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ON TWO–POINT BOUNDARY VALUE PROBLEMS FOR TWO–DIMENSIONAL NONLINEAR DIFFERENTIAL SYSTEMS WITH STRONG SINGULARITIES

Abstract. For two-dimensional nonlinear differential systems with strong singularities with respect to a time variable, unimprovable sufficient conditions for the solvability and unique solvability of two-point boundary value problems are established.

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Let $-\infty < a < b < +\infty$, and let $f_i :]a,b[\times R \to R$ $(i = 1,2)$ be continuous functions. In the open interval $]a,b[$, we consider the two-dimensional nonlinear differential system

$$\frac{du_1}{dt} = f_1(t,u_2), \quad \frac{du_2}{dt} = f_2(t,u_1) \tag{1}$$

with boundary conditions of one of the following two types:

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_1(t) = 0, \tag{2_1}$$

and

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_2(t) = 0. \tag{2_2}$$

A vector function (u_1, u_2) with continuously differentiable components $u_i :]a,b[\to R$ $(i = 1,2)$ is said to be a solution of the system (1) if it satisfies that system at each point of $]a,b[$.

A solution of the system (1), satisfying the boundary conditions (2_1) (the boundary conditions (2_2)), is said to be a solution of the problem $(1),(2_1)$ (a solution of the problem $(1),(2_2)$).

A solution of the problem $(1),(2_1)$ (of the problem $(1),(2_2)$), satisfying the condition

$$\int_a^b u_2^2(t)\,dt < +\infty, \tag{3}$$

is said to be a solution of the problem $(1), (2_1), (3)$ (a solution of the problem $(1), (2_2), (3)$).

Let

$$[f_2(t,x)]_- = \frac{1}{2}\left(|f_2(t,x)| - f_2(t,x)\right).$$

Theorems below on the solvability and unique solvability of the problem $(1), (2_1), (3)$ cover the case, where

$$\int_a^{t_0} (t-a)[f_2(t,x)]_- dt =$$

$$= \int_{t_0}^b (b-t)[f_2(t,x)]_- dt = +\infty \quad \text{for } t_0 \in]a,b[, \quad x \neq 0. \tag{4_1}$$

Analogous theorems for the problem $(1), (2_2), (3)$ cover the case, where

$$\int_a^b (t-a)[f_2(t,x)]_- dt = +\infty \quad \text{for } x \neq 0. \tag{4_2}$$

In the case, where the condition (4_1) (the condition (4_2)) is satisfied, we say that the system (1) has strong singularities at the points a and b (at the point a). In both cases, roughly speaking, the orders of singularity of the function f_2 with respect to the time variable are no less than 2, i.e., no less than the dimension of the considered differential system. Just because of that reason these singularities are said to be strong in the Agarwal–Kiguradze sence [1]. The above-mentioned cases essentially differ from so-called weak singular cases, where for arbitrary $t_0 \in]a,b[$ and $x \neq 0$, the following conditions

$$\int_a^{t_0} [f_2(t,x)]_- dt = \int_{t_0}^b [f_2(t,x)]_- dt = +\infty \quad \text{or} \quad \int_a^b [f_2(t,x)]_- dt = +\infty$$

hold but

$$\int_a^b (t-a)(b-t)[f_2(t,x)]_- dt < +\infty.$$

In the case of strong singularity, in contrast to the case of weak singularity, the problem $(1), (2_1), (3)$, generally speaking, is not equivalent to the problem $(1), (2_1)$. Analogously, the problem $(1), (2_2), (3)$ is not equivalent to the problem $(1), (2_2)$. To convince ourselves that this is so, let us consider the case, where the system (1) has the form

$$\frac{du_1}{dt} = u_2, \quad \frac{du_2}{dt} = -\frac{\mu}{(t-a)^2}\, u_1. \tag{$1'$}$$

If μ satisfies the inequality

$$0 < \mu < \frac{1}{4}\,,$$

then the problem $(1'), (2_1)$,(3) has only the trivial solution whereas the problem $(1'), (2_1)$ has infinite set of solutions

$$u_1(t) = c\big[(t-a)^{\lambda_1} - (b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2}\big], \quad u_2(t) =$$
$$= c\big[\lambda_1(t-a)^{\lambda_1-1} - \lambda_2(b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2-1}\big], \quad c \in R,$$

where

$$\lambda_1 = \frac{1+\sqrt{1-4\mu}}{2}\,, \quad \lambda_2 = \frac{1-\sqrt{1-4\mu}}{2}\,.$$

Analogously, the problem $(1'), (2_2), (3)$ has only the trivial solution, while the problem $(1'), (2_2)$ has infinite set of solutions

$$u_1(t) = c\Big[(t-a)^{\lambda_1} - \frac{\lambda_1}{\lambda_2}\,(b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2}\Big], \quad u_2(t) =$$
$$= c\lambda_1\big[(t-a)^{\lambda_1-1} - (b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2-1}\big], \quad c \in R.$$

For the weakly singular system (1) and its various particular cases, unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of problems of the type $(1), (2_1)$ and $(1), (2_2)$ are contained in [2]–[7], [11]–[14], [17]–[19]. Two-point boundary value problems for higher order differential equations with strong singularities are investigated in detail by I. Kiguradze and R. P. Agarwal (see, [1], [8]–[10]). Conditions, guaranteeing the existence of extremal solutions of two-point boundary value problems for second order nonlinear differential equations with strong singularities, are contained in [16]. The Agarwal–Kiguradze type theorems for two-dimensional linear differential systems are given in [15]. Below we give analogous results for the problems $(1), (2_1), (3)$ and $(1), (2_2), (3)$.

