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ON THE IMPROVEMENT OF CONVERGENCE RATE OF DIFFERENCE SCHEME FOR ONE MIXED BOUNDARY VALUE PROBLEM

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. A mixed problem with the third kind condition on one part of boundary and with the Dirichlet condition on the rest part of the boundary formulated for the Poisson equation, is considered in a unit square. To obtain an approximate solution, we suggest the two-stage finite-difference correction method. It is proved that the solution of the corrected scheme converges at the rate $O(h^m)$ in the discrete L_2-norm, when the solution of the initial problem belongs to the Sobolev space $W_2^m(\Omega)$ with exponent $m \in (2,4]$.

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რეზიუმე. ერთეულოვან კვადრატში განხილულია პუასონის განტოლებისათვის დასმული შერეული ამოცანა, მესამე გვარის პირობით საზღვრის ერთ ნაწილზე და დირიხლეს პირობით საზღვრის დარჩენილ ნაწილზე. მიახლოებითი ამოხსნისათვის შემოთავაზებულია სასრულ-სხვაობიანი ორსაფეხურიანი კორექციის მეთოდი. დამტკიცებულია კორექტირებული სქემის ამონახსნის კრებადობა $O(h^m)$ სიჩქარით დისკრეტული L_2 ნორმის მიმართ, თუ გამოსავალი სასაზღვრო ამოცანის ამონახსნი მიეკუთვნება $m \in (2,4]$ მაჩვენებლიან $W_2^m(\Omega)$ სობოლევის სივრცეს.

1. Introduction

For finite-difference schemes, just as for any numerical method, the question of accuracy is significant. One of the approaches for obtaining high accuracy solutions is the method of corrections by differences of higher order, offered empirically by L. Fox [4]. This idea is simple, but its theoretical foundation is connected with significant difficulties. This is evidenced in the works due to Volkov, in which the grounding of the method is given for the Laplace and Poisson equations (see e.g. [10, 11]); besides, the problem data are chosen in such a way that an exact solution belongs to the Holder class of functions $C_{6,\lambda}$.

When investigating difference schemes by the energetic method, it is desirable to take into account two points:

– the use of Taylor's formula for determination of an approximation error increases the requirement for the smoothness of an unknown solution;
– an unimprovable rate of convergence on the class W_2^m can be reached only by appropriate a priori estimates.

To overcome such difficulties in the last 30 years A. A. Samarskii and other authors (see e.g. [7, 5, 9]) worked out the methodology allowing one to obtain the estimates of convergence rate of difference schemes, in which the convergence rate is consistent with the smoothness of the solution sought for. For the elliptic problems such estimates have the form

$$\|U_h - u\|_{W_2^s(\omega)} \leq ch^{m-s}\|u\|_{W_2^m(\Omega)}.$$

In the present work we consider the Poisson's equation under the third kind boundary condition on one part of boundary and with the Dirichlet condition on the rest part of the boundary. As the first approximation, the solution of the difference scheme $\Lambda U = \varphi$ is considered which has the second order of approximation. Using the basic solution U of the first approximation, the correcting addend R for the right-hand side of the difference scheme is constructed. By means of the methodology for obtaining the consistent estimates, it is proved that the solution $\overline{U}$ of the corrected difference scheme $\Lambda\overline{U} = \varphi + R$ converges at rate $O(h^m)$ in the discrete L_2-norm, when the exact solution belongs to the Sobolev space $W_2^m(\Omega)$, $m \in (2, 4]$.

For determination of the convergence of the offered method we essentially use the convergence estimates obtained in the first and second stages with discrete W_2^2 and L_2-norms, respectively.

