Mem. Differential Equations Math. Phys. 58 (2013), 125–134

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ON THE CONTI–OPIAL TYPE EXISTENCE AND UNIQUENESS THEOREMS FOR GENERAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF IMPULSIVE EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES ACTIONS

Abstract. The general nonlocal boundary value problem is considered for systems of impulsive equations with finite and fixed points of impulses actions. Sufficient conditions are given for the solvability and unique solvability of the problem.

რეზიუმე. სასრულ და ფიქსირებულ იმპულსირ წერტილებიან განტოლებათა სისტემებისთვის განხილულია ზოგადი სახის არაწრფივი სასაზღვრო ამოცანა. მოყვანილია ამ ამოცანის ამოხსნადობისა და ცალსახად ამოხსნადობის საკმარისი პირობები.

2000 Mathematics Subject Classification: 34B37.
Key words and phrases: Nonlocal boundary value problems, nonlinear systems, impulsive equations, solvability, unique solvability, effective conditions.

In the present paper, we consider the system of nonlinear impulsive equations with a finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = I_l(x(\tau_l)) \ \ (l = 1, \dots, m_0), \tag{2}$$

with the general boundary value condition

$$h(x) = 0, \tag{3}$$

where $a < \tau_1 < \cdots < \tau_{m_0} \le b$ (we will assume $\tau_0 = a$ and $\tau_{m_0+1} = b$, if necessary), $-\infty < a < b < +\infty$, m_0 is a natural number, $f = (f_i)_{i=1}^n$ belongs to Carathéodory class $Car([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, $I_l = (I_{li})_{i=1}^n : R^n \to \mathbb{R}^n$ ($l = 1, \dots, m_0$) are continuous operators, and $h : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ is a continuous, nonlinear, in general, vector-functional.

In the paper, the sufficient (among them the effective sufficient) conditions are given for solvability and unique solvability of the general nonlinear impulsive boundary value problem $(1), (2); (3)$. We have established the Conti–Opial type theorems for the solvability and unique solvability of this problem. Analogous problems investigated in [8]– [11], [13] (see also the references therein) deal with the general nonlinear boundary value problems for ordinary differential and functional-differential systems.

Certain results obtained in the paper are more general than those already known even for ordinary differential case.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see e.g. [1]–[7], [12], [14] and the references therein). But the above-mentioned works, as we know, do not contain the results obtained in the present paper.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a,b]$ $(a,b \in R)$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$$|X| = (|x_{ij}|)_{i,j}^{n,m}, \ \ [X]_+ = \frac{|X| + X}{2};$$

$$\mathbb{R}_+^{n\times m} = \Big\{ (x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m) \Big\};$$

$$\mathbb{R}^{(n\times n)\times m} = \mathbb{R}^{n\times n} \times \dots \times \mathbb{R}^{n\times n} \ \ (m\text{-times}).$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix, inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n \times n$-matrix.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a,b] \to R^{n\times m}$, i.e., the sum of total variations of the latter components;

$$V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m},$$

where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \leq b$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limit of the matrix-function $X : [a,b] \to R^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a,b]\big\}.$$

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$);

$C([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all continuous matrix-functions $X : [a,b] \to D$;

$C([a,b], D; \tau_1, \dots, \tau_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$, having the one-sided limits $X(\tau_l-)$ $(l = 1, \dots, m_0)$ and $X(\tau_l+)$ $(l =$

$1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}l\}$ belong to $C([c,d],D)$;

$C_s([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})$ is the Banach space of all $X \in C([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})$ with the norm $\|X\|_s$.

