Memoirs on Differential Equations and Mathematical Physics
Volume 66, 2015, 103–117

Nahum Krupnik

INFLUENCE OF SOME B. V. KHVEDELIDZE'S RESULTS ON THE DEVELOPMENT OF FREDHOLM THEORY FOR SIOs WITH PC COEFFICIENTS IN $L_p^n(\Gamma, \rho)$

Dedicated to the 100th Anniversary of Boris Vladimirovich Khvedelidze

Abstract. A concise survey on the construction of the spectra, symbols and index-formulas for singular integral operators with piecewise continuous coefficients in the spaces $L_p^n(\Gamma, \rho)$ is given. Influence of some results by B. V. Khvedelidze on this research is shown. Several interesting associated results, obtained during this research, and their applications are discussed in appendix. An open question is stated. Some historical information, related to this paper is presented in the introduction.

2010 Mathematics Subject Classification. Primary 45E10, 45E05, 47B48; Secondary 47B35.

Key words and phrases. Banach algebras, Fredholm theory, spectrum, symbol, index.

რეზიუმე. ჩვენ წარმოგიდგენთ მოკლე მიმოხილვას სპექტრის, სიმბოლოს და ინდექსის აგების შესახებ უბან-უბან უწყვეტ კოეფიციენტებიანი სინგულარული ინტეგრალური ოპერატორებისათვის $L_p(\Gamma, \rho)$ სივრცეებში. ნაჩვენებია ბ. ხვედელიძის ზოგიერთი შედეგის გავლენა დასახელებულ კვლევებზე. დანართში განხილულია ზოგიერთი საინტერესო ასოცირებული შედეგი და მათი გამოყენება, რომლებიც მიღებულია ასეთი კვლევების დროს. დასმულია ზოგიერთი ამოცანა, რომელიც საჭიროებს გადაწყვეტას. შესავალში ჩართულია ისტორიული ინფორმაცია, რომელიც ეხება მოცემულ სტატიას.

1. Introduction

About eighty years ago S. G. Mikhlin [24] in solving the regularization problem for two-dimensional singular integral operators (SIOs) assigned to each such an operator A a function $\sigma(A)(x)$, which he called a *symbol*, and he showed that the regularization is possible if $\inf\limits_{x} |\sigma(A)(x)| > 0$. Thereafter (as widely known) the notion of the symbol was extended to multi-dimensional and one-dimensional SIOs by many authors. In particular, for one-dimensional singular operator $A = aI + bS + T$, where $a(t)$, $b(t)$ are continuous functions on a simple closed contour Γ, T is a compact operator and

$$Sf(t) := \frac{1}{\pi i} \int\limits_{\Gamma} \frac{f(\tau)}{\tau - t}\, d\tau \quad (t \in \Gamma), \tag{1.1}$$

the symbol in the space $L_p(\Gamma, \rho)$ $(1 < p < \infty)$ was defined by the equality

$$\sigma(aI + bS + T)(t, z) = a(t) + z\, b(t) \quad \big((t, z) \in \Gamma \times \{\pm 1\}\big). \tag{1.2}$$

For a long period of time, symbols of SIOs were used for the following (sufficient) conditions:

$$\text{If } \inf_{x} |\sigma(A)(x)| > 0, \text{ then } A \text{ is a Fredholm operator.}$$

An important role in raising the status of the symbols (for many classes of operators) was played by Gelfand's theory of maximal ideals in Banach algebras. Using this theory, I. Gohberg obtained the following important results.

Theorem 1.1 ([3]). *Let $A := aI+bS+T$ and $\sigma(A)(t, z)$ denote, respectively, the singular integral operator and its symbol, defined in* (1.2). *Then*

$$A \in F(L_2(\Gamma)) \Longleftrightarrow \sigma(A)(t, z) \neq 0, \quad \forall\, (t, z) \in \Gamma \times \{\pm 1\}, \tag{1.3}$$

where $F(\mathcal{B})$ denote the set of all Fredholm operators on Banach space $\mathcal{B}$.

To formulate a next theorem, we need the following notations. Let Ω denote the unit sphere in an n-dimensional space $\mathbb{R}^n$; $Y_n(\theta)$ $(\theta \in \Omega,\ n = 1, 2, \dots)$ the sequence of all n-dimensional spherical functions, numbered in some order; Y_n the simplest singular integral operator (see [24] or [5])

$$(Y_n f)(x) = \frac{1}{\gamma_n} \int\limits_{\mathbb{R}^n} \frac{y_n(\nu)}{|x - y|^n}\, f(y)\, dy$$

with the symbol $Y_n(\theta)$; $\mathcal{T}$ the ideal of all compact operators in the algebra L $(L_p(\mathbb{R}^n))$ $(1 < p < \infty)$; $\mathcal{A}_p$ the Banach subalgebra of L $(L_p(\mathbb{R}^n))$, generated by the operators

$$Af(x) := a_0(x) f(x) + \sum_{n=1}^{r} a_n(x)(Y_n f)(x) + T \quad (T \in \mathcal{T})$$

with continuous coefficients $a_n(x)$ and with the symbols

$$A(x,\theta) = a_0(x) + \sum_{n=1}^{r} a_n(x) Y_n(\theta).$$

Theorem 1.2 ([5]). *The quotient algebra $\widehat{\mathcal{A}}_2 = \mathcal{A}_2/\mathcal{T}$ is a commutative Banach algebra; the symbols $A(x,\theta)$ coincides with the functions of element $\widehat{A} \in \widehat{\mathcal{A}}_2$ on the compact space of maximal ideals of the algebra $\widehat{\mathcal{A}}_2$ and*

$$A \in F(L_2(\mathbb{R}^n)) \Longleftrightarrow A(x,\theta) \neq 0, \ \ \forall\,(x,\theta) \in \mathbb{R}^n \times \Omega. \tag{1.4}$$

Theorems 1.1, 1.2 were extended in [4,6] to systems of the corresponding SIOs.