2. Statement of the Problem

Let $\Omega = \{x = (x_1, x_2) :\ 0 < x_\alpha < 1\}$ be a unit square with boundary Γ. Let $\Gamma_{-1} = \{(0, x_2) :\ 0 < x_2 < 1\}$, $\Gamma_0 = \Gamma \setminus \Gamma_{-1}$. Let D^ν denote the differential operator $D^\nu = \partial^{|\nu|}/(\partial x_1^{\nu_1} \partial x_2^{\nu_2})$, where $\nu = (\nu_1, \nu_2)$ are multiindices with nonnegative integer components, and $|\nu| = \nu_1 + \nu_2$. By $W_2^s(\Omega)$, $s \geq 0$, we denote the Sobolev space with the norm defined by

$$\|u\|^2_{W_2^s(\Omega)} = \sum_{k=1}^{s} |u|^2_{W_2^k(\Omega)}, \quad |u|^2_{W_2^k(\Omega)} = \sum_{|\nu|=k} \|D^\nu u\|^2_{L_2(\Omega)},$$

when s is an integer. If s is a noninteger, let $s = \overline{s} + \varepsilon$, where $\overline{s}$ is the integer part of s, and $0 < \varepsilon < 1$. In this case, the norm is defined by

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\overline{s}}(\Omega)} + |u|^2_{W_2^s(\Omega)},$$

where

$$|u|^2_{W_2^s(\Omega)} = \int_\Omega \int_\Omega \frac{|D^\nu u(x) - D^\nu u(t)|^2}{|x-t|^{2+2\varepsilon}}\, dx\, dt.$$

In particular, for $s = 0$, we have $W_2^0 = L_2$.

In this paper, we investigate certain two-stage finite difference method for the following mixed boundary value problem:

$$\Delta u = -f, \quad x \in \Omega, \tag{2.1}$$

$$u = 0, \; x \in \Gamma_0, \quad \frac{\partial u}{\partial x_1} = \sigma u - g(x_2), \quad x \in \Gamma_{-1}. \tag{2.2}$$

We assume that the solution of the problem (2.1), (2.2) belongs to the space $W_2^m(\Omega)$, $m > 2$.

Let $h = 1/n$; $\hbar = h/2$ if $x_1 = 0$, $\hbar = h$ if $x_1 \neq 0$.

We introduce the mesh domains $\omega_\alpha = \{x_\alpha = i_\alpha : \; i_\alpha = 1, \dots, n-1\}$, $\omega = \omega_1 \times \omega_2$, $\omega_\alpha^- = \omega_\alpha \cup \{0\}$, $\omega_\alpha^+ = \omega_\alpha \cup \{1\}$, $\overline{\omega}_\alpha = \omega_\alpha \cup \{0; 1\}$, $\gamma_{-1} = \{(0, x_2) : \; x_2 \in \omega_2\}$, $\gamma_0 = \gamma \setminus \gamma_{-1}$, $\overline{\omega} = \overline{\omega}_1 \times \overline{\omega}_2$, $\gamma = \Gamma \cap \overline{\omega}$.

We define the difference quotients in x_α direction as follows:

$$v_{x_\alpha} = \frac{(I^{(+\alpha)} - I)v}{h}, \quad v_{\overline{x}_\alpha} = \frac{(I - I^{(-\alpha)})v}{h},$$

where $Iv := v$, $I^{(\pm\alpha)} = v(x \pm h r_\alpha)$ and r_α is the unit vector on the x_α axis.

On the set of mesh functions given on the mesh $\overline{\omega}$ and vanishing on γ_0, we define the inner product

$$(y, v) = \sum_{\omega \cup \gamma_{-1}} \hbar h y(x) v(x).$$

The norm $\|y\| = (y, y)^{1/2}$ turns this set into normalized space which we denote by $\mathcal{H}_h$.

Let

$$(y, v)_{\widetilde{\omega}} = \sum_{\widetilde{\omega}} h^2 y(x) v(x), \quad \|y\|_{\widetilde{\omega}} = (y, y)^{1/2}_{\widetilde{\omega}}, \quad \widetilde{\omega} \subseteq \overline{\omega}.$$

Denote

$$\|y\|^2_{W_2^2(\omega)} = \|y_{\overline{x}_1 x_1}\|^2 + \|y_{\overline{x}_2 x_2}\|^2 + 2\|y_{\overline{x}_1 \overline{x}_2}\|^2_{\omega_1^+ \times \omega_2^+}.$$

