\\
D_j &= e_{jkl}\varepsilon_{kl} + \varkappa_{jl}E_l + a_{jl}H_l + p_j\vartheta, \quad j = 1,2,3,\\
B_j &= q_{jkl}\varepsilon_{kl} + a_{jl}E_l + \mu_{jl}H_l + m_j\vartheta, \quad j = 1,2,3,\\
S &= \lambda_{kl}\varepsilon_{kl} + p_kE_k + m_kH_k + \gamma\vartheta.
\end{aligned}$$

Fourier Law: $q_j = -\eta_{jl}\partial_l\vartheta$, $j = 1,2,3$.

Equations of motion: $\partial_j\sigma_{rj} + X_r = \varrho\partial_t^2 u_r$, $r = 1,2,3$.

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free): $\partial_j D_j = \varrho_e$, $\partial_j B_j = 0$.

Linearised equation of the entropy balance: $T_0\partial_t S - Q = -\partial_j q_j$,

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjki} are the elastic constants, e_{jki} are the piezoelectric constants, q_{jki} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electro-magnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho c T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity. The constants involved in these equations satisfy the symmetry conditions

$$\begin{gathered}
c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \quad \varkappa_{kj} = \varkappa_{jk},\\
\lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk}, \quad r,j,k,l = 1,2,3.
\end{gathered} \tag{2.1}$$

From physical considerations it follows (see, e.g., [7], [12])

$$c_{rjkl}\xi_{rj}\xi_{kl} \geq c_0\xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq c_1|\xi|^2, \quad \mu_{kj}\xi_k\xi_j \geq c_2|\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq c_3|\xi|^2, \tag{2.2}$$

for all $\xi_{kj} = \xi_{jk} \in \mathbb{R}$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c_0, c_1, c_2 and c_3 are positive constants. More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure positive definiteness of the matrix

$$\Xi = \begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} & [p_j]_{3\times 1} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} & [m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & \gamma \end{bmatrix}_{7\times 7}. \tag{2.3}$$

Further we introduce the following generalised stress operator

$$\mathcal{T}(\partial, n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 3} & [q_{lrj}n_j\partial_l]_{3\times 1} & [-\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_jn_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}.$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -q_jn_j)^\top. \tag{2.4}$$

The components of the vector $\mathcal{T}U$ given by (2.4) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

From the above equations we derive the following equations of statics

$$A(\partial)U(x) = \Phi(x),$$

where $U = (u_1, \dots, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for vector function and $\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top$ is a given vector function; $A(\partial) = [A_{pq}(\partial)]_{6\times 6}$ is the matrix differential operator

$$A(\partial) = \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 3} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}.$$

From the symmetry conditions (2.1), inequalities (2.2) and positive definiteness of the matrix (2.3) it follows that $A(\partial)$ is a formally non-self adjoint strongly elliptic operator.

2.2. **Formulation of boundary value problems.** Let $\mathbb{R}^3$ be divided by some plane into two half-spaces. Without loss of generality we can assume that these half-spaces are

$$\mathbb{R}^3_1 := \{x \mid x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \text{ and } \ x_3 > 0\} \quad \text{and}$$
$$\mathbb{R}^3_2 := \{x \mid x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \text{ and } \ x_3 < 0\};$$

$n = (n_1, n_2, n_3) = (0, 0, -1)$ is the outward unit normal vector with respect to $\mathbb{R}^3_1$; $S := \partial\mathbb{R}^3_{1,2}$.

Now we formulate the basic boundary value problems of the thermo-electro-magneto-elasticity theory for a half-space.

Dirichlet problem $(D)^\pm$**.** Find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\mathbb{R}^3_{1,2}})]^6 \cap [C^2(\mathbb{R}^3_{1,2})]^6$ to the system of equations

$$A(\partial)U = 0 \quad \text{in} \quad \mathbb{R}^3_{1,2} \tag{2.5}$$

satisfying the Dirichlet type boundary condition

$$\{U\}^\pm = f \quad \text{on} \quad S. \tag{2.6}$$

The symbols $\{\cdot\}^\pm$ denote the one-sided limits on S from $\mathbb{R}^3_1$ (sign "+") and $\mathbb{R}^3_2$ (sign "−").

We require that the boundary data involved in the above setting possess the following smoothness property: $f \in \overset{\circ}{C}{}^\infty(\mathbb{R}^2)$, where $\overset{\circ}{C}{}^\infty(\mathbb{R}^2)$ is the space of infinitely differentiable functions with compact support.