$\widetilde{C}([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$;

$\widetilde{C}([a,b],D;\tau_1,\dots,\tau_{m_0})$ is the set of all matrix-functions $X:[a,b]\to D$, having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are the normed spaces, then the operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

The operator $\varphi: C([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})\to R^n$ is called nondecreasing if for every $x,y\in C([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})$ such that $x(t)\le y(t)$ for $t\in[a,b]$ the inequality $\varphi(x)(t)\le\varphi(y)(t)$ holds for $t\in[a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1\subset R^n$ and $D_2\subset R^{n\times m}$, then $Car([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

(b) the function $f_{kj}(t,\cdot):D_1\to D_2$ is continuous for almost all $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)|:\ x\in D_0\big\}\in L([a,b],R;g_{ik})\ \text{ for every compact }\ D_0\subset D_1.$$

$Car^0([a,b]\times D_1,D_2)$ is the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that the functions $f_{kj}(\cdot,x(\cdot))$ $(i=1,\dots,l;\ k=1,\dots,n)$ are measurable for every vector-function $x:[a,b]\to\mathbb{R}^n$ with a bounded total variation.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x\in\widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ satisfying both the system (1) a.e. on $[a,b]\setminus\{\tau_1\dots,\tau_{m_0}\}$ and the relation (2) for every $k\in\{1,\dots,m_0\}$.

Definition 1. Let $\ell: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ be a linear continuous operator, and let $\ell_0: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$ be a positive homogeneous operator. We say that a pair $(P,\{J_l\}_{l=1}^{m_0})$, consisting of a matrix-function $P\in Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a finite sequence of continuous operators $J_l=(J_{li})_{i=1}^n:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$, satisfy the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi \in L([a,b],\mathbb{R}^n_+)$ and constant matrices $\Psi_l \in \mathbb{R}^{n\times n} (l=1,\dots,m_0)$ such that

$$|P(t,x)| \le \Phi(t) \ \text{ a.e. on } \ [a,b], \ \ x \in \mathbb{R}^n$$

and

$$|J_l(x)| \le \Psi_l \ \text{ for } \ x \in \mathbb{R}^n \ \ (l=1,\dots,m_0);$$

(b)

$$\det(I_{n\times n} + G_l) \ne 0 \ \ (l=1,\dots,m_0) \tag{4}$$

and the problem

$$\frac{dx}{dt} = A(t)x \ \text{ a.e. on } \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{5}$$

$$x(\tau_l+) - x(\tau_l-) = G_l x(\tau_l) \ \ (l=1,\dots,m_0), \tag{6}$$

$$|\ell(x)| \le \ell_0(x) \tag{7}$$

has only the trivial solution for every matrix-function $A \in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices G_l $(l=1,\dots,m_0)$ for which there exists a sequence $y_k \in \widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty} \int\limits_a^t P(\tau, y_k(\tau))\,d\tau = \int\limits_a^t A(\tau)\,d\tau \ \text{ uniformly on } \ [a,b]$$

and

$$\lim_{k\to+\infty} J_l(y_k(\tau_l)) = G_l \ \ (l=1,\dots,m_0).$$

Remark 1. In particular, the condition (4) holds if

$$\|\Psi_l\| < 1 \ \ (l=1,\dots,m_0).$$

Below, we will assume that $f=(f_i)_{i=1}^n \in Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, in addition, $f(\tau_l,x)$ is arbitrary for $x \in \mathbb{R}^n$ $(l=1,\dots,m_0)$.

Theorem 1. *Let the conditions*

$$\big\|f(t,x)-P(t,x)x\big\| \le \alpha(t,\|x\|) \ \textit{ a.e. on } \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x\in\mathbb{R}^n, \tag{8}$$

$$\big\|I_l(x) - J_l(x)x\big\| \le \beta_l(\|x\|) \ \textit{ for } \ x \in \mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{9}$$

and

$$\big|h(x) - \ell(x)\big| \le \ell_0(x) + \ell_1(\|x\|_s) \ \textit{ for } \ x \in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{10}$$

hold, where

$$\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n \ \textit{ and } \ \ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$$

are, respectively, linear continuous and positive homogeneous continuous operators, the pair $(P,\{J_l\}_{l=1}^{m_0})$ *satisfies the Opial condition with respect to the pair* (ℓ,ℓ_0)*;* $\alpha \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the* *second variable, and* $\beta_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}\bigg(\|\ell_1(\rho)\| + \int\limits_a^b \alpha(t,\rho)\,dt + \sum_{l=1}^{m_0} \beta_l(\rho)\bigg) < 1. \tag{11}$$