With the appearance of the (revolutionary) results [3–6] the concept of the symbols of SIOs achieved a higher status: responsibility for the necessary and sufficient conditions of Fredholmness (see (1.3), (1.4)). This inspired many mathematicians, interested in the theory of symbol of SIOs, to generalize these results, obtained by Gohberg, to other Banach spaces[1].

The author of this survey was inspired, too. And in the papers [19,20] the main results from [3–6] were extended to spaces L_p and L_p^n $(1 < p < \infty)$.

Shortly thereafter, I. Gohberg invited me to join him for studying the Fredholm theory of one-dimensional SIOs *with piecewise continuous coefficients* on $L_p^n(\Gamma)$: to obtain the spectrum, symbols and formulas for computation the index. I gladly accepted this invitation.

The Fredholm theory for SIOs with PC coefficients, obtained in [7–10], is briefly described in Sections 2, 3. The influence of some results of B. V. Khvedelidze on this cycle of researches is described in Section 4. In Section 5, we construct a counterexample, related to a scalar symbol in algebra generated by SIOs with PC coefficients. In appendix (Section 6), some associated results and their applications, obtained in [19,20] and [8], are shown. An open question is stated.

It is my pleasure to thank my friend Prof. Roland Duduchava[2] for useful remarks and comments.

2. On the Spectrum and Index of SIOs with PC Coefficients

Recall (for convenience) several notations and definitions.

Let $L_p(\Gamma,\rho)$, $1 < p < \infty$ denote a weighted Banach space with

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \ \ -1 < \beta_k < p - 1,$$

$$\|f\|^p_{L_p(\Gamma,\rho)} = \int_\Gamma |f(t)|^p \rho(t)\,|dt|,$$

---

[1]Many results, obtained in this research, are described in the (encyclopedic) book [25].

[2]Note, that Roland Duduchava had a privilege to be the student of both: B. V. Khvedelidze and I. C. Gohberg!

PC (Γ) is the set of all piecewise continuous functions[3] on Γ; $A = aP + bQ$; $C = cP + Q$, where $a, b, c \in \mathrm{PC}$, $P := (I + S)/2$, $Q = (I - S)/2$, and the operator S is defined by (1.1).

In this section we assume, for simplicity, that Γ is a simple closed oriented Lyapunov contour, $0 \in D^+$ and the function $c(t)\,(\in \mathrm{PC}(\Gamma))$ has only one point t_0 of discontinuity:

$$c(t_0 - 0) = z_1, \quad c(t_0 + 0) = z_2. \tag{2.1}$$

Definition 2.1. We denote by $\nu(z_1, z_2, \delta)$ $(0 < \delta < \pi)$ the circular arc joining the points z_1 to z_2 and having the following properties:

1^0. Let $\delta \in (0, \pi)$. Then from any interior point $z \in \nu(z_1, z_2, \delta)$ one sees the straight line $[z_1, z_2]$ under the angle δ, and running through the arc from z_1 to z_2, this straight line is located in the left-hand side.

2^0. Let $\delta \in (\pi, 2\pi)$, then we define $\nu(z_1, z_2, \delta) := \nu(z_2, z_1, 2\pi - \delta)$.

3^0. Finally, $\nu(z_1, z_2, \pi)$ denotes the straight line $[z_1, z_2]$.

Next, we denote by $W_{p,\rho}(c)$ the plane curve which results from the range of the function $c(t)$ by adding the arc $\nu(c(t_0 - 0), c(t_0 + 0), \frac{2\pi(1+\beta)}{p})$. We orient the curve $W_{p,\rho}(c)$ in the natural manner. Also, we write $W_p(c)$ if $\rho(t) \equiv 1$.

Definition 2.2. The function $c(t)\,(\in \mathrm{PC}(\Gamma))$ is called $\{p, \rho\}$-non-singular, if the curve $W_{p,\rho}(c)$ does not contain the origin.

Definition 2.3. Let the function $c(t)$ be a $\{p, \rho\}$-non-singular. Then the winding number of the curve $W_{p,\rho}(c)$ around the point $z = 0$ is called $\{p, \rho\}$-*index* of the function $c(t)$. This index is abbreviated by $\mathrm{ind} c^{p,\rho}$.

Theorem 2.4. *The operator $C = cP + Q$ is at least one-side invertible on $L_p(\Gamma, \rho)$ if and only if the function $c(t)$ is $\{p, \rho\}$- non-singular. Let the function $c(t)$ be $\{p, \rho\}$-non-singular. Then the operator C is invertible, invertible only from the left or invertible only from the right, depending on whether the number $k := \mathrm{ind}\, c^{p,\rho}$ is equal to zero, positive or negative, respectively. If $k > 0$, then dim* $\mathrm{coker}(C) = k$, *and if $k < 0$, then dim* $\ker(C) = -k$.

Remark 2.5. If the function $c(t)$ has several points t_k of discontinuity, then $W_{p,\rho}(c)$ results from the range of the function $c(t)$ by adding several arcs $\nu(c(t_k - 0), c(t_k + 0), \delta)$.

Theorem 2.6. *The operator $A = aP + bQ$ is Fredholm on $L_p(\Gamma, \rho)$ if and only if $b(t \pm 0) \neq 0$ $(t \in \Gamma)$, and the function $c(t) = a(t)/b(t)$ is $\{p, \rho\}$-non-singular.*

Remark 2.7. A theorem similar to Theorem 2.6, was obtained by H. Widom [28] for the case where Γ is a measurable subset of $\mathbb{R}$.