Let $\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}$ and $\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}$ denote the direct and inverse generalized Fourier transforms in the space of tempered distributions (the Schwartz space $\mathcal{S}'(\mathbb{R}^2)$) which for regular summable functions g and h read as follows

$$\begin{aligned}\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g] &= \int\limits_{\mathbb{R}^2} g(\widetilde{x})e^{i\widetilde{x}\cdot\widetilde{\xi}}d\widetilde{x},\\ \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[h] &= \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2} h(\widetilde{\xi})e^{-i\widetilde{x}\cdot\widetilde{\xi}}d\widetilde{\xi},\end{aligned} \tag{2.7}$$

where $\widetilde{x} = (x_1, x_2)$, $\widetilde{\xi} = (\xi_1, \xi_2)$, $d\widetilde{x} = dx_1\,dx_2$, $\widetilde{x}\cdot\widetilde{\xi} = x_1\xi_1 + x_2\xi_2$.

Note that if $g(x) = g(x_1, x_2, x_3) = g(\widetilde{x}, x_3)$, then

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\partial_{x_j} g(x)] = -i\xi_j\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g] = -i\xi_j\widehat{g}(\widetilde{\xi}, x_3), \quad j = 1, 2,$$

and hence

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\nabla_x g(x)] = \begin{bmatrix}-i\xi_1\\ -i\xi_2\\ \partial_{x_3}\end{bmatrix}\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g(x)] = P(-i\widetilde{\xi}, \partial x_3)\widehat{g}(\widetilde{\xi}, x_3), \tag{2.8}$$

here $\widehat{g}(\widetilde{\xi}, x_3) = \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g]$ and

$$P = P(-i\widetilde{\xi}, \partial_{x_3}) = (-i\xi_1, -i\xi_2, \partial_{x_3})^\top. \tag{2.9}$$

Applying Fourier transform (2.7) in (2.5)–(2.6) and taking into account (2.9) we arrive at the problem:

$$A(P)\widehat{U}(\widetilde{\xi}, x_3) = 0, \quad x_3 \in (0; +\infty) \text{ or } x_3 \in (-\infty; 0), \tag{2.10}$$

$$\Big\{\widehat{U}(\widetilde{\xi}, x_3)\Big\}^{\pm}_{(x_3\to 0\pm)} = \widehat{f}(\widetilde{\xi}). \tag{2.11}$$

We see that (2.10) is the system of ordinary differential equations of second order for each $\widetilde{\xi} \in \mathbb{R}^2$.

3. Uniqueness Theorems

We start with constructing a system of linear independent solutions to the system (2.10).

Let us denote by $k_j = k_j(\widetilde{\xi})$, $j = \overline{1, 12}$, the roots of the equation

$$\det A(-i\xi) = 0 \tag{3.1}$$

with respect to ξ_3, where $A(-i\xi)$ is the symbol matrix of the operator $A(\partial)$.

Note that $\det A(-i\xi)$ is a homogeneous polynomial of order 12 and the equation (3.1) has no real roots, $\operatorname{Im} k_j \neq 0$, $j = \overline{1, 12}$. These roots are continuously dependent on the coefficients of (3.1) and the number of roots with positive and negative imaginary parts are equal. Denote by $k_1, k_2, \dots, k_6$ roots with positive imaginary parts and by $k_7, \dots, k_{12}$ with negative ones.

Let us construct the following matrices:

$$\Phi^{(+)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^+} A^{-1}(-i\xi)e^{-i\xi_3 x_3}\,d\xi_3, \tag{3.2}$$

$$\Phi^{(-)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^-} A^{-1}(-i\xi)e^{-i\xi_3 x_3}\,d\xi_3, \tag{3.3}$$

where ℓ^+ (respectively, ℓ^-) is a closed simple curve of positive counterclockwise orientation (respectively, negative clockwise orientation) in the upper (respectively, lower) complex half-plane $\operatorname{Re}\xi_3 > 0$ (respectively, $\operatorname{Re}\xi_3 < 0$) enclosing all the roots with respect to ξ_3 of the equation $\det A(-i\xi) = 0$ with positive (respectively, negative) imaginary parts (see Fig. 1). Clearly, (3.2) and (3.3) do not depend on the shape of ℓ^+ (respectively, ℓ^-).

