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ON ASYMPTOTIC STABILITY OF SOLUTIONS OF SECOND ORDER LINEAR NONAUTONOMOUS DIFFERENTIAL EQUATIONS

Abstract. The sufficient conditions for asymptotic stability of solutions of second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

with continuously differentiable coefficients $p : [0, +\infty) \to \mathbb{R}$ and $q : [0, +\infty) \to \mathbb{R}$ are established in the case where the roots of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

satisfy conditions

$$\operatorname{Re}\lambda_i(t) < 0 \ \text{ for } \ t \geq 0, \qquad \int\limits_{t_0}^{+\infty} \operatorname{Re}\lambda_i(t)\, dt = -\infty \ \ (i = 1, 2).$$

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$$y'' + p(t)y' + q(t)y = 0$$

უწყვეტად წარმოებადი $p : [0, +\infty) \to \mathbb{R}$ და $q : [0, +\infty) \to \mathbb{R}$ კოეფიციენტებით დადგენილია ამონახსნების ასიმპტოტური მდგრადობის საკმარისი პირობები იმ შემთხვევაში, როცა მახასიათებელი

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

განტოლების ფესვები აკმაყოფილებენ პირობებს

$$\operatorname{Re}\lambda_i(t) < 0, \ \ \text{როცა} \ \ t \geq 0, \qquad \int\limits_{t_0}^{+\infty} \operatorname{Re}\lambda_i(t)\, dt = -\infty \ \ (i = 1, 2).$$

1. Introduction

This present paper is a continuation of the article *Sufficiency conditions for asymptotic stability of solutions of a linear homogeneous nonautonomous differential equation of second-order.*

In the theory of stability of linear homogeneous on-line systems (LHS) of ordinary differential equations

$$\frac{dY}{dt} = P(t)Y, \quad t \in [t_0; +\infty) = I,$$

where the matrix $P(t)$ is, in a general case, complex, of great importance is the study of the LHS stability depending on the roots $\lambda_i(t)$ $(i = \overline{1,n})$ of the characteristic equation

$$\det(P(t) - \lambda E) = 0.$$

L. Cesáro [1] has considered the system of differential equations of n-th order

$$\frac{dY}{dt} = \big[A + B(t) + C(t)\big]Y,$$

where A is a constant matrix, whose roots of the characteristic equation λ_i $(i = \overline{1,n})$ are distinct and satisfy the condition $\operatorname{Re}\lambda_i \le 0$ $(i = \overline{1,n})$; $B(t) \to 0$ as $t \to +\infty$,

$$\int\limits_{t_0}^{+\infty} \Big\|\frac{dB(t)}{dt}\Big\|\, dt < +\infty, \quad \int\limits_{t_0}^{+\infty} \|C(t)\|\, dt < +\infty,$$

the roots of the characteristic equation of the matrix $A + B(t)$ have non-positive material parts.

In his work, C. P. Persidsky [2] considers the case in which elements of the matrix $P(t)$ are the functions of weak variation, that is, every function can be represented in the form

$$f(t) = f_1(t) + f_2(t),$$

where $f_1(t) \in C_I$, and there exists $\lim\limits_{t\to+\infty} f_1(t) \in \mathbb{R}$, and $f_2(t)$ is such that

$$\sup_{t\in I} |f_2(t)| < +\infty, \quad \lim_{t\to+\infty} f_2'(t) = 0,$$

and the condition $\operatorname{Re}\lambda_i(t) \le a \in \mathbb{R}_-$ $(i = \overline{1,n})$ is fulfilled.

N. Y. Lyaschenko [3] has considered the case $\operatorname{Re}\lambda_i(t) < a \in \mathbb{R}_-$ $(i = \overline{1,n})$, $t \in I$,

$$\sup_{t\in I} \|A'(t)\| \le \varepsilon.$$

The case $n = 2$ is thoroughly studied by N. I. Izobov.

I. K. Hale [4] investigated asymptotic behavior of LHS by comparing the roots of the characteristic equation with exponential functions

$$\operatorname{Re}\lambda_i(t) \le -gt^\beta, \quad g > 0, \ \ \beta > -1 \ \ (i = \overline{1,n}).$$

Then there are the constants $K > 0$ and $0 < \rho < 1$ such that for solving the system

$$\frac{dy}{dt} = A(t)y$$

the estimate

$$\|y(t)\| \leq Ke^{-\frac{\rho g}{1+\beta}t^{1+\beta}}\|y(0)\|$$

is fulfilled.

In this paper we consider the problem of stability of a real linear homogeneous differential equation (LHDE) of second order

$$y'' + p(t)y' + q(t)y = 0 \quad t \in I \tag{1}$$

provided the roots $\lambda_i(t)$ $(i = 1, 2)$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\operatorname{Re}\lambda_i(t) < 0, \;\; t \in I, \quad \int\limits_{t_0}^{+\infty} \operatorname{Re}\lambda_i(t)\,dt = -\infty \;\; (i = 1, 2) \tag{2}$$

and there exist finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = 1, 2)$. We have not yet encountered with the problems in such a formulation. The case where at least one of the roots satisfies the condition

$$0 < \int\limits_{t_0}^{+\infty} \big|\operatorname{Re}\lambda_i(t)\big|\,dt < +\infty \;\; (i = 1, 2)$$

should be considered separately.

Under the term "almost triangular LHS" we agree to understand each LHS

$$\frac{dy_i(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t)y_k \;\; (i = \overline{1, n}) \tag{3}$$

with $p_{ik}(t) \in C_I$ $(i, k = \overline{1, n})$, which differs little from a linear triangular system

$$\frac{dy_i^*(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t)y_k^* \;\; (i = \overline{1, n}), \tag{4}$$

and the conditions either of Theorem 0.1 or of Theorem 0.2 due to A. V. Kostin [5] are fulfilled.