3. Finite Difference Method

We need the following averaging operators for functions defined on Ω:

$$T_1 v(x) = \frac{1}{h^2} \int\limits_{x_1-h}^{x_1+h} \big(h - |x_1 - \xi_1|\big) v(\xi_1, x_2)\, d\xi_1, \quad x \in \omega,$$

$$T_1 v(x) = \frac{2}{h^2} \int\limits_{x_1}^{x_1+h} (h + x_1 - \xi_1) v(\xi_1, x_2)\, d\xi_1, \quad x \in \gamma_{-1},$$

$$T_2 v(x) = \frac{1}{h^2} \int\limits_{x_2-h}^{x_2+h} \big(h - |x_2 - \xi_2|\big) v(x_1, \xi_2)\, d\xi_2, \quad x \in \omega \cup \gamma_{-1}.$$

In the Hilbert space $\mathcal{H}_h$ we define the difference operators:

$$\partial_{x_1} y = y_{x_1}, \quad \Lambda_1 y = \begin{cases} y_{\overline{x}_1 x_1}, & x \in \omega \\ \dfrac{2}{h}\,(y_{x_1} - \sigma y), & x \in \gamma_{-1}, \end{cases}$$

$$\Lambda_2 y = \Big(1 + \sigma\, \frac{h}{3}\Big) y_{\overline{x}_2 x_2}, \quad \mathring{\Lambda}_2 y = y_{\overline{x}_2 x_2}.$$

We approximate problem (2.1), (2.2) by the following finite-difference scheme

$$\Lambda U := \Lambda_1 U + \Lambda_2 U = -\varphi, \quad x \in \omega \cup \gamma_{-1}, \tag{3.1}$$

where

$$\varphi := T_1 T_2 f + \delta(x_1) T_2 g - \frac{h^2}{4}\, \delta(x_1) g_{\overline{x}_2 x_2},$$

$$\delta(x_1) = \begin{cases} \dfrac{2}{h}, & x_1 = 0, \\ 0, & x_1 \neq 0. \end{cases}$$

Using obtained solution U on the second stage of the method we correct the right-hand side of the scheme and then we solve on the same mesh the following difference scheme

$$\Lambda \overline{U} = -\overline{\varphi}, \quad x \in \omega \cup \gamma_{-1}, \tag{3.2}$$

where

$$\overline{\varphi} = \varphi + \frac{h^2}{6}\, \big(\Lambda_1 \mathring{\Lambda}_2 U + \delta(x_1) g_{\overline{x}_2 x_2}\big).$$

The following theorem represents the main result of this paper.

Theorem 3.1. *Let the solution of problem* (2.2) *belong to the space* $W_2^m(\Omega)$, $m > 2$. *Then the convergence rate of the corrected difference scheme* (3.2) *in the discrete* L_2*-norm is defined by the estimate*

$$\|\overline{U} - u\|_{L_2(\omega)} \leq c h^m \|u\|_{W_2^m(\Omega)}, \quad 2 < m \leq 4, \tag{3.3}$$

where the positive constant c *does not depend on* u *and* h.

4. Auxiliary Results

Let $Z = U - u$, where U is a solution of the difference scheme (3.1), while u is a solution of the differential problem (2.1), (2.2).

Lemma 4.1. *The error of the difference scheme* (3.1) $Z = U - u$ *represents a solution of the following problem*

$$\Lambda Z = \eta_1 + \eta_2, \quad Z \in \mathcal{H}_h, \tag{4.1}$$

where

$$\eta_1 = \begin{cases} \Lambda_1(T_2 u - u), & x \in \omega, \\ \Lambda_1\Big(T_2 u - u - \dfrac{h^2}{12}\, u_{\overline{x}_2 x_2}\Big), & x \in \gamma_{-1}, \end{cases}$$

$$\eta_2 = \begin{cases} (T_1 u - u)_{\overline{x}_2 x_2}, & x \in \omega, \\ \Big(T_1 u - u - \dfrac{h}{2}\, \dfrac{\partial u}{\partial x_1} + \dfrac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2}, & x \in \gamma_{-1}. \end{cases}$$