With the help of the Cauchy integral theorem for analytic functions, we conclude that the entries of the matrix $\Phi^{(+)}(\widetilde{\xi}, x_3) = [\Phi^{(+)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to +\infty$ and are decreasing exponentially as $x_3 \to -\infty$ $(-i\xi_3 x_3 = -i(\xi_3' + i\xi_3'')x_3 = -i\xi_3' x_3 + \xi_3'' x_3)$.

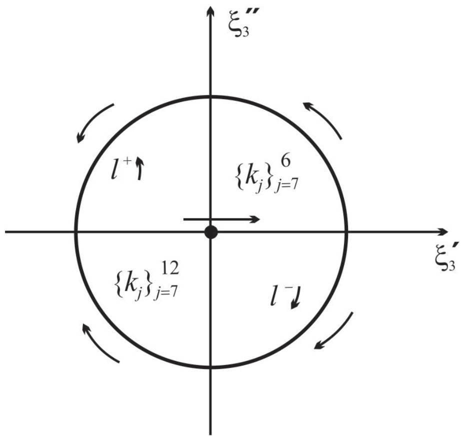

Figure 1

Analogously, the entries of the matrix $\Phi^{(-)}(\widetilde{\xi}, x_3) = [\Phi^{(-)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to -\infty$ and vanish exponentially as $x_3 \to +\infty$.

Lemma 3.1. *The columns of* $\Phi^{(\pm)}(\widetilde{\xi}, x_3)$ *are linearly independent solutions to system* (2.10).

Proof. Applying the Cauchy integral theorem we have

$$A(P)\Phi^{(\pm)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^\pm} A(-i\xi)A^{-1}(-i\xi)e^{-i\xi_3 x_3}d\xi_3$$
$$= \int\limits_{\ell^\pm} I_6 e^{-\xi_3 x_3} d\xi_3 = 0,$$

where I_6 is the 6×6 unit matrix. Now we prove that the columns of the matric $\Phi^{(+)}(\widetilde{\xi}, x_3)$

$$\overset{(1)}{\Phi}{}^{(+)}, \overset{(2)}{\Phi}{}^{(+)}, \dots, \overset{(6)}{\Phi}{}^{(+)}$$

are linearly independent vector functions.

Assume that there exists a complex vector $(C_1, C_2, \dots, C_6) =: C \in \mathbb{C}^6$ ($C = C(\widetilde{\xi})$) such that $\sum\limits_{j=1}^{6} C_j \overset{(j)}{\Phi}{}^{(+)}(\widetilde{\xi}, x_3) = 0$ or

$$\Phi^{(+)}(\widetilde{\xi}, x_3)C = 0. \tag{3.4}$$

If $x_3 = 0$, then from (3.2) and (3.4) we get

$$\Phi^{(+)}(\widetilde{\xi}, 0)C = \int\limits_{\ell^+} A^{-1}(-i\xi)\, d\xi_3\; C = 0. \tag{3.5}$$

Taking into account that (see (3.27) in [11])

$$\int\limits_{\ell^+} A^{-1}(-i\xi)\, d\xi_3 = \int\limits_{-\infty}^{\infty} A^{-1}(-i\xi)\, d\xi_3,$$

one can rewrite (3.5) as follows

$$\int\limits_{-\infty}^{\infty} A^{-1}(-i\xi)C\, d\xi_3 = \int\limits_{-\infty}^{\infty} A^{-1}_{kj}(-i\xi)C_j\, d\xi_3 = 0, \quad k = \overline{1,6},$$

or

$$\int\limits_{-\infty}^{+\infty}\begin{bmatrix} [c_{rjkl}\xi_j\xi_l]_{3\times 3} & [e_{lrj}\xi_j\xi_l]_{3\times 1} & [q_{lrj}\xi_j\xi_l]_{3\times 1} & [-\lambda_{rj}\xi_j]_{3\times 1} \\ [-e_{jkl}\xi_j\xi_l]_{1\times 3} & \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & -ip_j\xi_j \\ [-q_{jkl}\xi_j\xi_l]_{1\times 3} & a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & -im_j\xi_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\xi_j\xi_l \end{bmatrix}_{6\times 6}^{-1} C\,d\xi_3 = 0. \tag{3.6}$$