---

[3]See (for details) the definition of PC(Γ) in [13, p. 62].

Remark 2.8. For the space $L_2(\Gamma)$, the results of Theorems 2.4 and 2.6 were obtained in [7]. For the spaces $L_p(\Gamma)$ and $L_p(\Gamma,\rho)$ in [8, 9].

After the papers [8, 9] were published, we (the authors) were periodically asked (at seminars and conferences) various questions related to these papers. Most often we were asked the following

Question 2.9. *How did you guess (or, how did you come) to adding these special circular arcs, depending on p,ρ and joining the points $c(t_k \pm 0)$?*

In Sections 4, we show a way, paved by B. V. Khvedelidze, on which we came to the idea of these circular arcs.

3. SIOs with Matrix PC-Coefficients in $L_p^n(\Gamma,\rho)$

Let $R := AP + BQ$ denote a singular integral operator with piecewise continuous matrix coefficients $A := [a_{ik}]_{i,k=1}^n$ and $B := [b_{ik}]_{i,k=1}^n$. Sufficient conditions for the operator R to be Fredholm in $L_p^n(\Gamma,\rho)$ was first obtained by B. V. Khvedelidze (see [17, Chapter 2]). Then the Fredholm criterion was obtained in our work [10]. See also [25, Chapter 5, Section 6] for some additional historical details.

Let $C := [c_{ik}]_{i,k=1}^n$ be a piecewise continuous matrix function, and let $t_1,\dots,t_r$ be the points of discontinuity of the matrix C. To each point t_s $(s = 1,\dots,r)$ we attach a matrix-valued arc

$$\nu(t_s,\mu) := \frac{e^{i\mu\theta_s}\sin(1-\mu)\theta_s}{\sin\theta_s}\,G(t_s-0) + \frac{e^{i(\mu-1)\theta_s}\sin\mu\theta_s}{\sin\theta_s}\,G(t_s+0), \quad (3.1)$$

where $\theta_s = \pi - \frac{2\pi(1+\beta_s)}{p}$, and we assume that

$$\rho(t) = \prod_{k=1}^{m} |t-t_k|^{\beta} \quad (m \geq r). \quad (3.2)$$

We associate with the matrix C a continuous matrix curve $C^{p,\rho}(t,\mu)$, obtained by adding r arcs $\nu_(t_s,\mu)$ to the range of the matrix C.

Definition 3.1. The matrix function $C := [c_{ik}]_{i,k=1}^n$ is called $\{p,\rho\}$-nonsingular if $0 \notin \det C^{p,\rho}(t,\mu)$. Let $C(t)$ be $\{p,\rho\}$-nonsingular matrix function, then its $\{p,\rho\}$ index is defined by the equality $\operatorname{ind} C^{p,\rho} := \operatorname{ind}\det C^{p,\rho}(t,\mu)$.

Theorem 3.2. *The operator $R = AP + BQ$ is a Fredholm operator on $L_p^n(\Gamma,\rho)$ if and only if $\det B(t\pm 0) \neq 0$ for all $t \in \Gamma$ and the matrix function $C(t) := B(t)^{-1}A(t)$ is $\{p,\rho\}$-nonsingular. If these conditions are fulfilled, then the index of operator R in the space $L_p^n(\Gamma,\rho)$ is defined by the equality $\operatorname{ind} R = -\operatorname{ind} C^{p,\rho}$.*

4. Influence of Some Results by B. V. Khvedelidze

In this section we assume, for simplicity, that Γ is a simple closed oriented Lyapunov contour, $0 \in D^+$ and $1 \in \Gamma$.

By the time we (Gohberg–Krupnik) started to work on the Fredholm theory of SIOs with piecewise continuous coefficients on $L_p(\Gamma)$ $(1 < p < \infty)$, the following statement was well known:

Proposition 4.1. *The spectrum and Fredholm spectrum for one-dimensional SIOs with continuous coefficients in the spaces $L_p(\Gamma)$ do not depend on $p \in (1, \infty)$.*

Naturally, there arose the following

Question 4.2. *Is Proposition* 4.1 *true in the case of piecewise continuous coefficients?*

In order to get the answer to this question (as well as to some other questions), we referred to Khvedelidze's works [16–18]. First we turned our attention to the following important statements.

Theorem 4.3 ([16]). *Let $1 < p < \infty$ and $\rho = |t - t_0|^\beta$ $(t_0 \in \Gamma)$. If $-1 < \beta < p - 1$, then the singular operator S is bounded in $L_p(\Gamma, \rho)$.*

Corollary 4.4. *The operator $(t - t_0)^\delta S(t - t_0)^{-\delta}$ $(t_0 \in \Gamma)$ is bounded in $L_p(\Gamma)$ if and only if*

$$-\frac{1}{p} < \operatorname{Re} \delta < 1 - \frac{1}{p}\,.$$

Next, using suitable ideas and results from [16–18], we have proved the following

Theorem 4.5. *The operator $A = t^\gamma P + Q$ with $\operatorname{Re} \gamma \in (0, 1)$ is a Fredholm operator in $L_p(\Gamma)$ for all $p \neq 1/\operatorname{Re} \gamma$.*

Proof. Following [17, 18], we considered the following two factorizations of the function $\psi(t) = t^\gamma$ $(\operatorname{Re} \gamma \in (0, 1))$:

$$\psi(t) = (t - 1)^\gamma \Big(\frac{t - 1}{t}\Big)^{-\gamma} = \psi_+(t)\psi_-(t)$$

and

$$\psi(t) = (t - 1)^{\gamma - 1} t \Big(\frac{t - 1}{t}\Big)^{1 - \gamma} = \xi_+(t)\, t\, \xi_-(t). \tag{4.1}$$

We assumed that Γ satisfies the conditions, formulated above (before (2.1)). Without loss of generality, we also assumed that $t_0 = 1$ and $0 \in D^+$.