Theorem 1. *Let the conditions*

1) *LHS* (4) *is stable when* $t \in I$*;*
2) *for a partial solution* $\sigma_i(t)$ $(i = \overline{1, n})$ *of a linear inhomogeneous triangular system*

$$\frac{d\sigma_i(t)}{dt} = \sum_{k=1}^{i-1} |p_{ik}(t)| + \operatorname{Re} p_{ii}(t)\sigma_i(t) + \sum_{k=i+1}^{n} |p_{ik}(t)|\sigma_k(t) \;\; (i = \overline{1, n}) \tag{5}$$

with the initial conditions $\sigma_i(t_0) = 0$ $(i = \overline{1,n})$ *the estimate of the form* $0 < \sigma_i(t) < 1 - \gamma$ $(i = \overline{1,n})$, $\gamma = const$, $\gamma \in (0,1)$ *holds for all* $t \in I$.

Then the zero solution of the system (3) *is a fortiori stable for* $t \in I$.

Theorem 2. *Let the system* (3) *satisfy all the conditions of Theorem* 1 *and, moreover,*

1) *triangular linear system* (4) *is asymptotically stable for* $t \in I$;
2) $\lim\limits_{t \to +\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is asymptotically stable for* $t \in I$.

Theorem 3. *Let the system* (3) *satisfy all the conditions of Theorem* 1 *and, moreover,*

1) *none of the functions*
$$\psi_i(t) = \sum_{k=1}^{i-1} |p_{ik}(t)| \quad (i = \overline{2,n}) \not\equiv 0 \ \textit{ for } \ t \in I;$$
2) $\lim\limits_{t \to +\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is stable for* $t \in I$.

We will also use the following lemma [5]:

Lemma 1. *If the functions* $p(t), q(t) \in C_I$, $\operatorname{Re} p(t) < 0$, $t \in I$,
$$\int\limits_{t_0}^{+\infty} \operatorname{Re} p(\tau)\, d\tau = -\infty, \quad \lim_{t \to +\infty} \frac{q(t)}{\operatorname{Re} p(t)} = 0,$$
then
$$e^{\int\limits_{t_0}^{t} \operatorname{Re} p(\tau)\, d\tau} \int\limits_{t_0}^{t} q(\tau) e^{-\int\limits_{\tau_0}^{\tau} \operatorname{Re} p(\tau_1)\, d\tau_1} d\tau = o(1), \ \ t \to +\infty.$$

Further, it will be assumed that all limits and characters o, O are considered as $t \to +\infty$.

In case equation (1) has the form
$$y'' + p(t) y = 0, \tag{6}$$
where $p(t) \in C_I^2$, $p(t) > 0$ in I, $\lambda_1(t) = -i\sqrt{p}$, $\lambda_2(t) = i\sqrt{p}$, $p = p(t)$, there is the well-known I. T. Kiguradze's theorem [6]:

Theorem 4. *Let equation* (6) *be such that*
$$p(+\infty) = +\infty, \ \ p' p^{-\frac{3}{2}} = o(1), \quad (\ln p)^{-1} \int\limits_{a}^{t} \left| (p' p^{-\frac{3}{2}})' \right| d\tau = o(1), \ \ t \to t_0.$$

Then there take place the property of asymptotic stability.

2. The Main Results

2.1. **Reduction of equation** (1) **to the system of the form** (5)**.** Consider the real second order LHDE (1):

$$y'' + p(t)y' + q(t)y = 0, \;\; t \in I,$$

where $p(t), q(t) \in C_I^1$. Let $y = y_1$, $y' = y_2$. We reduce the equation to an equivalent system

$$\begin{cases} y_1' = 0 \cdot y_1 + 1 \cdot y_2, \\ y_2' = -q \cdot y_1 - p \cdot y_2. \end{cases} \tag{7}$$

Consider the characteristic equation of LHS (6):

$$\begin{vmatrix} 0-\lambda & 1 \\ -q & -p-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + p\lambda + q = 0, \tag{8}$$

and assume that $\frac{p^2}{2} - q < 0$ at I. Then this equation has two complex-conjugate roots:

$$\lambda_1 = \alpha - i\beta, \quad \lambda_2 = \alpha + i\beta,$$

where $\lambda_i = \lambda_i(t)$ $(i = 1, 2)$, $\alpha = \alpha(t) \in C_I^1$, $\beta = \beta(t) \in C_I^1$. Given (2), we will consider the case

$$\alpha(t) < 0, \qquad \int\limits_{t_0}^{+\infty} \alpha(t)\, dt = -\infty. \tag{9}$$

There is the question on the sufficient conditions for stability of the trivial solution of the system (7). Consider the following transformation for the system (7):

$$Y = C(t)Z, \quad C(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix}, \quad Z = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix},$$

where $z_i(t)$ are new unknown functions $(i = 1, 2)$.

$$Z' = (C^{-1}AC - C^{-1}C')Z,$$
$$\det C(t) = \lambda_2(t) - \lambda_1(1),$$
$$C^{-1}(t) = \frac{1}{\lambda_2(t) - \lambda_1(1)} \begin{pmatrix} \lambda_2(t) & -1 \\ -\lambda_1(t) & 1 \end{pmatrix},$$
$$C'(t) = \begin{pmatrix} 0 & 0 \\ \lambda_1'(t) & \lambda_2'(t) \end{pmatrix}, \quad C^{-1}C' = \frac{1}{\lambda_2(t) - \lambda_1(1)} \begin{pmatrix} -\lambda_1'(t) & -\lambda_2'(t) \\ \lambda_1'(t) & \lambda_2'(t) \end{pmatrix},$$
$$C^{-1}AC = \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix}.$$