The integrand in (3.6) is $\Psi := -A^{-1}(-i\xi)C$ and hence $C = -A(-i\xi)\Psi$. Using these notation we can write

$$\int\limits_{-\infty}^{+\infty}\Psi_k\,d\xi_3 = 0, \quad \int\limits_{-\infty}^{+\infty}\overline{\Psi}_k\,d\xi_3 = 0, \quad k=\overline{1,6}, \quad \text{and}$$

$$\begin{gathered}\int\limits_{-\infty}^{+\infty}\sum_{r=1}^{3} C_r\overline{\Psi}_r\,d\xi_3 = 0, \quad \int\limits_{-\infty}^{+\infty}\overline{C}_4\Psi_4\,d\xi_3 = 0, \\ \int\limits_{-\infty}^{+\infty}\overline{C}_5\Psi_5\,d\xi_3 = 0, \quad \int\limits_{-\infty}^{+\infty}\overline{C}_6\Psi_6\,d\xi_3 = 0.\end{gathered} \tag{3.7}$$

Using (2.2) and the last equality of (3.7) we conclude that $\Psi_6 = 0$.

Taking the sum of the first five equalities of (3.7) we obtain

$$\begin{aligned}&\int\limits_{-\infty}^{+\infty}\bigg\{\sum_{r=1}^{3}(-A(-i\xi)\Psi)_r\,\overline{\Psi}_r + \Big(-\overline{A(-i\xi)\Psi}\Big)_4\Psi_4 + \Big(-\overline{A(-i\xi)\Psi}\Big)_5\Psi_5\bigg\}d\xi_3 \\ &= \int\limits_{-\infty}^{+\infty}\big\{c_{rjkl}\xi_j\xi_l\Psi_k\overline{\Psi}_r + e_{lrj}\xi_j\xi_l\Psi_4\overline{\Psi}_r + q_{lrj}\xi_j\xi_l\Psi_5\overline{\Psi}_r - e_{jkl}\xi_j\xi_l\overline{\Psi}_r\Psi_4 \\ &\quad + \varkappa_{jl}\xi_j\xi_l\overline{\Psi}_4\Psi_4 + a_{jl}\xi_j\xi_l\overline{\Psi}_5\Psi_4 - q_{jkl}\xi_j\xi_l\overline{\Psi}_r\Psi_5 + a_{jl}\xi_j\xi_l\overline{\Psi}_4\Psi_5 + \mu_{jl}\xi_j\xi_l\overline{\Psi}_5\Psi_5\big\}d\xi_3 = 0,\end{aligned}$$

i.e.

$$\int\limits_{-\infty}^{+\infty}\big\{c_{rjkl}\xi_j\xi_l\Psi_k\overline{\Psi}_r + \varkappa_{jl}\xi_j\xi_l\overline{\Psi}_4\Psi_4 + a_{jl}\xi_j\xi_l(\overline{\Psi}_5\Psi_4 + \Psi_5\overline{\Psi}_4) + \mu_{jl}\xi_j\xi_l\overline{\Psi}_5\Psi_5\big\}d\xi_3 = 0.$$

De to (2.1), (2.2) and positive definiteness of the matrix (2.3) from the last equality we conclude that $\Psi_k = 0$, $k=\overline{1,5}$, and therefore together with $\Psi_6 = 0$ we have $C_k = 0$, $k = \overline{1,6}$.

Hence the columns of the matrix $\Phi^{(+)}(\widetilde{\xi}, x_3)$ are linearly independent vector functions. Similarly, it can be proved that the columns of the matrix $\Phi^{(-)}(\widetilde{\xi}, x_3)$ defined by (3.3) are linearly independent vector functions. □

Theorem 3.2. *The boundary value problems* (2.10)–(2.11) *have only one solution in the space of functions vanishing at infinity.*

Proof. If $x_3 \in (0; +\infty)$, then we look for a solution of the Dirichlet problem in the following form

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(-)}(\widetilde{\xi}, x_3)C, \quad x_3 > 0,$$

where $C = (C_1, \dots, C_6)$ is unknown vector depending only on $\widetilde{\xi}$.