Let $A = \psi(t)P + Q = \psi_-(\psi_+ P + \psi_-^{-1} Q)$ and $B = (\psi_+^{-1} P + \psi_- Q)\psi_-^{-1}$. It is not difficult to check that $AB = BA = I$. Therefore, the operator A is invertible in some space $L_p(\Gamma)$, if and only if the operator B is bounded in L_p. Using the representation

$$B = \frac{1}{2}\left[(\psi^{-1} + 1)I + (\psi^{-1} - 1)\psi_- S \psi_-^{-1} I\right]$$

and Corollary 4.4, it was obtained in [17] that the operator A is invertible in L_p for all p:

$$\frac{1 - p}{p} < \operatorname{Re} \gamma < \frac{1}{p}\,, \ \text{ i.e., for } \ p < \frac{1}{\operatorname{Re} \gamma}\,.$$

Next, we used factorization (4.1) and represented the operator A in the form $A = A_1 T$, where $A_1 = \xi_+(t)\xi_-(t)P + Q$ and $T = tP + Q$. The operator T is Fredholm in L_p for all $p \in (1, \infty)$. Like in the first factorization, one can obtain here that the operator A_1 is invertible in $L_p(\Gamma)$ for all p such that

$$\frac{1-p}{p} < \operatorname{Re}\gamma - 1 < \frac{1}{p}, \quad \text{i.e., for} \quad p > \frac{1}{\operatorname{Re}\gamma}.$$

Thus, for all $p > 1/\operatorname{Re}\gamma$, the operator A is a Fredholm one with $\operatorname{ind}^p A = 1$. $\square$

Example 4.6. Let $\gamma = 1/2$. The operator $A = t^{1/2}P + Q$ is invertible in L_p for all $p < 2$ and it is a Fredholm with $\operatorname{ind}^p = 1$ for all $p > 2$. This follows from Theorem 4.5. For $p = 2$, the operator A is not Fredholm. This does not follow from Theorem 4.5, but it follows from the paper [7] in which the Fredholm theory for SIOs with PC coefficients in $L_2(\Gamma)$ was developed.

Remark 4.7. Example 4.6 shows the spectral behavior of the point $\lambda = 0$ of the operator $A - \lambda I$ and, in particular, gives the negative answer to Question 4.2.

In order to analyze the spectral behavior of other points λ, we consider $\psi(t) = t^{1/2}$, $A = \psi P + Q$, and $\lambda \notin \{\psi(t) : \ t \in \Gamma\}$. We represent operator $A - \lambda I$ in the form

$$A - \lambda I = (1-\lambda)R, \quad \text{where} \ R := \left(\frac{\psi(t)-\lambda}{1-\lambda}P + Q\right) := g(t)P + Q. \tag{4.2}$$

It follows from (4.2) that

$$\frac{g(1-0)}{g(1+0)} = \frac{\lambda+1}{\lambda-1} := z = re^{i\theta} = e^{i\theta + \ln r}. \tag{4.3}$$

Following [17], we consider such a function $h(t) = t^\gamma$, that

$$\frac{h(1-0)}{h(1+0)} = e^{2\pi i\gamma} = e^{i\theta+\ln r} \Longrightarrow \operatorname{Re}\gamma = \frac{\theta}{2\pi}. \tag{4.4}$$

Now we can prove the following

Theorem 4.8. *Let* $\psi(t) = t^{1/2}$, $A = \psi P + Q$,

$$\lambda \notin \big\{\psi(t) : \ t \in \Gamma\big\}, \quad \text{and} \quad \frac{\lambda+1}{\lambda-1} \neq r\exp\frac{2\pi i}{p} \quad (0 \le r < \infty). \tag{4.5}$$

Then the operator $A - \lambda I$ *is a Fredholm operator in* $L_p(\Gamma)$.

Proof. It follows from (4.5) and (4.3) that $\theta \neq 2\pi/p$ and from (4.4) that $\operatorname{Re}\gamma \neq 1/p$. Thus (see Theorem 4.5), operator $H = hP + Q$ is Fredholm in $L_p(\Gamma)$. Equalities (4.3), (4.4) provide that the function $h(t)/g(t)$ is continuous on Γ, and hence operators R (as well as operator $A - \lambda I$) under the condition $\theta \neq 2\pi/p$ is a Fredholm operator in L_p, too. This proves the theorem. $\square$

Remark 4.9. It remains to describe the set ℓ of the points $\lambda \in \mathbb{C} \setminus \{\psi(t) : t \in \Gamma\}$ (candidates for "non-Fredholm points"), for which the second condition in (4.5) is not satisfied. This is not difficult.

Let $z = (\lambda + 1)/(\lambda - 1) = r \exp(2\pi i/p)$ $(0 \leq r < \infty)$. If $r = 0$, then $\lambda = -1$, if $r = \infty$, then $\lambda = 1$. If $r = 1$, then, $\lambda = -i \cot \frac{\pi}{p}$. Thus, ℓ is a circular arc with the chord $[-1, 1]$. The point $-i \cot \frac{\pi}{p}$ is located on the circular arc ℓ, and from this point one sees the segment $[-1, 1]$ under the angle $\delta = \frac{2\pi}{p}$.