The system withe respect to new unknowns $z_i(t)$ $(i = 1, 2)$ in a scalar form is

$$\begin{cases} z_1'(t) = \Big(\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)}\Big) z_1(t) + \dfrac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)} z_2(t), \\ z_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)} z_1(t) + \Big(\lambda_2(t) - \dfrac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)}\Big) z_2(t). \end{cases} \tag{10}$$

It is not difficult to see that

$$\operatorname{Re} \frac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)} = -\frac{1}{2} \frac{\beta'}{\beta}, \quad \operatorname{Re} \frac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)} = \frac{1}{2} \frac{\beta'}{\beta},$$

$$h(t) = \Big|\frac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)}\Big| = \Big|\frac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)}\Big| = \frac{1}{2} \sqrt{\Big(\frac{\beta'}{\beta}\Big)^2 + \Big(\frac{\alpha'}{\beta}\Big)^2}.$$

In accordance with Theorem 1 we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \Big(\alpha - \dfrac{1}{2} \dfrac{\beta'}{\beta}\Big) \sigma_1(t) + h(t) \sigma_2(t), \\ \sigma_2'(t) = h(t) + \Big(\alpha - \dfrac{1}{2} \dfrac{\beta'}{\beta}\Big) \sigma_2(t). \end{cases} \tag{11}$$

Consider a particular solution with initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$. This solution has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int_{t_0}^{t} (\alpha - \frac{1}{2} \frac{\beta'}{\beta})\, d\tau} \displaystyle\int_{t_0}^{t} h(\tau) e^{-\int_{\tau_0}^{\tau} (\alpha - \frac{1}{2} \frac{\beta'}{\beta})\, d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int_{t_0}^{t} (\alpha - \frac{1}{2} \frac{\beta'}{\beta})\, d\tau} \displaystyle\int_{t_0}^{t} h(\tau) \widetilde{\sigma}_2(\tau) e^{-\int_{\tau_0}^{\tau} (\alpha - \frac{1}{2} \frac{\beta'}{\beta})\, d\tau_1} d\tau. \end{cases} \tag{12}$$

Assume also that there exists a finite or an infinite limit

$$\lim_{t \to +\infty} \frac{\alpha}{\beta}.$$

2.2. **Various cases of behavior of the roots $\lambda_i(t)$ $(i = 1, 2)$.** We consider the following cases of behavior of the roots of the characteristic equation, assuming that the condition (9) is fulfilled:

1) $\alpha(+\infty) \in \mathbb{R}_-$, $\beta(+\infty) \in \mathbb{R}$;
2) $\alpha = o(1)$, $\beta = o(1)$, $\frac{\alpha}{\beta} \to const \neq 0$;
3) $\alpha = o(1)$, $\beta = o(1)$, $\frac{\alpha}{\beta} \to \infty$;
4) $\alpha = o(1)$, $\beta(+\infty) \in \mathbb{R} \setminus \{0\}$;
5) $\alpha = o(1)$, $\beta = o(1)$, $\frac{\alpha}{\beta} \to 0$;
6) $\alpha(+\infty) = -\infty$, $\beta(+\infty) = \infty$, $\frac{\alpha}{\beta} \to \infty$;
7) $\alpha(+\infty) = -\infty$, $\beta(+\infty) \in \mathbb{R} \setminus \{0\}$;

8) $\alpha(+\infty) = -\infty$, $\beta = o(1)$;

9) $\alpha(+\infty) = -\infty$, $\beta(+\infty) = \infty$, $\frac{\alpha}{\beta} \to const \neq 0$;

10) $\alpha = o(1)$, $\beta(+\infty) = \infty$;

11) $\alpha(+\infty) \in \mathbb{R}_-$, $\beta(+\infty) = \infty$;

12) $\alpha(+\infty) = -\infty$, $\beta(+\infty) = \infty$, $\frac{\alpha}{\beta} \to 0$.

Theorems 5–16 correspond to the above cases 1)–12).

Theorem 5. *Let the condition* (9) *be fulfilled and*

$$\alpha(+\infty) \in \mathbb{R}_-, \quad \beta(+\infty) \in \mathbb{R}.$$

Then the trivial solution of equation (1) *is asymptotically stable.*

This case is well known.

Theorem 6. *Let the condition* (9) *and the following conditions*

$$\alpha = o(1), \quad \beta = o(1), \quad \frac{\alpha}{\beta} \to const \neq 0,$$
$$\frac{\alpha'}{\alpha^2} = o(1), \quad \frac{\beta'}{\beta^2} = o(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. We consider the system (10), auxiliary system of differential equations (11) and its particular solution (12).

In this case

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta}} = \frac{1}{2}\lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta^2})^2 + (\frac{\alpha'}{\beta^2})^2}}{\frac{\alpha}{\beta} - \frac{1}{2}\frac{\beta'}{\beta^2}} =$$
$$= \frac{1}{2}\lim_{t\to+\infty} \left|\frac{\alpha'}{\beta^2}\right| \frac{\beta}{\alpha} = \frac{1}{2}\lim_{t\to+\infty} \frac{|\alpha'|}{\alpha^2}\,\frac{\alpha}{\beta} = 0.$$

Consequently, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta}}\,\widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. Obviously, $\psi(t) = h(t) \not\equiv 0$ for $t \in I$. All the conditions of Theorem 3 are fulfilled and thus Theorem 6 is complete. To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$ we make in the system (10) the following substitution:

$$z_i(t) = e^{\delta \int_{t_0}^{t} \alpha\, d\tau}\,\eta_i(t) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \tag{13}$$