From (2.11) we have

$$\Phi^{(-)}(\widetilde{\xi}, 0)C = \widehat{f}(\widetilde{\xi})$$

and since $\det\Phi^{(-)}(\widetilde{\xi}, 0) \neq 0$, $|\widetilde{\xi}| \neq 0$, due to Lemma 3.1 we obtain

$$C = [\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1}\widehat{f}(\widetilde{\xi}).$$

Therefore the unique solution has the following form

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(-)}(\widetilde{\xi}, x_3)[\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1}\widehat{f}(\widetilde{\xi}), \quad x_3 > 0. \tag{3.8}$$

Similarly, if $x_3 \in (-\infty; 0)$, then the unique solution of the Dirichlet problem has the form

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(+)}(\widetilde{\xi}, x_3)[\Phi^{(+)}(\widetilde{\xi}, 0)]^{-1}\widehat{f}(\widetilde{\xi}), \quad x_3 < 0. \tag{3.9}$$

The theorem is proved. $\square$

Lemma 3.3. *There hold the following relations*

$$[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\begin{bmatrix}[O(|\widetilde{\xi}|)]_{5\times 5} & [O(1)]_{5\times 1}\\ [0]_{1\times 5} & O(|\widetilde{\xi}|)\end{bmatrix}_{6\times 6}. \tag{3.10}$$

Proof. It can be shown (see [11]) that the entries of the matrix $A^{-1}(-i\xi)$ are homogeneous functions in ξ and

$$A^{-1}(-i\xi)=\begin{bmatrix}[O(|\xi|^{-2})]_{5\times 5} & [O(|\xi|^{-3})]_{5\times 1}\\ [0]_{1\times 5} & O(|\xi|^{-2})\end{bmatrix}_{6\times 6}. \tag{3.11}$$

Assume that $\xi_1=t_1|\widetilde{\xi}|$, $\xi_2=t_2|\widetilde{\xi}|$, $\xi_3=t_3|\widetilde{\xi}|$, where $\xi=(\xi_1,\xi_2,\xi_3)=(\widetilde{\xi},\xi_3)$, $t_1^2+t_2^2=1$. If $|\widetilde{\xi}|\neq 0$, from (3.11) we obtain

$$\Phi^{(-)}_{kj}(\widetilde{\xi},0)=\int\limits_{-\infty}^{+\infty}A^{-1}_{kj}(-i\xi)\,d\xi_3=\int\limits_{-\infty}^{+\infty}O(|\xi|^{-m})\,d\xi_3,\quad m=2\ \text{ or }\ m=3.$$

Hence

$$|\Phi^{(-)}_{kj}(\widetilde{\xi},0)|\leq\int\limits_{-\infty}^{+\infty}\frac{\widetilde{c}}{|\xi|^m}\,d\xi_3=\int\limits_{-\infty}^{+\infty}\frac{\widetilde{c}}{\Big(\sqrt{t_1^2|\widetilde{\xi}|^2+t_2^2|\widetilde{\xi}|^2+t_3^2|\widetilde{\xi}|^2}\Big)^m}|\widetilde{\xi}|\,dt_3$$

$$=\frac{\widetilde{c}}{|\widetilde{\xi}|^{m-1}}\int\limits_{-\infty}^{+\infty}\frac{dt_3}{(1+t_3^2)^{m/2}}=\frac{\widetilde{c}_1}{|\widetilde{\xi}|^{m-1}};$$

here $\widetilde{c}>0$ and $\widetilde{c}_1>0$ are some constants.

We derive the following relations

$$\Phi^{(-)}(\widetilde{\xi},0)=\begin{bmatrix}[O(|\widetilde{\xi}|^{-1})]_{5\times 5} & [O(|\widetilde{\xi}|^{-2})]_{5\times 1}\\ [0]_{1\times 5} & O(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times 6}. \tag{3.12}$$

It can easily be checked that $\det\Phi^{(-)}(\widetilde{\xi},0)=O(|\widetilde{\xi}|^{-6})$ and there exist constants $c_1^*>0$ and $c_2^*>0$ such that

$$c_1^*|\widetilde{\xi}|^{-6}\leq|\det\Phi^{(-)}(\widetilde{\xi},0)|\leq c_2^*|\widetilde{\xi}|^{-6}. \tag{3.13}$$

If $\Phi_c^{(-)}(\widetilde{\xi},0)$ is the corresponding matrix of cofactors, then

$$[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\frac{1}{\det\Phi^{(-)}(\widetilde{\xi},0)}\,\Phi_c^{(-)}(\widetilde{\xi},0).$$