Proof. From equation (2.1) we have:

$$(T_2 u)_{\overline{x}_1 x_1} + (T_1 u)_{\overline{x}_2 x_2} = -T_1 T_2 f, \quad x \in \omega, \tag{4.2}$$

or, the same,

$$u_{\overline{x}_1 x_1} + u_{\overline{x}_2 x_2} + \eta_1 + \eta_2 = -T_1 T_2 f, \quad x \in \omega. \tag{4.3}$$

Acting on the equation (2.1) by operator $T_1 T_2$ we obtain

$$\frac{2}{h}\, T_2\Big(u_{x_1} - \frac{\partial u}{\partial x_1}\Big) + (T_1 u)_{\overline{x}_2 x_2} = -T_1 T_2 f, \quad x \in \gamma_{-1}. \tag{4.4}$$

Rewriting the addend of the left-hand side of this equality we get

$$\begin{aligned} \frac{2}{h}\, T_2\Big(u_{x_1} - \frac{\partial u}{\partial x_1}\Big) &= \frac{2}{h}\, T_2\Big(u_{x_1} - \sigma u\Big) + \frac{2}{h}\, T_2 g = \Lambda_1 T_2 u + \frac{2}{h}\, T_2 g \\ &= \Lambda_1 u + \eta_1 + \frac{h}{6}\,(u_{x_1 \overline{x}_2 x_2} - \sigma u_{\overline{x}_2 x_2}) + \frac{2}{h}\, T_2 g, \end{aligned} \tag{4.5}$$

$$\begin{aligned} (T_1 u)_{\overline{x}_2 x_2} &= \Big(1 + \sigma\, \frac{h}{3}\Big) u_{\overline{x}_2 x_2} - \frac{\sigma h}{3}\, u_{\overline{x}_2 x_2} \\ &\quad + \Big(T_1 u - u - \frac{h}{2}\, \frac{\partial u}{\partial x_1} + \frac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2} \\ &\quad + \Big(\frac{h}{2}\, \frac{\partial u}{\partial x_1} - \frac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2} \\ &= \Lambda_2 u + \eta_2 - \frac{\sigma h}{3}\, u_{\overline{x}_2 x_2} + \Big(\frac{h}{2}\, \frac{\partial u}{\partial x_1} - \frac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2}. \end{aligned} \tag{4.6}$$

Summing up equalities (4.5), (4.6) we find

$$\begin{aligned} &\frac{2}{h}\, T_2\Big(u_{x_1} - \frac{\partial u}{\partial x_1}\Big) + (T_1 u)_{\overline{x}_2 x_2} \\ &\qquad = \Lambda_1 u + \Lambda_2 u + \eta_1 + \eta_2 + \frac{2}{h}\, T_2 g + \frac{h}{2}\, \Big(\frac{\partial u}{\partial x_1} - \sigma u\Big)_{\overline{x}_2 x_2} \end{aligned}$$

and according to (4.4) we have

$$\Lambda_1 u + \Lambda_2 u + \eta_1 + \eta_2 + \frac{2}{h}\, T_2 g - \frac{h}{2}\, g_{\overline{x}_2 x_2} = -T_1 T_2 f, \quad x \in \gamma_{-1}. \tag{4.7}$$

The equalities (4.3), (4.7) can be rewritten as follows

$$\Lambda_1 u + \Lambda_2 u + \eta_1 + \eta_2 = -\varphi, \quad x \in \omega \cup \gamma_{-1}. \tag{4.8}$$

Subtraction of (4.8) from (3.1) proves (4.1). □

Let $\overline{Z} = \overline{U} - u$, where U is a solution of the problem (3.2), and u is a solution of the differential problem (2.1), (2.2).