Conclusion 4.10. *Let $\psi(t) = t^{1/2}$ and $A = \psi P + Q$. Then the set of the points $\lambda \in \mathbb{C} \setminus \{\psi(t) : t \in \Gamma\}$, which are candidates for "non-Fredholm points" of operator $A - \lambda I$ in $L_p(\Gamma)$, coincides with the circular arc $\nu(-1, 1, \frac{2\pi}{p})$.*

This is the way on which we came to the idea of circular arc, and it gives the answer to Question 2.9.

5. Symbols for Algebras of SIOs with PC Coefficients

Let $\mathcal{E}$ denote a subalgebra of the algebra $\mathcal{A} := L(\mathcal{B})$, where B is a Banach space. We say that algebra $\mathcal{E}$ is with a (scalar) Fredholm symbol if there exists a collection $\{h_y\}_{y \in Y}$, of multiplicative functionals $h_y : \mathcal{E} \to \mathbb{C}$ such that

$$A \in \mathcal{E} \cap F(\mathcal{B}) \Longleftrightarrow h_y(A) \neq 0, \quad \forall\, y \in Y. \tag{5.1}$$

Compare (5.1) with scalar symbols in (1.2) and (1.4), where the sets Y_1, Y_2 are defined, respectively, by the equalities:

$$Y_1 = \Gamma \times \{\pm 1\} \quad \text{and} \quad Y_2 = \mathbb{R}^n \times \Omega.$$

After the results in [7–10] were obtained a natural question arose:

Question 5.1. *Is algebra $\mathcal{E}$, generated by SIOs with piecewise continuous coefficients on $L_p(\Gamma, \rho)$, with a scalar symbol?*

We (I. Gohberg and N. Krupnik) tried to get a positive answer to this question. But (instead), we constructed a counterexample (see below). After some thought, we decided to construct a *matrix symbol* for algebras, generated by (scalar) SIOs with PC coefficients. This idea opened a next cycle of our common research, review of which is beyond the scope of this article.

We conclude this section with a counterexample, mentioned above.[4]

Lemma 5.2. *Let $\mathcal{E}$ denote the algebra generated by SIOs with PC coefficients on $L_p(\Gamma)$, where Γ is a unite circle, and let $G = \lambda I + CP - PC$, where $C := c(t)I$. If algebra $\mathcal{E}$ is with a scalar symbol, then G is a Fredholm operator for each $\lambda \neq 0$.*

[4]To my knowledge, such a counterexample has never been published.

Proof. Let algebra $\mathcal{A}$ be with a scalar symbol. Then

$$h_x(G) = \lambda + h_x(C)h_x(P) - h_x(P)h_x(C) = \lambda \neq 0, \ \ \forall\, \lambda \neq 0.$$

From the definition of scalar symbol it follows that $G \in F(L_p(\Gamma))$ for all $\lambda \neq 0$. □

Lemma 5.3. *Let $p = 2$, $c(t) = t^{1/2}$ and $c(1 \pm 0) = \pm 1$. Then there exists $\lambda \neq 0$ such that the operator G, defined in Lemma 5.2, is not Fredholm.*

Proof. It follows from Shur's representation

$$R := \begin{bmatrix} I & C \\ P & \lambda I + CP \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}$$

that the operator $G \in F(L_2(\Gamma))$ if and only if the operator $R \in F(L_2^2(\Gamma))$. The operator R can be represented in the form

$$R = \begin{bmatrix} 1 & c(t) \\ 1 & \lambda + c(t) \end{bmatrix} P + \begin{bmatrix} 1 & c(t) \\ 0 & \lambda \end{bmatrix} Q = AP + BQ.$$

Since $\det B(t) \neq 0$, the operator R is Fredholm if and only if the matrix $M_\lambda := B^{-1}A$ is 2-nonsingular. In particular (see Theorem 3.2 and equalities (3.1), (3.2)), this means that

$$0 \notin \det \nu_\lambda(1,\mu), \ \ \text{where} \ \ \nu_\lambda(t,\mu) := (1-\mu)M_\lambda(1-0) + \mu M_\lambda(t+0).$$

But for $\mu = 1/2$, we have the equality

$$\nu_\lambda\Big(1, \frac{1}{2}\Big) = \frac{1}{2\lambda}\left(\begin{bmatrix} \lambda - 1 & -1 \\ 1 & \lambda + 1 \end{bmatrix} + \begin{bmatrix} \lambda + 1 & -1 \\ 1 & \lambda - 1 \end{bmatrix} \right) = \frac{1}{\lambda} \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$$

and for $\lambda_0 = i$, we receive $\det \nu_{\lambda_0}(1, 1/2) = 0$. This proves that the operator $G = iI + t^{1/2}P - Pt^{1/2}I$ is not an F-operator in $L_2(\Gamma)$. □

Corollary 5.4. *Combining these two lemmas, we obtain the negative answer to Question* 5.1.

6. Appendix: Several (Side) Results Associated with the Main Results in [19, 20] and [8]

6.1. **Banach spaces versus Hilbert spaces.** Let $L(\mathcal{B})$ $(L(\mathcal{H}))$ denote the algebra of all linear bounded operators in the Banach (Hilbert) space $\mathcal{B}$ $(\mathcal{H})$ and $GL(B)$ be the group of invertible operators. By $\mathcal{T}(\mathcal{B})$ we denote the ideal of all compact operators on $\mathcal{B}$ and by $F(\mathcal{B})$ the set of all Fedholm operators on $\mathcal{B}$.