Taking into account (3.12) and (3.13) we arrive at the relation

$$[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\frac{1}{\det\Phi^{(-)}(\widetilde{\xi},0)}\begin{bmatrix}[O(|\widetilde{\xi}|^{-5})]_{5\times 5} & [O(|\widetilde{\xi}|^{-6})]_{5\times 1}\\ [0]_{1\times 5} & O(|\widetilde{\xi}|^{-5})\end{bmatrix}_{6\times 6}$$

$$=\begin{bmatrix}[O(|\widetilde{\xi}|)]_{5\times 5} & [O(1)]_{5\times 1}\\ [0]_{1\times 5} & O(|\widetilde{\xi}|)\end{bmatrix}_{6\times 6}.$$ $\square$

Remark 3.4. Note that $\Phi^{(-)}(\widetilde{\xi},x_3)$ has the same behaviour (3.12) as $\Phi^{(-)}(\widetilde{\xi},0)$ for arbitrary x_3 and due to (3.10)

$$\Phi^{(-)}(\widetilde{\xi},x_3)[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\begin{bmatrix}[O(1)]_{5\times 5} & [O(|\widetilde{\xi}|^{-1})]_{5\times 1}\\ [0]_{1\times 5} & O(1)\end{bmatrix}_{6\times 6}. \tag{3.14}$$

Theorem 3.5. *The Dirichlet boundary value problems* (2.5)–(2.6) *have at most one solution* $U=(u,\varphi,\psi,\vartheta)^\top$ *in the space* $[C^1(\overline{\mathbb{R}^3_{1,2}})]^6\cap[C^2(\mathbb{R}^3_{1,2})]^6$ *provided*

$$\partial^\alpha\vartheta(x)=O(|x|^{-1-|\alpha|}), \tag{3.15}$$

$$\partial^\alpha\widetilde{U}(x)=O(|x|^{-1-|\alpha|}\ln|x|)\quad\text{as}\quad|x|\to\infty \tag{3.16}$$

for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$*. Here* $\widetilde{U} = (u, \varphi, \psi)^\top$.

Proof. Let $U^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})$ be two solutions of the problem under consideration with properties indicated in the theorem for $\mathbb{R}^3_1$. It is evident that the difference

$$V = (u', \varphi', \psi', \vartheta') = U^{(1)} - U^{(2)}$$

solves the corresponding homogeneous problem.

Therefore for the temperature function we get the separated homogeneous Dirichlet problem

$$[A(\partial)V]_6 = \eta_{jl}\partial_j\partial_l\vartheta' = 0 \quad \text{in} \quad \mathbb{R}^3_1, \tag{3.17}$$

$$\{\vartheta'\}^+ = 0 \quad \text{on} \quad S. \tag{3.18}$$

By Green's formula (see (2.83) in [11]) for $B^+(0;R) := \{(x_1,x_2,x_3) \mid x_1^2+x_2^2+x_3^2 \le R^2 \text{ and } x_3 > 0\}$ and (3.17)–(3.18) we have

$$\begin{aligned}\int\limits_{B^+(0;R)} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx &= \int\limits_{\partial B^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+ dS\\ &= \int\limits_{\Sigma^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+\,d\Sigma.\end{aligned} \tag{3.19}$$

Here $\Sigma^+(0;R)$ is the upper half sphere.

Taking the limit as $R \to \infty$ in (3.19) according to (3.15) we get

$$\int\limits_{\mathbb{R}^3_1} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx = 0.$$

Due to (2.2) $\vartheta' = const$ and from (3.15) we conclude that $\vartheta' = 0$.

Therefore the five dimensional vector $\widetilde{V} = (u', \varphi', \psi')^\top$ constructed by the first five components of the solution vector V, solves the following homogeneous boundary value problem

$$\begin{aligned}\widetilde{A}(\partial)\widetilde{V} &= 0 \quad \text{in} \quad \mathbb{R}^3_1,\\ \{\widetilde{V}\}^+ &= 0 \quad \text{on} \quad S,\end{aligned} \tag{3.20}$$

where $\widetilde{A}(\partial)$ is the 5×5 differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects (see (2.85) in [11]).