Lemma 4.2. *The error of the solution of difference scheme* (3.2) $\overline{Z} = \overline{U} - u$ *represents a solution of the following problem*

$$\Lambda \overline{Z} = \Lambda_1 \zeta_1 + \Lambda_2 \zeta_2 + \frac{h^2}{6}\, \Lambda_1 \mathring{\Lambda}_2 (u - U), \tag{4.9}$$

where

$$\zeta_1 = T_2 u - u - \frac{h^2}{12}\, u_{\overline{x}_2 x_2} + \frac{h^5}{720}\, \delta(x_1) \Lambda_2 \Big(\frac{\partial u}{\partial x_1}\Big)_{x_1}, \quad x \in \omega \cup \gamma_{-1},$$

$$\zeta_2 = \begin{cases} T_1 u - u - \dfrac{h^2}{12}\, u_{\overline{x}_1 x_1}, & x \in \omega, \\ T_1 u - u - \dfrac{h}{6}\, \dfrac{\partial u}{\partial x_1} - \dfrac{h}{6}\, u_{x_1} - \dfrac{h^3}{180} \Big(\dfrac{\partial u}{\partial x_1}\Big)_{x_1 x_1}, & x \in \gamma_{-1}. \end{cases}$$

Proof. (4.2) can be easily rewritten as follows

$$u_{\overline{x}_1 x_1} + u_{\overline{x}_2 x_2} + \frac{h^2}{6}\, u_{\overline{x}_1 x_1 \overline{x}_2 x_2} + \Lambda_1 \zeta_1 + \Lambda_2 \zeta_2 = -T_1 T_2 f, \quad x \in \omega. \tag{4.10}$$

Summing up (4.7) and identity

$$\mathring{\Lambda}_2 \Big(\frac{2h}{6}\, \frac{\partial u}{\partial x_1} - \frac{2h}{6}\, u_{x_1}\Big) + \frac{h^2}{6}\, \Lambda_1 \mathring{\Lambda}_2 u = -\frac{2h}{6}\, g_{\overline{x}_2 x_2}$$

we obtain

$$\Lambda_1 u + \Lambda_1 \zeta_1 + \Lambda_2 u + \mathring{\Lambda}_2 \zeta_2 + \frac{h^2}{6}\, \Lambda_1 \mathring{\Lambda}_2 u = -T_1 T_2 f - \frac{2}{h}\, T_2 g + \frac{h}{6}\, g_{\overline{x}_2 x_2}, \quad x \in \gamma_{-1}. \tag{4.11}$$

Then (4.10), (4.11) can be rewritten as follows

$$\Lambda_1 u + \Lambda_2 u + \frac{h^2}{6}\, \Lambda_1 u_{\overline{x}_2 x_2} + \Lambda_1 \zeta_1 + \Lambda_2 \zeta_2 = -T_1 T_2 f - \delta(x_1) T_2 g + \frac{h^2}{12}\, \delta(x_1) g_{\overline{x}_2 x_2}, \quad x \in \omega \cup \gamma_{-1}. \tag{4.12}$$

Subtracting (4.12) from (3.2) we conclude that the lemma is valid.

Lemma 4.3. *For solutions of the problems* (4.1) *and* (4.9) *the following a priori estimates*

$$\|Z\|_{W_2^2(\omega)} \le c\big(\|\eta_1\| + \|\eta_2\|\big), \tag{4.13}$$

$$\|\overline{Z}\| \le c\big(\|\zeta_1\| + \|\zeta_2\| + \|Z_{\overline{x}_2 x_2}\|\big) \tag{4.14}$$

are valid.

The proof follows from the facts that $\Lambda_1,\ \Lambda_2$ and, therefore, Λ are self-adjoint and negative definite (see e.g. [8, Ch. IV, § 2]):

$$\|\Lambda Z\| \ge c\|Z\|_{W_2^2(\omega)},$$
$$\|\Lambda^{-1}\Lambda_1\| \le 1, \quad \|\Lambda^{-1}\Lambda_2\| \le 1.$$

To determine the rate of convergence of the two-stage finite difference method with the help of Lemma 4.3, it is sufficient to estimate the terms on the right-hand sides of (4.13), (4.14).

Lemma 4.4. *Assume that the linear functional $l(u)$ is bounded in $W_2^s(E)$, where $s = \overline{s} + \varepsilon$, $\overline{s}$ is an integer, $0 < \varepsilon \le 1$, and $l(P) = 0$ for every polynomial P of degree $\le \overline{s}$ in two variables. Then, there exists a constant c, independent of u, such that $|l(u)| \le c|u|_{W_2^s(E)}$.*

This lemma is a particular case of the Dupont–Scott approximation theorem [3] and represents a generalization of the Bramble–Hilbert lemma [2] (see also [8]).