Analyzing the proofs of Theorems 1.1 and 1.2 for the purpose of transferring them to Banach spaces, an idea appeared to find a replacement of the following well known Proposition 6.1 (so that it would work in Banach spaces):

Proposition 6.1. *For any operator $A \in L(\mathcal{H})$, there exist two operators $A_1, A_2 \in L(\mathcal{H})$ such that $A = A_1 + iA_2$, $\operatorname{spec}(A_i) \subset \mathbb{R}$ $(i = 1, 2)$ and the relation*

$$A = A_1 + iA_2 \in GL(\mathcal{H}) \Longleftrightarrow \overline{A} := A_1 - iA_2 \in GL(\mathcal{H})$$

holds.

Indeed, one can take $A_1 = (A + A^*)/2$, and $A_2 = (A - A^*)/2i$.

In the paper [19], the following version of substitution was proposed.

Theorem 6.2. *Let operators $A_1, A_2 \in L(\mathcal{B})$, $A_1A_2 = A_2A_1$ and $\operatorname{spec}(A_i) \subset \mathbb{R}$ $(i = 1, 2)$. Then*

$$A := A_1 + iA \in GL(\mathcal{B}) \Longleftrightarrow \overline{A} := A_1 - iA_2 \in GL(\mathcal{B}).$$

Corollary 6.3. *Let operators $A_1, A_2 \in L(\mathcal{B})$, $A_1A_2 - A_2A_1 \in \mathcal{T}(\mathcal{B})$ and $\operatorname{spec}(A_i) \subset \mathbb{R}$ $(i = 1, 2)$. Then*

$$A := A_1 + iA \in F(\mathcal{B}) \Longleftrightarrow \overline{A} := A_1 - iA_2 \in F(\mathcal{B}).$$

Remark 6.4. These (side) results were first used in [19] for extending Gohberg's Theorem 1.2 from L_2 to L_p. Thereafter, Theorem 6.2 and Corollary 6.3 were used for different purposes by many authors. For illustration we consider two examples.

In 1962 Kharazov and Khvedelidze proved the following statement [15]:

Theorem 6.5. *Let $A = a(t)I + b(t)S$ be a SIO with continuous coefficients on a closed contour in $L_p(\Gamma)$. If A is a Fredholm operator in both $L_p(\Gamma)$ and $L_q(\Gamma)$, $(p^{-1} + q^{-1} = 1)$, then $a(t)^2 - b(t)^2 \neq 0$ on Γ.*

Let us show (for illustration) how Theorem 6.5 and Corollary 6.3 could be combined for a simple extension of Gohberg's Theorem 1.1 from $L_2(\Gamma)$ to $L_p(\Gamma)$.

Theorem 6.6. *The operator $A = aI + bS$ with continuous coefficients on a closed contour Γ is Fredholm in $L_p(\Gamma)$ if and only if $a(t)^2 - b(t)^2 \neq 0$ on Γ.*

Proof. The sufficiency of this condition was proved earlier by B. V. Khvedelidze [17]. Now, let $A \in F(L_p)$. It follows from Corollary 6.3 that $\overline{A} = \overline{a}I + \overline{b}S \in F(L_p)$, too. Therefore, the operator $\overline{A}^* = aI + bS + T$, $T \in \mathcal{T}(L_p(\Gamma))$ is Fredholm in L_p^*. Thus, the operator A is a Fredholm operator in both $L_p(\Gamma)$ and $L_q(\Gamma)$. Using Theorem 6.5, we obtain $a(t)^2 - b(t)^2 \neq 0$. □

For a second illustration, consider the following theorem which is proved by using Theorem 6.2.

Theorem 6.7. *Let $\mathcal{K}$ be a Banach algebra and let $\mathcal{K}_0$ be commutative subalgebra of $\mathcal{K}$, which possesses a symmetric sufficient family of multiplicative functionals. Then $\mathcal{K}_0$ is inverse closed in $\mathcal{K}$. See* [21, *Theorem* 13.3] *for details.*

We conclude this subsection with an **open**

Question 6.8. *Can we replace in Theorem* 6.2 *the Banach space* $\mathcal{B}$ *with a normed or topological* (*or even with non-topological*) *space?*

6.2. **The circular arc $\nu_p(c)$ and exact values of the norms of operators S, P, Q on $L_p(\Gamma)$.** It is well known (especially now) that the norms of SIOs play an important role in various applications. But, by the time we were working on the paper [8], almost nothing was known about these norms. We decided to illustrate the results (we just received) for this paper with possible estimation of the norms of operators S, P, Q. We started with the following experiment:

It is evident that for any operator R on the Banach space $\mathcal{B}$ the relation $I+R \notin GL(B) \Longrightarrow \|R\| \geq 1$ holds. We considered the operator $A := cP+Q$, where the function $c(t)$ ($|t| = 1$) takes only two values : $r \exp(\pm\pi i/p)$, $r > 0$, $p \geq 2$. In this case, $\nu_p(c)$ is a circular arc which connects these two points, and from the point $0 \in \nu_p(c)$ the segment $[r \exp(-\pi i/p), r \exp(\pi i/p)]$ is seen at the angle $2\pi/p$.