Then the system (10) takes the form

$$\begin{cases} \eta_1'(t) = \Big(\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)} - \delta\alpha\Big)\eta_1(t) + \dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}\,\eta_2(t), \\ \eta_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}\,\eta_1(t) + \Big(\lambda_2(t) - \dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)} - \delta\alpha\Big)\eta_2(t). \end{cases} \tag{14}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\Big)\sigma_1(t) + h(t)\sigma_2(t), \\ \sigma_2'(t) = h(t) + \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\Big)\sigma_2(t). \end{cases} \tag{15}$$

It's particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$ has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau} \displaystyle\int_{t_0}^{t} h(\tau) e^{-\int\limits_{\tau_0}^{\tau}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau} \displaystyle\int_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau_1} d\tau. \end{cases} \tag{16}$$

It is not difficult to see that the replacement (13) does not affect the asymptotic stability. Taking into account the transformation $C(t)$,

$$\begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = \lambda_1(t) z_1(t) + \lambda_2(t) z_2(t). \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} y_1(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big), \\ y_2(t) = o\Big(\lambda_1(t) e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau} + \lambda_2(t) e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big). \end{cases}$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t}\left(\delta\alpha + \frac{\lambda_1'(t)}{\lambda_1(t)}\right)d\tau} + e^{\int\limits_{t_0}^{t}\left(\delta\alpha + \frac{\lambda_2'(t)}{\lambda_2(t)}\right)d\tau}\Big),$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t}\alpha\left(\delta + \frac{1}{\alpha}\frac{\lambda_1'(t)}{\lambda_1(t)}\right)d\tau} + e^{\int\limits_{t_0}^{t}\alpha\left(\delta + \frac{1}{\alpha}\frac{\lambda_2'(t)}{\lambda_2(t)}\right)d\tau}\Big).$$

It is easy to see that

$$R(t) = \operatorname{Re}\frac{\lambda_1'(t)}{\lambda_1(t)} = \operatorname{Re}\frac{\lambda_2'(t)}{\lambda_2(t)} = \frac{\alpha'\alpha + \beta'\beta}{\alpha^2 + \beta^2}\,,$$
$$I(t) = \operatorname{Im}\frac{\lambda_1'(t)}{\lambda_1(t)} = -\operatorname{Im}\frac{\lambda_2'(t)}{\lambda_2(t)} = \frac{\alpha'\beta - \alpha\beta'}{\alpha^2 + \beta^2}\,.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,\mathrm{Re}\,\frac{\lambda_1'(t)}{\lambda_1(t)}=\lim_{t\to+\infty}\frac{\alpha'\alpha+\beta'\beta}{\alpha(\alpha^2+\beta^2)}=$$

$$=\lim_{t\to+\infty}\left(\frac{\frac{\alpha'}{\alpha^2}}{1+(\frac{\beta}{\alpha})^2}+\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3+\frac{\alpha}{\beta}}\right)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,\mathrm{Im}\,\frac{\lambda_1'(t)}{\lambda_1(t)}=\lim_{t\to+\infty}\frac{\alpha'\beta-\alpha\beta'}{\alpha(\alpha^2+\beta^2)}=$$

$$=\lim_{t\to+\infty}\left(\frac{\frac{\alpha'}{\alpha^2}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}-\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\right)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\quad(i=1,2).$$

Therefore,

$$y_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\quad(i=1,2),\quad\delta\in(0,1).\qquad\square$$

Theorem 7. *Let the condition* (9) *and the following conditions*

$$\alpha=o(1),\quad\beta=o(1),\quad\frac{\alpha}{\beta}\to\infty,$$

$$\frac{\alpha'}{\alpha}=o(\beta),\quad\frac{\beta'}{\beta^2}=O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replace (13). We obtain the system (14), an auxiliary system of differential equations (15) and its particular solution (16).

In this case,

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac12\frac{\beta'}{\beta}}=\frac12\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\beta})^2+(\frac{\alpha'}{\beta})^2}}{\alpha(1-\delta-\frac12\frac{\beta'}{\alpha\beta})}=$$

$$=\frac12\lim_{t\to+\infty}\sqrt{\left(\frac{\frac{\beta'}{\alpha\beta}}{1-\delta-\frac12\frac{\beta'}{\alpha\beta}}\right)^2+\left(\frac{\frac{\alpha'}{\alpha\beta}}{1-\delta-\frac12\frac{\beta'}{\alpha\beta}}\right)^2}=$$

$$=\frac12\lim_{t\to+\infty}\sqrt{\left(\frac{\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}{1-\delta-\frac12\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}\right)^2+\left(\frac{\frac{\alpha'}{\alpha\beta}}{1-\delta-\frac12\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}\right)^2}=0.$$

Consequently, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac12\frac{\beta'}{\beta}}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 7 is valid. Moreover,

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Next,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \left(\frac{\frac{\alpha'}{\alpha\beta}}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \right) = 0,$$

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \left(\frac{\frac{\alpha'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2 + 1} - \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \right) = 0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2).$$

Therefore, just as in Theorem 6:

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 8. *Let the condition* (9) *and the following conditions*

$$\alpha = o(1), \quad \beta(+\infty) \in \mathbb{R} \setminus \{0\},$$
$$\frac{\alpha'}{\alpha} = o(1), \quad \frac{\beta'}{\beta} = o(\alpha)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (13). We obtain the system (14), an auxiliary system of differential equations (15) and its particular solution (16).