Then the problem (1), (2); (3) *is solvable.*

Theorem 2. *Let the conditions* (8)–(10),

$$P_1(t) \le P(t,x) \le P_2(t) \ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n,$$

and

$$J_{1l} \le I_k(x) \le J_{2l} \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0)$$

hold, where $P \in Car^0([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i \in L([a,b],\mathbb{R}^{n\times n})$ $(i = 1,2)$, $J_{il} \in \mathbb{R}^{n\times n}$ $(i = 1,2;\ l = 1,\dots,m_0)$, $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators;* $\alpha \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the second variable, and* $\beta_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover, the condition* (4) *hold and the problem* (5), (6); (7) *have only the trivial solution for every matrix-function* $A \in L([a,b],\mathbb{R}^{n\times n})$ *and constant matrices* $G_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *such that*

$$P_1(t) \le A(t) \le P_2(t) \ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n$$

and

$$J_{1l} \le G_l \le J_{2l} \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0).$$

Then the problem (1), (2); (3) *is solvable.*

Remark 2. Theorem 1.2 is of interest only in the case where $P \notin Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem follows immediately from Theorem 1.1 in the case where $P \in Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 3. *Let the conditions* (10),

$$\begin{aligned} &|f(t,x) - P_0(t)x| \le \\ &\qquad \le Q(t)|x| + q(t,\|x\|) \ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n, \end{aligned}$$

and

$$\big|I_l(x) - J_{0l}\cdot x\big| \le H_l|x| + h_l(\|x\|) \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0)$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $Q \in L([a,b],\mathbb{R}^{n\times n}_+)$, J_{0l} *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices,* $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, the linear continuous and positive homogeneous continuous operators;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and*

$h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function such that

$$\det(I_{n\times n} + J_{0l}) \neq 0 \ \ (l = 1,\dots,m_0) \tag{12}$$

and

$$\|H_l\| \cdot \|(I_{n\times n} + J_{0l})^{-1}\| < 1 \ \ (j = 1,2;\ l = 1,\dots,m_0) \tag{13}$$

hold, and the system of impulsive inequalities

$$\Big|\frac{dx}{dt} - P_0(t)x\Big| \leq Q(t)x \ \ \text{a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{14}$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_{0l}x(\tau_l)\big| \leq H_l \cdot x(\tau_l) \ \ (l = 1,\dots,m_0) \tag{15}$$

have only the trivial solution under the condition (7)*. Then the problem* (1), (2)*;* (3) *is solvable.*

Corollary 1. *Let the conditions* (12)

$$\big|f(t,x) - P_0(t)x\big| \leq q(t,\|x\|) \ \ \text{a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n, \tag{16}$$

and

$$\big|I_l(x) - J_{0l} \cdot x\big| \leq h_l(\|x\|) \ \ \text{for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0) \tag{17}$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are the constant matrices, $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators; $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ is a vector-function, nondecreasing in the second variable, and $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function such that the condition (11) *holds. Let, moreover,*

$$\big|h(x) - \ell(x)\big| \leq \ell_1(\|x\|_s) \ \ \text{for } x \in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{18}$$

and the impulsive system

$$\frac{dx}{dt} = P_0(t)x \ \ \text{a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\},$$
$$x(\tau_l+) - x(\tau_l-) = J_{0l}x(\tau_l) \ \ (l = 1,\dots,m_0)$$

have only the trivial solution under the condition

$$\ell(x) = 0.$$

Then the problem (1), (2)*;* (3) *is solvable.*

For every matrix-function $X \in L([a,b],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k=1,\dots,m_0)$ we introduce the operators

$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \text{ for } \ a \le t \le b,$$

$$\big[(X,Y_1,\dots,Y_{m_0})(a)\big]_i = O_{n\times n} \ \ (i=1,2,\dots),$$

$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int\limits_a^t X(\tau)\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau +$$

$$+\sum_{a\le\tau_l<t} Y_l\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \text{ for } \ a<t\le b \ \ (i=1,2,\dots). \tag{19}$$