It follows from Theorem 2.4 that the operator A is not invertible. But $A = I + (c-1)P$, $|c(t) - 1| = r^2 + 1 - 2r \cos\frac{\pi}{p}$ does not depend on t, and its minimal value (for a fixed number p) equals $\sin\frac{\pi}{p}$ (when $r = \cos\frac{\pi}{p}$). Taking $r = cos\frac{\pi}{p}$, we obtain

$$1 \leq \left\| \sin\frac{\pi}{p}\, P \right\|, \text{ therefore } \|P\| \geq \left(\sin\frac{\pi}{p}\right)^{-1}.$$

This was the best estimation we could extract from our experiment. Using same approach, we obtained the following estimates:

$$\|Q\|_p \geq |Q|_p \geq \frac{1}{\sin(\pi/p)}\,, \quad \|P\|_p \geq |P|_p \geq \frac{1}{\sin(\pi/p)}\,, \tag{6.1}$$

$$\|S\|_p \geq |S|_p \geq \cot\frac{\pi}{2p^*}\,, \tag{6.2}$$

where $|A| := \inf\limits_T \|A+T\|$, T are compact operators, and $p^* = \max(p, p/(p-1))$.

These estimates acquired greater significance (for us) when we were able to prove the accuracy of some estimates. For example,

$$\|S\|_p = \begin{cases} \cot\dfrac{\pi}{2p} & \text{if } p = 2^n, \\[2mm] \tan\dfrac{\pi}{2p} & \text{if } p = \dfrac{2^n}{2^n - 1}\,, \end{cases} \qquad n = 1, 2, \ldots \tag{6.3}$$

(see [8, Section 3] for details). And we formulated the following

Conjecture 6.9. *Inequalities* (6.1), (6.2) *can be replaced by equalities.*

These results (associated with the main part of results in [8]) and Conjecture 6.9 gave rise to a large number of publications dedicated to the best constants, and such publications continue to appear. Almost all new result related to best constants required new ideas and methods for their proofs.

Some problems turned out to be very complicated. For example, it took more than 30 years of attempts of many authors to confirm Conjecture 6.9 for analytical projections P and Q. This was done by B. Hollenbeck and I. Verbitsky (see [14] and the list of references in this paper). The operator S was more lucky. Conjecture 6.9 was confirmed by S. K. Pichorides [26] in 1972. Some addendum to his paper was obtained in [23]. A survey related to best constant in the theory of one-dimensional SIO is written in the paper [22].

6.3. **One more associated result.** Denote by $\mathcal{E}$ a subalgebra of the Banach algebra $\mathcal{A} = L(\mathcal{B})$, where $\mathcal{B}$ is a Banach space, and by $M_n(\mathcal{E})$ the algebra of all $n \times n$-matrices with the entries from $\mathcal{E}$. Comparing the results in articles [3, 5] and [4, 6] related, respectively, to the symbols of SIOs with scalar and matrix coefficients, the following statement was predicted:

Theorem 6.10. *Let the algebra $\mathcal{E}$ be commutative modulo compact operators, and let $R \in M_n(\mathcal{E})$. Then*

$$R \in F(L(\mathcal{B}^n)) \Longleftrightarrow \det(R) \in F(L(\mathcal{B})). \tag{6.4}$$

Remark 6.11. When one writes the determinant $\det(R)$, the order of the factors is irrelevant, since the possible determinants differ from one another by a compact term.

In [20], a following statement, associated with Theorem 6.10, was obtained:

Theorem 6.12. *Let $\mathcal{K}$ be an associative and, generally speaking, noncommutative ring with identity e. Assume that $a_{mk} \in \mathcal{K}$ $(m, k \le n)$ for some $n \in \mathbb{N}$, and $a_{mk}a_{pq} = a_{pq}a_{mk}$, $\forall\, m, k, p, q = 1, \dots, n$. Then the matrix $A := [a_{mk}]_{m,k=1}^{n}$ is invertible in $M_n(\mathcal{K})$ if and only if the element $\Delta := \det A$ is invertible in $\mathcal{K}$.*

The proof of Theorem 6.10 was represented in [20], as a corollary from the general Theorem 6.12.

These two theorems (6.10 and 6.12) proved to be useful for many classes of equations and they were included in many publications, even in the publications of the current millennium (see, for example, Lemma 1.2.34 and related statements in [27]). Theorem 6.10 was first used in the proof of Theorem 6 from [19].

Consider one more example of application of Theorem 6.10. Let $T_a := [a_{i-k}]_{i,k=1}^{\infty}$ denote the Toeplitz operator, generated by a function $a(t) = \sum\limits_{j=-\infty}^{\infty} a_j t^j \in L_\infty(S^1)$. The following statement is proved in [11, Section 3].

Theorem 6.13. *Algebra $\mathcal{E} \subset L(\ell_2)$, generated by Toeplitz operators $T_a := [a_{i-k}]$, where $a(t)$ are piecewise continuous functions on the unite circle , is with a scalar Fredholm symbol. In particular, the symbol of operator Ta is defined by the equality*

$$a(t, \mu) = \mu a(t + 0) + (1 - \mu)a(t - 0) \quad \big(|t| = 1,\ \ 0 \le \mu \le 1\big). \tag{6.5}$$

The following corollary follows directly from Theorems 6.13 and 6.10:

Corollary 6.14. *Let* $A := [A_{i,k}]_{i,k=1}^n$ $(A_{i,k} \in \mathcal{E})$*. Then*

$$A \in F(L(\ell_2^n)) \Longleftrightarrow \det A \in F(L(l_2)).$$

Remark 6.15. In order to get the analogue of Theorem 6.13 and Corollary 6.14 for ℓ_p spaces with $p \neq 2$, it was necessary to obtain some additional results, related to Toeplitz operators on ℓ_p. In contrast with the space ℓ_2, here the Khvedelidze and Gohberg–Krupnik approaches did not work. But, Rolland Duduchava proposed a new approach and succeeded in solving the necessary problems (see [1, 2]). This made it possible to obtain in [12] the analogues of Theorem 6.13 for ℓ_p $(1 < p < \infty)$ and to use (automatically) Theorem 6.10 in ℓ_p^n.