In this case,

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta}} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta})^2 + (\frac{\alpha'}{\beta})^2}}{\alpha(1 - \delta - \frac{1}{2}\frac{\beta'}{\alpha\beta})} =$$

$$= \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2 + (\frac{\alpha'}{\alpha\beta})^2}}{1 - \delta - \frac{1}{2}\frac{\beta'}{\alpha\beta}} = \frac{1}{2(1-\delta)} \lim_{t\to+\infty} \left| \frac{\alpha'}{\alpha\beta} \right| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 8 is valid. Moreover,

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, \tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha+\frac{\beta}{\alpha}\,\beta}+\frac{\frac{\beta'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2+1}\Big)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\frac{\alpha}{\beta}\,\alpha+\beta}-\frac{\frac{\beta'}{\alpha\beta}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2).$$

Therefore, just as in Theorem 6,

$$y_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).\qquad\square$$

Theorem 9. *Let the condition* (9) *and the following conditions*

$$\alpha=o(1),\quad \beta=o(1),\quad \frac{\alpha}{\beta}\to 0,$$
$$\frac{\alpha'}{\alpha^2}=O(1),\quad \frac{\beta'}{\beta}=o(\alpha)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (13). We obtain the system (14), an auxiliary system of differential equations (15) and its particular solution (16).

In this case,

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\,\frac{\beta'}{\beta}}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2+(\frac{\alpha'}{\alpha\beta})^2}}{1-\delta-\frac{1}{2}\,\frac{\beta'}{\alpha\beta}}=$$
$$=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha\beta}\Big|=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha^2}\,\frac{\alpha}{\beta}\Big|=0.$$

Consequently, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\,\frac{\beta'}{\beta}}\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 9 is valid. Moreover,

$$z_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha^2}}{1+(\frac{\beta}{\alpha})^2}+\frac{\frac{\beta'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2+1}\Big)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha^2}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}-\frac{\frac{\beta'}{\alpha\beta}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2).$$

Therefore, just as in Theorem 6,

$$y_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).\qquad\square$$

Theorem 10. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty)=-\infty,\quad \beta(+\infty)=\infty,\quad \frac{\alpha}{\beta}\to\infty,$$

$$\frac{\alpha'}{\alpha}=O(1),\quad \frac{\beta'}{\beta^2}=O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the following replacement:

$$z_1(t)\lambda_1(t)=\xi_1(t),\quad z_2(t)\lambda_2(t)=\xi_2(t). \tag{17}$$

Then the system (10) takes the form

$$\begin{cases}\xi_1'(t)=\Big(\lambda_1(t)+\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}-\dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\xi_1(t)+\\ \qquad +\dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}\,\dfrac{\lambda_1(t)}{\lambda_2(t)}\xi_2(t),\\ \xi_2'(t)=-\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}\,\dfrac{\lambda_2(t)}{\lambda_1(t)}\xi_1(t)+\\ \qquad +\Big(\lambda_2(t)-\dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}-\dfrac{\lambda_2'(t)}{\lambda_2(t)}\Big)\xi_2(t).\end{cases} \tag{18}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases}\sigma_1'(t)=\Big(\alpha-\dfrac{1}{2}\,\dfrac{\beta'}{\beta}-R(t)\Big)\sigma_1(t)+h(t)\sigma_2(t),\\ \sigma_2'(t)=h(t)+\Big(\alpha-\dfrac{1}{2}\,\dfrac{\beta'}{\beta}-R(t)\Big)\sigma_2(t).\end{cases}$$

Consider a particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau} \displaystyle\int\limits_{t_0}^{t} h(\tau) e^{-\int\limits_{\tau_0}^{\tau} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau} \displaystyle\int\limits_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau_1} d\tau. \end{cases}$$

In this case,

$$\lim_{t\to+\infty} \frac{1}{\beta} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(1 + (\frac{\beta}{\alpha})^2)} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta^2})^2 + (\frac{\alpha'}{\beta^2})^2}}{\frac{\alpha}{\beta} - \frac{1}{2}\frac{\beta'}{\beta^2} - \frac{1}{\beta} R(t)} =$$

$$= \frac{1}{2} \lim_{t\to+\infty} \sqrt{\Big(\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}\Big)^2 + \Big(\frac{\alpha'}{\alpha\beta}\Big)^2} = \frac{1}{2} \lim_{t\to+\infty} \Big|\frac{\alpha'}{\alpha\beta}\Big| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Next,

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 10 is valid. To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$, we make in the system (18) the following replacement:

$$\xi_i(t) = e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau} \eta_i(t) \;\; (i = 1, 2), \;\; \delta \in (0, 1). \tag{19}$$

Then system (18) takes the form

$$\begin{cases} \eta_1'(t) = \Big(\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)} - \dfrac{\lambda_1'(t)}{\lambda_1(t)} - \delta\alpha\Big)\eta_1(t) + \\ \qquad + \dfrac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)} \dfrac{\lambda_1(t)}{\lambda_2(t)} \eta_2(t), \\ \eta_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)} \dfrac{\lambda_2(t)}{\lambda_1(t)} \eta_1(t) + \\ \qquad + \Big(\lambda_2(t) - \dfrac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)} - \dfrac{\lambda_2'(t)}{\lambda_2(t)} - \delta\alpha\Big)\eta_2(t). \end{cases} \tag{20}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta} - R(t)\Big)\sigma_1(t) + h(t)\sigma_2(t), \\ \sigma_2'(t) = h(t) + \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta} - R(t)\Big)\sigma_2(t). \end{cases} \tag{21}$$

Let us consider a particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t}\big((1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)\big)\,d\tau} \times \\ \qquad \times \int\limits_{t_0}^{t} h(\tau) e^{-\int\limits_{\tau_0}^{\tau}\big((1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)\big)\,d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t}\big((1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)\big)\,d\tau} \times \\ \qquad \times \int\limits_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau}\big((1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)\big)\,d\tau_1}\,d\tau. \end{cases} \tag{22}$$