Corollary 2. *Let the conditions* (12)*,* (16)–(18) *hold, where*

$$\ell(x) \equiv \int\limits_a^b dL(t)\cdot x(t),$$

$P_0 \in L([a,b],\mathbb{R}^{n\times n})$*,* $J_{0l} \in \mathbb{R}^{n\times n}(l=1,\dots,m_0)$ *are constant matrices,* $L \in L([a,b],\mathbb{R}^{n\times n})$*,* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int\limits_a^b dL(t)\cdot\big[(P_0,G_l,\dots,G_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{20}$$

where the operators $[(P_0,G_1,\dots,G_{m_0})(t)]_i$ $(i=0,1,\dots)$ *are defined by* (19)*, and*

$$M_{k,m} = \Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(b)\Big]_m + \\ +\sum_{i=0}^{m-1}\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(b)\Big]_i\times \\ \times\int\limits_a^b dV(M_k^{-1}L)(t)\cdot\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(t)\Big]_k.$$

Then the problem (1)*,* (2)*;* (3) *is solvable.*

Corollary 3. *Let the conditions* (12)*,* (16)–(18) *and*

$$\ell(x) \equiv \sum_{j=1}^{n_0} L_j x(t_j) \tag{21}$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices,* $t_j \in [a,b]$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$, $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover, there exist natural numbers* l *and* m *such that the matrix*

$$M_k = \sum_{j=1}^{n_0}\sum_{i=0}^{k-1} L_j\big[(P_0,G_l,\dots,G_{m_0})(t_j)\big]_i$$

is nonsingular and the inequality (20) *holds, where*

$$M_{k,m} = \Big[\big(|P_0|,|G_l|,\dots,|G_{m_0}|\big)(b)\Big]_m + \\ + \Big(\sum_{i=0}^{m-1}\Big[\big(|P_0|,|G_l|,\dots,|G_{m_0}|\big)(b)\Big]_i\Big)\times \\ \times\sum_{j=1}^{n_0}|M_k^{-1}L_j|\cdot\Big[\big(|P_0|,|G_l|,\dots,|G_{m_0}|\big)(t_j)\Big]_k.$$

Then the problem (1), (2); (3) *is solvable.*

Corollary 4. *Let the conditions* (12), (16)–(18) *and* (21) *hold, where* $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices,* $t_j \in [a,b]$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$, $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0} L_j\Big) \neq 0 \ \ \text{and} \ \ r(L_0\cdot V(A)(b)) < 1,$$

where

$$L_0 = I_{n\times n} + \Big|\Big(\sum_{j=1}^{n_0} L_j\Big)^{-1}\Big|\cdot\sum_{j=1}^{n_0}|L_j| \ \ \text{and} \ \ A_0 = \int_a^b |P_0(t)|\,dt + \sum_{l=1}^{m_0}|G_l|.$$

Then the problem (1), (2); (3) *is solvable.*

Theorem 4. *Let the conditions* (12), (13),

$$\big|f(t,x) - f(t,y) - P_0(t)(x-y)\big| \le Q(t)|x-y| \\ \text{a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x,y\in\mathbb{R}^n, \\ \big|I_l(x)-I_l(y)-J_{0l}\cdot(x-y)\big| \le H_k\cdot|x-y| \ \ \text{for} \ \ x,y\in\mathbb{R}^n \ \ (k=l,\dots,m_0)$$

and

$$\big|h(x)-h(y)-\ell(x-y)\big|\le \ell_0(x-y) \ \ \textit{for}\ \ x,y\in \mathrm{BV}([a,b],\mathbb{R}^n)$$

hold, where $P_0\in L([a,b],\mathbb{R}^{n\times n})$, $Q\in L([a,b],\mathbb{R}_+^{n\times n})$, J_{0k} *and* $H_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices,* $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}_+^n$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities* (14), (15) *have only the trivial solution under the condition* (7). *Then the problem* (1), (2); (3) *is solvable.*

Acknowledgement

The present paper was supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

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(Received 14.03.2013)

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