Acknowledgement

The research was partially supported by Retalon Inc., Toronto, ON, Canada.

References

1. R. Duduchava, Discrete Wiener–Hopf equations that are composed of the Fourier coefficients of piecewise Wiener functions. (Russian) *Dokl. Akad. Nauk SSSR* **207** (1972), 1273–1276; translation in *Sov. Math., Dokl.* **13** (1972), 1903–1907.
2. R. Duduchava, Discrete Wiener-Hopf equations in l_p spaces with weight. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **67** (1972), 17–20.
3. I. C. Gohberg, On an application of the theory of normed rings to singular integral equations. (Russian) *Uspehi Matem. Nauk (N.S.)* **7** (1952), no. 2(48), 149–156.
4. I. C. Gohberg, On systems of singular integral equations. (Russian) *Kišinev. Gos. Univ. Uč. Zap.* **11** (1954), 55–60.
5. I. C. Gohberg, On the theory of multidimensional singular integral equations. (Russian) *Dokl. Akad. Nauk SSSR* **133** (1960), 1279–1282; translated in *Soviet Math. Dokl.* **1** (1960), 960–963.
6. I. Gohberg, Some topics of the theory of multidimensional singular integral equations. (Russian) *Izv. Mold. Akad. Nauk* No. 10 (76), (1960), 35–50.
7. I. C. Gohberg and N. Ja. Krupnik, The spectrum of one-dimensional singular integral operators with piece-wise continuous coefficients. (Russian) *Mat. Issled.* **3** (1968), vyp. 1 (7), 16–30.
8. I. C. Gohberg and N. Ja. Krupnik, The spectrum of singular integral operators in L_p spaces. (Russian) *Studia Math.* **31** (1968), 347–362.
9. I. C. Gohberg and N. Ja. Krupnik, The spectrum of singular integral operators in L_p spaces with weight. (Russian) *Dokl. Akad. Nauk SSSR* **185** (1969), 745–748.
10. I. C. Gohberg and N. Ja. Krupnik, Systems of singular integral equations in L_p spaces with weight. (Russian) *Dokl. Akad. Nauk SSSR* **186** (1969), 998–1001.
11. I. C. Gohberg and N. Ja. Krupnik, The algebra generated by the Toeplitz matrices. (Russian) *Funkcional. Anal. i Priložen.* **3** (1969), no. 2, 46–56.
12. I. C. Gohberg and N. Ja. Krupnik, On a local principle and algebras generated by Toeplitz matrices. (Russian) *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)* **19** (1973), 43–71.
13. I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. Vol. II. General theory and applications. Translated from the 1979 German translation by S. Roch and revised by the authors. *Operator Theory: Advances and Applications,* 54. *Birkhäuser Verlag, Basel,* 1992.

14. B. Hollenbeck and I. E. Verbitsky, Best constants for the Riesz projection. *J. Funct. Anal.* **175** (2000), no. 2, 370–392.
15. D. F. Kharazov and B. V. Khvedelidze, Some remarks on the theory of singular integral equations with Cauchy kernel. (Russian) *Soobşč. Akad. Nauk Gruzin. SSR* **28** (1962), 129–135.
16. B. V. Khvedelidze, On a discontinuous problem of Riemann–Privalov in the theory of analytic functions. (Russian) *Dokl. Akad. Nauk SSSR (N.S.)* **102** (1955), 1081–1084.
17. B. V. Khvedelidze, Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications. (Russian) *Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze* **23** (1956), 3–158.
18. B. V. Khvedelidze, The Riemann-Privalov boundary-value problem with a piecewise continuous coefficient. (Russian) *Gruzin. Politehn. Inst. Trudy* **1962**, no. 1 (81) 11–29.
19. N. Ja. Krupnik, On multidimensional singular integral equations. (Russian) *Uspehi Mat. Nauk* **20** (1965), no. 6 (126), 119–123.
20. N. Ja. Krupnik, On the question of normal solvability and the index of singular integral equations. (Russian) *Kišinev. Gos. Univ. Učen. Zap.* **82** (1965), 3–7.
21. N. Krupnik, Banach algebras with symbol and singular integral operators. (Translated from the Russian) *Operator Theory: Advances and Applications*, 26. *Birkhäuser Verlag, Basel*, 1987.
22. N. Krupnik, Survey on the best constants in the theory of one-dimensional singular integral operators. In: *Topics in operator theory.* Volume 1. *Operators, matrices and analytic functions*, 365–393, *Oper. Theory Adv. Appl.*, 202, *Birkhäuser Verlag, Basel*, 2010.
23. N. Ja. Krupnik and E. P. Polonskiĭ, The norm of a singular integration operator. (Russian) *Funkcional. Anal. i Priložen.* **9** (1975), no. 4, 73–74.
24. S. G. Mikhlin, Composition of singular integrals. *Dokl. Akad. Nauk SSSR* **2** (1936), no. 11, 3–6.
25. S. G. Mikhlin and S. Prössdorf, Singular integral operators. (Translated from the German) Springer-Verlag, Berlin, 1986.
26. S. K. Pichorides, On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, II. *Studia Math.* **44** (1972), 165–179.
27. S. Roch, P. A. Santos, and B. Silbermann, Non-commutative Gelfand theories. A tool-kit for operator theorists and numerical analysts. Universitext. *Springer-Verlag London, Ltd., London,* 2011.
28. H. Widom, Singular integral equations in L_p. *Trans. Amer. Math. Soc.* **97** (1960), 131–160.

(Received 31.08.2015)

Author's address:

208–7460, Bathurst St. Thornhill, L4J 7K9, Ontario, Canada.
E-mail: `krupnik13@rogers.com`