Proof of Theorem 3.1. Functionals η_α, ζ_α, $\alpha = 1,2$, are bounded when $u \in W_2^m(\Omega)$, $m > 2$, and they vanish on polynomials up to the third order. Using the well-known methodology (see e.g. [8, 1]), which is based on the Lemma 4.4, we have for them the following estimates

$$|\eta_\alpha| \le ch^{m-3}|u|_{W_2^m(e)}, \ \ 2 < m \le 4,$$
$$|\zeta_\alpha| \le ch^{m-1}|u|_{W_2^m(e)}, \ 2 < m \le 4,$$

where symbol e denotes those elementary cells on which functionals $\eta_\alpha,\ \zeta_\alpha$, are defined:

$$e = e(x) = \begin{cases} \big\{(\xi_1,\xi_2):\ |x_\alpha - \xi_\alpha| < h,\ \alpha = 1,2\big\}, & \text{if } x \in \omega, \\ \big\{(\xi_1,\xi_2):\ 0 < \xi_1 < 2h,\ |x_2 - \xi_2| < h\big\}, & \text{if } x \in \gamma_{-1}. \end{cases}$$

As a result we have

$$\begin{aligned} \|\eta_\alpha\|^2 &= \sum_{\omega\cup\gamma_{-1}} \hbar h |\eta_\alpha|^2 \\ &\le c \sum_{\omega\cup\gamma_{-1}} h^{2m-4}|u|^2_{W_2^m(e)} \le ch^{2m-4}|u|^2_{W_2^m(\Omega)}, \ \ 2 < m \le 4, \end{aligned}$$

$$\|\zeta_\alpha\|^2 = \sum_{\omega\cup\gamma_{-1}} \hbar h|\zeta_\alpha|^2$$
$$\leq c \sum_{\omega\cup\gamma_{-1}} h^{2m}|u|^2_{W_2^m(e)} \leq ch^{2m}|u|^2_{W_2^m(\Omega)}, \ \ 2 < m \leq 4.$$

These estimates with the Lemma 4.3 accomplish the proof of the Theorem 3.1. □

5. Numerical Experiments

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete L_2 and maximum norms are computed by formulas

$$Ord(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|}\,, \ \ Ord(Y) = \log_2 \frac{\|Y_h - u\|_\infty}{\|Y_{h/2} - u\|_\infty}\,,$$

where u is the exact solution of original problem, while Y_h denotes the solution of the difference scheme on the grid with step h.

Below, in the examples the symbols U , $\overline{U}$ denote solutions of the difference schemes (3.1), (3.2), respectively.

Let $\Omega = \{x = (x_1, x_2) : \ |x_1| < 1, \ 0 < x_2 < 1\}$ and Γ be its boundary; $\Gamma_{-1} = \{(-1, x_2) : \ 0 < x_2 < 1\}$, $\Gamma_0 = \Gamma \setminus \Gamma_{-1}$.

Consider the problem

$$\Delta u = -f, \ \ x \in \Omega,$$
$$u = 0, \ \ x \in \Gamma_0, \quad \frac{\partial u}{\partial x_1} = 3u - g(x_2), \ \ x \in \Gamma_{-1},$$

where

$$f(x) = \begin{cases} \big(\pi^2(x_1^3 - x_1 + 1) - 6x_1\big)\sin(\pi x_2), & x \in (-1,0)\times(0,1),\\ \pi^2(1 - x_1)\sin(\pi x_2), & x \in [0,1)\times(0,1), \end{cases}$$

$g(x_2) = \sin(\pi x_2)$.

The exact solution is

$$u(x) = \begin{cases} (x_1^3 - x_1 + 1)\sin(\pi x_2), & x \in [-1,0)\times[0,1],\\ (1 - x_1)\sin(\pi x_2), & x \in [0,1]\times[0,1]. \end{cases} \tag{5.1}$$

The right-hand side is calculated by the computer algebra system (CAS) MuPAD.