First we consider the problem $(1), (2_1), (3)$. The following theorems are valid.

Theorem 1. *Let in the domain* $]a,b[\times R$ *the inequalities*

$$\delta|x| \le \big[f_1(t,x) - f_1(t,0)\big]\operatorname{sgn} x \le \ell_0|x|, \tag{5}$$

$$\big[f_2(t,x) - f_2(t,0)\big]\operatorname{sgn} x \ge -\ell\bigg(\frac{1}{(t-a)^2} + \frac{1}{(b-t)^2}\bigg)|x| \tag{6}$$

be fulfilled, where δ, ℓ_0, *and* ℓ *are positive constants such that*

$$4\ell\ell_0 < 1. \tag{7}$$

If, moreover,

$$\int_a^b f_1^2(t,0)dt < 0, \quad \int_a^b (t-a)^{1/2}(b-t)^{1/2}|f_2(t,0)|dt < +\infty, \tag{8}$$

then the problem $(1), (2_1), (3)$ *has at least one solution.*

Theorem 2. *Let in the domain* $]a,b[\times R$ *the conditions*

$$\delta|x-y| \le \big[f_1(t,x)-f_1(t,y)\big]\operatorname{sgn}(x-y) \le \ell_0|x-y|, \tag{9}$$

$$\big[f_2(t,x)-f_2(t,y)\big]\operatorname{sgn}(x-y) \ge -\ell\bigg(\frac{1}{(t-a)^2}+\frac{1}{(b-t)^2}\bigg)|x-y| \tag{10}$$

be fulfilled, where δ, ℓ_0, *and* ℓ *are positive constants, satisfying the inequality* (7)*. If, moreover, the condition* (8) *holds, then the problem* (1), (2_1), (3) *has one and only one solution.*

Note that the condition (7) in Theorems 1 and 2 is unimprovable in the sense that it cannot be replaced by the non-strict inequality

$$4\ell\ell_0 \le 1. \tag{7$'$}$$

Indeed, consider the case, where

$$f_1(t,x)=x, \quad f_2(t,x)=-\frac{1}{4(t-a)^2}\,x+9.$$

Then the conditions (5), (6), (8)–(10) are satisfied, where $\delta=\ell_0=1$ and $\ell=\frac{1}{4}$. Consequently, all the conditions of Theorems 1 and 2 are fulfilled except the condition (7), instead of which the inequality (7$'$) is satisfied. Nevertheless, in the considered case the problem (1), (2_1), (3) has no solution. The fact is that in that case an arbitrary solution of the system (1) admits the representation

$$u_1(t)=c_1(t-a)^{1/2}+c_2(t-a)^{1/2}\ln(t-a)+4(t-a)^2,$$

$$u_2(t)=\frac{1}{2}\,c_1(t-a)^{-1/2}+c_2(t-a)^{-1/2}\bigg(\frac{1}{2}\,\ln(t-a)+1\bigg)+8(t-a),$$

where c_1 and c_2 are arbitrary real numbers, and consequently,

$$\int\limits_a^b u_2^2(t)dt=+\infty \quad \text{for } |c_1|+|c_2|\ne 0.$$

Consider now the problem (1), (2_2), (3). Suppose

$$f_2^*(t,x)=\max\big\{|f_2(t,y)|:\ |y|\le x\big\} \quad \text{for } a<t<b,\ \ x>0.$$

Theorem 3. *Let in the domain* $]a,b[\times R$ *the inequalities* (5) *and*

$$\big[f_2(t,x)-f_2(t,0)\big]\operatorname{sgn} x \ge -\frac{\ell}{(t-a)^2}\,|x|$$

be fulfilled, where δ, ℓ_0, *and* ℓ *are positive constants, satisfying the condition* (7). *If, moreover,*

$$\int_a^b f_1^2(s,0)ds < +\infty, \quad \int_a^b (s-a)^{1/2}|f_2(s,0)|ds <$$

$$< +\infty, \quad \int_t^b f_2^*(s,x)ds < +\infty \quad \text{for } a < t < b, \ \ x > 0, \tag{11}$$

then the problem (1), (2_2), (3) *has at least one solution.*

Theorem 4. *Let in the domain* $]a,b[\times R$ *the conditions* (5) *and*

$$\big[f_2(t,x) - f_2(t,y)\big] \operatorname{sgn}(x-y) \geq -\frac{\ell}{(t-a)^2}\,|x-y|$$

be fulfilled, where δ, ℓ_0, *and* ℓ *are positive constants, satisfying the inequality* (7). *If, moreover, the conditions* (11) *hold, then the problem* (1), (2_2), (3) *has one and only one solution.*

Note that the condition (7) in Theorems 3 and 4 is unimprovable and it cannot be replaced by the condition $(7')$.

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