It is not difficult to see that the replacement (19) does not affect the stability. At the same time,

$$\xi_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int\limits_{t_0}^{t}(\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Further,

$$\lim_{t\to+\infty} \frac{1}{\alpha}\, R(t) = \lim_{t\to+\infty} \Bigg(\frac{\frac{\alpha'}{\alpha}}{\alpha(1 + (\frac{\beta}{\alpha})^2)} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}}\Bigg) = 0,$$

$$\lim_{t\to+\infty} \frac{1}{\alpha}\, I(t) = \lim_{t\to+\infty} \Bigg(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} - \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1}\Bigg) = 0.$$

Consequently,

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2)$$

and

$$z_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = \lambda_1(t) z_1(t) + \lambda_2(t) z_2(t) \end{cases} \Longrightarrow \begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = \xi_1(t) + \xi_2(t) \end{cases} \Longrightarrow$$

$$\Longrightarrow y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha \, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1). \qquad \square$$

Theorem 11. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty) = -\infty, \quad \beta(+\infty) \in \mathbb{R} \setminus \{0\},$$
$$\frac{\alpha'}{\alpha} = o(1), \quad \frac{\beta'}{\beta^2} = O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t \to +\infty} \frac{1}{\beta} R(t) = \lim_{t \to +\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(1 + (\frac{\beta}{\alpha})^2)} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Then

$$\lim_{t \to +\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} = \frac{1}{2} \lim_{t \to +\infty} \frac{\sqrt{(\frac{\beta'}{\beta^2})^2 + (\frac{\alpha'}{\beta^2})^2}}{(1-\delta)\frac{\alpha}{\beta} - \frac{1}{2}\frac{\beta'}{\beta^2} - \frac{1}{\beta} R(t)} =$$

$$= \frac{1}{2(1-\delta)} \lim_{t \to +\infty} \sqrt{\Big(\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}\Big)^2 + \Big(\frac{\alpha'}{\alpha\beta}\Big)^2} = \frac{1}{2(1-\delta)} \lim_{t \to +\infty} \Big|\frac{\alpha'}{\alpha\beta}\Big| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Next,

$$\lim_{t \to +\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 11 is valid. Thus

$$\xi_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha \, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)}) \, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

Further,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(1+(\frac{\beta}{\alpha})^2)}+\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3+\frac{\alpha}{\beta}}\Big)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}-\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1). \qquad \square$$

Theorem 12. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty)=-\infty,\quad \beta=o(1),$$
$$\frac{\alpha'}{\alpha}=o(\beta),\quad \frac{\beta'}{\beta^2}=O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty}\frac{1}{\beta}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha\beta}}{1+(\frac{\beta}{\alpha})^2}+\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\beta^2})^2+(\frac{\alpha'}{\beta^2})^2}}{(1-\delta)\frac{\alpha}{\beta}-\frac{1}{2}\frac{\beta'}{\beta^2}-\frac{1}{\beta}\,R(t)}=$$
$$=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\sqrt{\Big(\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}\Big)^2+\Big(\frac{\alpha'}{\alpha\beta}\Big)^2}=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha\beta}\Big|=0.$$

Therefore, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Next,

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 12 is valid. Thus

$$\xi_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha\beta}}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \Big) = 0,$$

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2 + 1} - \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Hence

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2)$$

and

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 13. *Let the condition* (9) *and thr following conditions*

$$\alpha(+\infty) = -\infty, \quad \beta(+\infty) = \infty, \quad \frac{\alpha}{\beta} \to const \neq 0,$$

$$\frac{\alpha'}{\alpha} = O(1), \quad \frac{\beta'}{\beta^2} = o(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \Big) = 0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\,\frac{\beta'}{\beta}-R(t)}=\lim_{t\to+\infty}\frac{h(t)}{\alpha(1-\delta-\frac{1}{2}\,\frac{\beta'}{\alpha\beta}-\frac{1}{\alpha}R(t))}=$$

$$=\frac{1}{2}\lim_{t\to+\infty}\sqrt{\Big(\frac{\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}}{1-\delta-\frac{1}{2}\,\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}}\Big)^2+\Big(\frac{\frac{\alpha'}{\alpha\beta}}{1-\delta-\frac{1}{2}\,\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}}\Big)^2}=$$

$$=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha\beta}\Big|=0.$$

Therefore, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\,\frac{\beta'}{\beta}-R(t)}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 13 is valid. Thus

$$\xi_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, according to (17),

$$z_i(t)=o\Big(e^{\int\limits_{t_0}^{t}(\delta\alpha-\frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

As is shown above,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=0.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}-\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

□

Theorem 14. *Let the condition* (9) *and the following conditions*

$$\alpha=o(1),\quad \beta(+\infty)=\infty,$$

$$\frac{\alpha'}{\alpha}=O(1),\quad \frac{\beta'}{\beta}=o(\alpha)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\beta(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}+\frac{\frac{\beta'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2+(\frac{\alpha'}{\alpha\beta})^2}}{(1-\delta-\frac{1}{2}\,\frac{\beta'}{\alpha\beta}-\frac{1}{\alpha}\,R(t))}=0.$$

Therefore, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\,\frac{\beta'}{\beta}-R(t)}\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 14 is valid. Moreover,

$$\xi_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, according to (17),

$$z_i(t)=o\Big(e^{\int_{t_0}^{t}(\delta\alpha-\frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

As is shown above,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=0.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\frac{\alpha}{\beta}\,\alpha+\beta}-\frac{\frac{\beta'}{\alpha\beta}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).\qquad\square$$