Using the limiting procedure as above in the corresponding Green's identity for vectors satisfying decay conditions (3.16) we obtain

$$\int\limits_{\mathbb{R}^3_1} [\widetilde{A}(\partial)\widetilde{V}\cdot\widetilde{V} + \widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})]\,dx = \lim_{R\to\infty} \int\limits_{\Sigma^+(0;R)} [\widetilde{\mathcal{T}}\widetilde{V}]^+\cdot[\widetilde{V}]^+\,d\Sigma. \tag{3.21}$$

Here $\widetilde{\mathcal{T}}(\partial, n)$ is the corresponding 5×5 generalized stress operator (see (2.86) in [11]) and

$$\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V}) = c_{rjkl}\partial_l u'_k\partial_j u'_r + \varkappa_{jl}\partial_l\varphi'\partial_j\varphi' + a_{jl}(\partial_l\varphi'\partial_j\psi' + \partial_j\psi'\partial_l\varphi') + \mu_{jl}\partial_l\psi'\partial_j\psi'. \tag{3.22}$$

If $\widetilde{V}$ is a solution of (3.20) satisfying (3.16), then from (3.21) we have

$$\int\limits_{\mathbb{R}^3_1} \widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})\,dx = 0. \tag{3.23}$$

From (3.20), (3.22) and (3.23) along with (2.2) we get

$$u'(x) = a\times x + b, \quad \varphi'(x) = b_4, \quad \psi' = b_5,$$

where $a = (a_2, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are arbitrary constant vectors and b_4, b_5 are arbitrary constants. Now, in view of (3.16) we arrive at the equalities $u'(x) = 0$, $\varphi'(x) = 0$, $\psi'(x) = 0$ for all $x \in \mathbb{R}^3_1$, consequently, $U^{(1)} = U^{(2)}$ in $\mathbb{R}^3_1$.

The proof is similar for the domain $\mathbb{R}^3_2$. □

Theorem 3.6. *Let* $f \in \overset{\circ}{C}^{\infty}(\mathbb{R}^2)$ *and*

$$\int\limits_{\mathbb{R}^2} f(\widetilde{x})\, d\widetilde{x} = 0, \quad \int\limits_{\mathbb{R}^2} f(\widetilde{x}) x_j \, d\widetilde{x} = 0, \quad j = 1, 2.$$

Then the Dirichlet boundary value problems (2.5)–(2.6) *possess unique solutions which can be represented in the following form*

$$U(x) = \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}} \Big[\Phi^{(-)}(\widetilde{\xi}, x_3) [\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1} \widehat{f}(\widetilde{\xi}) \Big], \quad x_3 > 0, \tag{3.24}$$

or

$$U(x) = \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}} \Big[\Phi^{(+)}(\widetilde{\xi}, x_3) [\Phi^{(+)}(\widetilde{\xi}, 0)]^{-1} \widehat{f}(\widetilde{\xi}) \Big], \quad x_3 < 0. \tag{3.25}$$

Proof. It suffices to show that the vector functions (3.24) and (3.25) satisfy the conditions (3.15)–(3.16). This will be done if we prove that the following relations hold for all $x \in \mathbb{R}^3$:

$$x_j \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}} [\widehat{U}(\widetilde{\xi}, x_3)] \le O(1), \quad j = 1, 2, 3, \tag{3.26}$$

and

$$x_j^2 \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}} [\widehat{U}(\widetilde{\xi}, x_3)] \le O(1), \quad j = 1, 2, 3, \tag{3.27}$$

where $\widehat{U}(\widetilde{\xi}, x_3)$ is defined by (3.8) or (3.9). For $j = 1$ or $j = 2$, we find

$$\begin{aligned} x_j \int\limits_{\mathbb{R}^2} \widehat{U}(\widetilde{\xi}, x_3) e^{-i\widetilde{\xi}\cdot\widetilde{x}} \, d\widetilde{\xi} &= i \int\limits_{\mathbb{R}^2} \widehat{U}(\widetilde{\xi}, x_3) \, \frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial \xi_j} \, d\widetilde{\xi} = i \lim_{R\to\infty} \int\limits_{K(0;R)} \widehat{U}(\widetilde{\xi}, x_3) \, \frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial \xi_j} \, d\widetilde{\xi} \\ &= -i \lim_{R\to\infty} \bigg(\int\limits_{K(0;R)} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} \, e^{-i\widetilde{\xi}\cdot\widetilde{x}} \, d\widetilde{\xi} - \int\limits_{\partial K(0;R)} \widehat{U}(\widetilde{\xi}, x_3) \, e^{-i\widetilde{\xi}\cdot\widetilde{x}} \, \frac{\xi_j}{R} \, ds \bigg) \\ &= -i \lim_{R\to\infty} \int\limits_{K(0;R)} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} \, e^{-i\widetilde{\xi}\cdot\widetilde{x}} \, d\widetilde{\xi} = -i \int\limits_{\mathbb{R}^2} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} \, e^{-i\widetilde{\xi}\cdot\widetilde{x}} \, d\widetilde{\xi}, \end{aligned} \tag{3.28}$$

where $K(0, R)$ is the circle of radius R centered at the origin.