For $x_1 = 0$:

$$\varphi = T_1T_2 f = \Big(\pi^2 - \frac{\pi^2 h^3}{20} + h\Big)\lambda^2 \sin(\pi x_2).$$

For $x_1 = h, 2h, 3h, \ldots$:

$$\varphi = T_1T_2 f = \pi^2(1 - x_1)\lambda^2 \sin(\pi x_2).$$

For $x_1 = -h, -2h, -3h, \dots, -(n-1)h$:

$$T_1T_2f = [\pi^2(x_1^3 + 1 - x_1) - 6x_1 + \frac{\pi^2 h^2}{2}\, x_1]\lambda^2 \sin(\pi x_2).$$

For $x = -1$:

$$T_1T_2f = \Big(\pi^2 h\Big(\frac{h^2}{10} - \frac{h}{2} + \frac{2}{3}\Big) - 2h + \pi^2 + 6\Big)\lambda^2 \sin(\pi x_2),$$
$$T_2 g = \lambda^2 \sin(\pi x_2), \quad g_{\overline{x}_2 x_2} = -\pi^2\lambda^2 \sin(\pi x_2).$$

The results of calculations are given by Tables 1, 2.

Table 1. Experimental order of convergence with respect to the norm of L_2.

h	$\\|U_h - u\\|$	$\\|\widetilde{U}_h - u\\|$	$Ord(U)$	$Ord(\widetilde{U})$
$\frac{1}{4}$	1.6881 e−02	9.2278 e−04		
			2.0151	4.0074
$\frac{1}{8}$	4.1762 e−03	5.7377 e−05		
			2.0140	4.0245
$\frac{1}{16}$	1.0340 e−03	3.5256 e−06		
			2.0087	4.0178
$\frac{1}{32}$	2.5695 e−04	2.1765 e−07		
			2.0048	4.0103
$\frac{1}{64}$	6.4024 e−05	1.3507 e−08		
			2.0025	4.0055
$\frac{1}{128}$	1.5978 e−05	8.4099 e−10		

Remark. The function defined by formula (5.1) belongs to the class $W_2^{3.5}(\Omega)$. The order of convergence obtained experimentally, and equaled 4, may point at the fact that condition $u \in W_2^m(\Omega)$ in the Theorem 3.1 is sufficient, not necessary.

6. Conclusion

We consider a mixed boundary-value problem for the 2D Poisson's equation in a square which is solved by the finite-difference scheme with approximation of order $O(h^2)$ based on a 5-point stencil. Using the obtained solution, we correct the right-hand side of the scheme and repeatedly solve the scheme on the same mesh with the same stencil. Using the methodology of obtaining the consistent estimates, worked by Samarskiĭ et al., it is

Table 2. Experimental order of convergence with respect to the maximum norm.

h	$\|U_h - u\|_\infty$	$\|\widetilde{U}_h - u\|_\infty$	$Ord(U)$	$Ord(\widetilde{U})$
$\frac{1}{4}$	$2.8838\,e{-}02$	$1.6708\,e{-}03$		
			1.9843	3.8432
$\frac{1}{8}$	$7.2884\,e{-}03$	$1.1641\,e{-}04$		
			1.9960	3.9825
$\frac{1}{16}$	$1.8271\,e{-}03$	$7.3647\,e{-}06$		
			1.9990	3.9902
$\frac{1}{32}$	$4.5710\,e{-}04$	$4.6344\,e{-}07$		
			1.9997	3.9989
$\frac{1}{64}$	$1.1430\,e{-}04$	$2.8988\,e{-}08$		
			1.9997	3.9997
$\frac{1}{128}$	$2.8579\,e{-}05$	$1.8121\,e{-}09$		

proved that the solution of the corrected difference scheme converges at rate $O(h^m)$ in the discrete $L_2(\omega)$-norm, when the exact solution belongs to the Sobolev space $W_2^m(\Omega)$, $m \in (2, 4]$. For determination of the convergence of the offered method we essentially use the convergence estimates obtained in the first and second stages with discrete W_2^2 and L_2 - norms, respectively.

The method can be generalized for an elliptic differential equation with mixed derivatives and a system of equations, and also for the case of other type boundary conditions.

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