Theorem 15. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty) \in \mathbb{R}_-, \quad \beta(+\infty) = \infty,$$
$$\frac{\alpha'}{\alpha} = O(1), \quad \frac{\beta'}{\beta} = o(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty} \frac{1}{\alpha}\,R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})} + \frac{\frac{\beta'}{\beta}}{\alpha((\frac{\alpha}{\beta})^2+1)}\Big) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)} = \frac{1}{2}\lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2 + (\frac{\alpha'}{\alpha\beta})^2}}{(1-\delta-\frac{1}{2}\,\frac{\beta'}{\alpha\beta} - \frac{1}{\alpha}\,R(t))} = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)}\,\widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 15 is valid. Hence

$$\xi_i(t) = o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big) \ \ (i=1,2), \ \ \delta\in(0,1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int_{t_0}^{t}(\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big) \ \ (i=1,2), \ \ \delta\in(0,1).$$

As is shown above,

$$\lim_{t\to+\infty} \frac{1}{\alpha}\,R(t) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{1}{\alpha}\,I(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})} - \frac{\frac{\beta'}{\beta}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}\Big) = 0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i=1,2)$$

and

$$z_i(t) = o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big) \ \ (i=1,2), \ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 16. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty) = -\infty, \quad \beta(+\infty) = \infty, \quad \frac{\alpha}{\beta} \to 0,$$

$$\frac{\alpha'}{\alpha^2} = O(1), \quad \frac{\beta'}{\beta} = O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha^2}}{1 + (\frac{\beta}{\alpha})^2} + \frac{\frac{\beta'}{\beta}}{\alpha((\frac{\alpha}{\beta})^2 + 1)} \Big) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2 + (\frac{\alpha'}{\alpha\beta})^2}}{(1 - \delta - \frac{1}{2}\frac{\beta'}{\alpha\beta} - \frac{1}{\alpha} R(t))} = 0.$$

Consequently, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 16 is valid. Moreover,

$$\xi_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)}) \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

As is shown above,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha^2}}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} - \frac{\frac{\beta'}{\beta}}{\alpha(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} \Big) = 0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2)$$

and

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1). \qquad \square$$

2.3. **The case of purely imaginary roots** $\lambda_i(t)$ $(i = 1, 2)$**.** Let us analyze equation (6):

$$y'' + p(t)y = 0$$

where $p(t) \in C_I^2$, $p(t) > 0$ in I. Then $\lambda_1(t) = -i\sqrt{p}$, $\lambda_2(t) = i\sqrt{p}$, $p = p(t)$.

Theorem 17. *Let the conditions*

$$\beta(+\infty) = +\infty, \quad \frac{\beta'}{\beta^2} = o(1), \quad \frac{\big(\frac{\beta'}{\beta^2}\big)'}{\frac{\beta'}{\beta}} = o(1)$$

be fulfilled. Then the trivial solution of equation (6) *is asymptotically stable.*

Proof. In this case the system (10) takes the form

$$\begin{cases} z_1'(t) = \Big(-\dfrac{1}{2}\,\dfrac{\beta'}{\beta} - i\beta\Big) z_1(t) + \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, z_2(t), \\[2mm] z_2'(t) = \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, z_1(t) + \Big(-\dfrac{1}{2}\,\dfrac{\beta'}{\beta} + i\beta\Big) z_2(t). \end{cases}$$

In this system we make the following replacement:

$$z_i(t) = e^{\delta \int\limits_{t_0}^{t} (-\frac{1}{2}\frac{\beta'}{\beta})\, d\tau} \varphi_i(t) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

As a result, we obtain the following system:

$$\begin{cases} \varphi_1'(t) = \Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta} - i\beta\Big) \varphi_1(t) + \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, \varphi_2(t), \\[2mm] \varphi_2'(t) = \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, \varphi_1(t) + \Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta} + i\beta\Big) \varphi_2(t). \end{cases} \tag{23}$$

Then in the system (23) we make the following replacement:

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r(t) & 1 \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix},$$

where $\eta_i(t)$ are the new unknown functions $(i = 1, 2)$. Then the system (23) takes the form

$$\begin{cases} \eta_1'(t) = \Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta} + \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, r(t) - i\beta\Big) \eta_1(t) + \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, \eta_2(t), \\[2mm] \eta_2'(t) = \Big(\dfrac{1}{2}\,\dfrac{\beta'}{\beta} + 2i\beta r(t) - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, r^2(t) - r'(t)\Big) \eta_1(t) + \\[2mm] \qquad + \Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta} - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\, r(t) + i\beta\Big) \eta_2(t). \end{cases} \tag{24}$$

Suppose

$$\frac{1}{2}\,\frac{\beta'}{\beta} + 2i\beta r(t) = 0.$$

Then

$$r(t) = \frac{1}{4}\,\frac{\beta'}{\beta^2}\,i = o(1).$$

Then the system (24) takes the form

$$\begin{cases} \eta_1'(t) = \Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta} + \dfrac{1}{8}\,\dfrac{\beta'}{\beta}\,\dfrac{\beta'}{\beta^2}\,i - i\beta\Big)\eta_1(t) + \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\,\eta_2(t), \\ \eta_2'(t) = \Big(-\dfrac{1}{4}\,\Big(\dfrac{\beta'}{\beta^2}\Big)' i + \dfrac{1}{8}\,\dfrac{\beta'}{\beta}\,\Big(\dfrac{\beta'}{\beta^2}\Big)^2\Big)\eta_1(t) + \\ \qquad + \Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta} - \dfrac{1}{8}\,\dfrac{\beta'}{\beta}\,\dfrac{\beta'}{\beta^2}\,i + i\beta\Big)\eta_2(t). \end{cases} \tag{25}$$