Under the restriction on f we conclude that $\widehat{f} \in \mathcal{S}(\mathbb{R}^2)$ and $\widehat{f}(\widetilde{\xi}) = O(|\widetilde{\xi}|^2)$ as $|\widetilde{\xi}| \to 0$, where $\mathcal{S}$ is the space of rapidly decreasing functions. Therefore in view of (3.14) we have

$$\begin{aligned} &\frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} = O(1), \quad |\widetilde{\xi}| \to 0 \quad \text{and} \\ &\frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} = O(|\widetilde{\xi}|^{-k}), \quad |\widetilde{\xi}| \to \infty, \quad k \ge 2, \end{aligned} \tag{3.29}$$

uniformly for all $x \in \mathbb{R}^3$. Then the relations (3.28) and (3.29) imply (3.26). The condition (3.27) can be proved similarly if we note that

$$\begin{aligned} &\frac{\partial^2 \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j^2} = O\bigg(\frac{1}{|\widetilde{\xi}|}\bigg), \quad |\widetilde{\xi}| \to 0 \quad \text{and} \\ &\frac{\partial^2 \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j^2} = O(|\widetilde{\xi}|^{-k-1}), \quad |\widetilde{\xi}| \to \infty, \quad k \ge 2, \end{aligned}$$

uniformly for all $x \in \mathbb{R}^3$.

Note that

$$x_3 \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}} [\widehat{U}(\widetilde{\xi}, x_3)] = x_3 \int\limits_{\mathbb{R}^2} \bigg(\int\limits_{\ell^-} A^{-1}(-\xi) e^{-i\xi_3 x_3} d\xi_3 \bigg) [\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1} \widehat{f}(\widetilde{\xi}) e^{-i\widetilde{\xi}\cdot\widetilde{x}} d\widetilde{\xi}. \tag{3.30}$$

Using the Cauchy integral theorem for analytic functions and the relations (3.10), (3.11), from (3.30) we get

$$\begin{aligned} & x_3 \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] \\ & = x_3 \int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3} \begin{bmatrix} [O(|\widetilde{\xi}|^{-1})]_{5\times 5} & [O(|\widetilde{\xi}|^{-2})]_{5\times 1} \\ [0]_{1\times 5} & O(|\widetilde{\xi}|^{-1}) \end{bmatrix} \begin{bmatrix} [O(|\widetilde{\xi}|)]_{5\times 5} & [O(1)]_{5\times 1} \\ [0]_{1\times 5} & O(|\widetilde{\xi}|) \end{bmatrix} \widehat{f}(\widetilde{\xi})\, d\widetilde{\xi} \\ & = x_3 \int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3} [O(1)]_{6\times 6} \widehat{f}(\widetilde{\xi})\, d\widetilde{\xi} = I_1 + I_2, \end{aligned} \tag{3.31}$$

where

$$I_1 = x_3 \int\limits_{|\xi| \le M} e^{-|\widetilde{\xi}|x_3} [O(1)]_{6\times 6} \widehat{f}(\widetilde{\xi})\, d\widetilde{\xi} \quad \text{and} \quad I_2 = x_3 \int\limits_{|\xi| > M} e^{-|\widetilde{\xi}|x_3} [O(1)]_{6\times 6} \widehat{f}(\widetilde{\xi})\, d\widetilde{\xi}.$$

Since $\widehat{f}(\widetilde{\xi}) \in S(\mathbb{R}^2)$, it is easy to check that $I_1 = O(1)$ and $I_2 = O(1)$ and hence (3.26) holds.

We can prove the boundedness of the vector function $x_3^2 \mathcal{F}^{-1}_{\widetilde{\xi} \to \widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)]$ quite similarly taking into account that $\widehat{f}(\widetilde{\xi}) = O(|\widetilde{\xi}|^2)$ as $|\widetilde{\xi}| \to 0$. $\square$

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