In accordance with Theorem 1, for the system (25) we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = -\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta}\,\sigma_1(t) + \dfrac{1}{2}\,\Big|\dfrac{\beta'}{\beta}\Big|\,\sigma_2(t), \\ \sigma_2'(t) = \dfrac{1}{8}\sqrt{4\Big(\Big(\dfrac{\beta'}{\beta^2}\Big)'\Big)^2 + \Big(\dfrac{\beta'}{\beta}\Big)^2\Big(\dfrac{\beta'}{\beta^2}\Big)^4} - \dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta}\,\sigma_2(t). \end{cases} \tag{26}$$

We denote

$$g(t) = \frac{1}{8}\sqrt{4\Big(\Big(\frac{\beta'}{\beta^2}\Big)'\Big)^2 + \Big(\frac{\beta'}{\beta}\Big)^2\Big(\frac{\beta'}{\beta^2}\Big)^4}.$$

Consider a particular solution of the system (26) with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int_{t_0}^{t}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau} \displaystyle\int_{t_0}^{t} g(\tau)e^{-\int_{\tau_0}^{\tau}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int_{t_0}^{t}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau} \displaystyle\int_{t_0}^{t} \frac{1}{2}\,\Big|\frac{\beta'}{\beta}\Big|\widetilde{\sigma}_2(\tau)e^{-\int_{\tau_0}^{\tau}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau_1}\,d\tau. \end{cases}$$

In this case,

$$\lim_{t\to+\infty}\frac{g(t)}{-\frac{1-\delta}{2}\,\frac{\beta'}{\beta}} = -\frac{1}{4(1-\delta)}\lim_{t\to+\infty}\sqrt{4\Big(\frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}}\Big)^2 + \Big(\frac{\beta'}{\beta^2}\Big)^4} =$$

$$= -\frac{1}{1-\delta}\lim_{t\to+\infty}\Big|\frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}}\Big| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{\frac{1}{2}\,|\frac{\beta'}{\beta}|}{-\frac{1-\delta}{2}\,\frac{\beta'}{\beta}}\,\widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 17 is valid. Moreover, $\eta_i(t) = o(1)$ $(i = 1, 2)$. Then $\varphi_i(t) = o(1)$ $(i = 1, 2)$. Then we have obtained

$$z_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}(-\frac{1}{2}\,\frac{\beta'}{\beta})\,d\tau}\Big) \ \ (i = 1, 2), \quad \delta \in (0, 1).$$

Then

$$\begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = -i\beta z_1(t) + i\beta z_2(t) \end{cases} \Longrightarrow \begin{cases} y_1(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}(-\frac{1}{2}\,\frac{\beta'}{\beta})\,d\tau}\Big), \\ y_2(t) = -i\beta z_1(t) + i\beta z_2(t) \end{cases} \Longrightarrow$$

$$\Longrightarrow |y_2(t)| = o\Big(e^{\int\limits_{t_0}^{t}(-\frac{1}{2}\,\delta\,\frac{\beta'}{\beta}+\frac{\beta'}{\beta^2})\,d\tau}\Big) \Longrightarrow$$

$$\Longrightarrow \begin{cases} |y_1(t)| = o\Big(e^{\delta\int\limits_{t_0}^{t}(-\frac{1}{2}\,\frac{\beta'}{\beta})\,d\tau}\Big), \\ |y_2(t)| = o\Big(e^{\int\limits_{t_0}^{t}\delta(-\frac{1}{2}\,\frac{\beta'}{\beta}+o(1))\,d\tau}\Big), \end{cases} \qquad \delta \in (0, 1). \qquad \square$$

Remark 1. The condition

$$\frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}} = o(1)$$

is satisfied if there exists the corresponding limit.

Proof.

$$\lim_{t\to+\infty} \frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}} = \Big[\text{we use the inverse de L'Hospital's rule}\Big] =$$

$$= -\frac{1}{2}\lim_{t\to+\infty} \frac{\beta'}{\beta^2 \ln\beta} = 0. \qquad \square$$

Remark 2. The conditions of I. T. Kiguradze's Theorem 4 are equivalent to those of Theorem 17. But, in addition to Theorem 17, we have obtained the estimate of solutions of equation (6).

Proof.

$$\lim_{t\to+\infty} p'p^{-\frac{3}{2}} = \lim_{t\to+\infty} (\beta^2)'\beta^{-3} = \lim_{t\to+\infty} \frac{2\beta\beta'}{\beta^3} = 2\lim_{t\to+\infty} \frac{\beta'}{\beta^2} = 0,$$

$$\lim_{t\to+\infty} (\ln p)^{-1} \int\limits_a^t \left|(p'p^{-\frac{3}{2}})'\right| d\tau = \Big[\text{we use de L'Hospital's rule}\Big] =$$

$$= \lim_{t\to+\infty} \frac{|(p'p^{-\frac{3}{2}})'|}{\frac{p'}{p}} = \lim_{t\to+\infty} \frac{|(2\beta\beta'\beta^{-3})'|}{\frac{2\beta\beta'}{\beta^2}} = \lim_{t\to+\infty} \frac{|(\frac{\beta'}{\beta^2})'|}{\frac{\beta'}{\beta}} = 0. \quad \square$$

Conclusion

In the present paper we have revealed the sufficient conditions for asymptotic stability, as well as the estimate of solutions of the homogeneous linear non-autonomous second order differential equation in terms of the behavior of roots of the characteristic equation in the case of complex roots. The results of the work allow one to proceed both to investigating equations of higher order and to considering the problems on a simple stability and instability.

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