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ON TWO–POINT BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STRONG SINGULARITIES

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. For higher order linear singular functional differential equations, the Agarwal–Kiguradze type theorems on the unique solvability of two-point boundary value problems are proved.

რეზიუმე. მაღალი რიგის წრფივი სინგულარული ფუნქციონალურ-დიფერენციალური განტოლებებისათვის დამტკიცებულია აგარვალ-კიღურაძის ტიპის თეორემები ორწერტილოვანი სასაზღვრო ამოცანების ცალსახად ამოხსნადობის შესახებ.

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Consider the functional differential equation

$$u^{(2m)}(t)=p(t)u(\tau(t))+q(t) \tag{1}$$

with the boundary conditions

$$u^{(i-1)}(a)=0, \quad u^{(i-1)}(b)=0 \ \ (i=1,\dots,m), \quad \int_a^b |u^{(m)}(s)|^2 ds<+\infty \tag{2}$$

or

$$u^{(i-1)}(a)=0, \ \ u^{(m+i-1)}(b)=0 \ \ (i=1,\dots,m), \quad \int_a^b |u^{(m)}(s)|^2 ds<+\infty. \tag{3}$$

Here m is a natural number, $-\infty<a<b<+\infty$, $\tau:[a,b]\to[a,b]$ is a measurable function, and the functions p and $q:]a,b[\to\mathbb{R}$ are Lebesgue integrable on $[a+\varepsilon,b-\varepsilon]$ for arbitrarily small $\varepsilon>0$. However, these functions may be non-integrable on $[a,b]$, having singularities at the endpoints of that interval. In that sense, the equation (1) is singular.

For $\tau\equiv t$, the equation (1) has the form

$$u^{(n)}(t)=p(t)u(t)+q(t). \tag{4}$$

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From the results of the monographs [1, 4] and the papers [3, 5, 7–15] it follow rather delicate conditions guaranteeing the existence of a unique solution of the singular differential equation (4), satisfying the boundary conditions

$$u^{(i-1)}(a)=0, \quad u^{(i-1)}(b)=0 \ \ (i=1,\dots,m) \tag{5}$$

or

$$u^{(i-1)}(a)=0, \quad u^{(m+i-1)}(b)=0 \ \ (i=1,\dots,m). \tag{6}$$

However, all these results concern the cases, where the function p satisfies either the condition

$$\int_a^b (t-a)^{2m-1}(b-t)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt<+\infty, \tag{7}$$

or the condition

$$\int_a^b (t-a)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt<+\infty. \tag{8}$$

Note that if the condition (7) (the condition (8)) is satisfied, then (1), (2) and (4), (5) ((1), (3) and (4), (6)) are equivalent problems. However, if

$$\int_a^b (t-a)^{2m-1}(b-t)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt=+\infty \tag{9}$$

or

$$\int_a^b (t-a)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt=+\infty, \tag{10}$$

then the above-mentioned problems are not equivalent. More precisely, from the unique solvability of the problem (1), (2) (of the problem (1), (3)) it does not follow the unique solvability of the problem (4), (5) (of the problem (4), (6)). In that case we will say that the function p has strong singularities.

By I. Kiguradze and R. P. Agarwal [2, 6], unimprovable sufficient conditions are found for the unique solvability of the problem (4), (2) (of the problem (4), (3)), which cover the cases when the function p has strong singularities. In the present paper, the Agarwal–Kiguradze type results are established for the equation (1).

Throughout the paper we use the following notation.

$[x]_+$ is the positive part of a number x, i.e.,

$$[x]_+=\frac{x+|x|}{2}.$$

$L_{loc}(\,]a,b[\,)$ ($L_{loc}(\,]a,b]\,)$) is the space of functions $y:]a,b[\to\mathbb{R}$ which are integrable on $[a+\varepsilon,b-\varepsilon]$ (on $[a+\varepsilon,b]$) for arbitrarily small $\varepsilon>0$.

$L_{\alpha,\beta}(\,]a,b[)$ is the space of integrable with the weight $(t-a)^{\alpha}(b-t)^{\beta}$ functions $y :]a,b[\to \mathbb{R}$ with the norm

$$\|y\|_{L_{\alpha,\beta}} = \int_a^b (t-a)^{\alpha}(b-t)^{\beta}|y(t)|dt.$$

$$h_1(p)(t) = (2m-1)\Big|\int_c^t [(-1)^m p(s)]_+ ds\Big| \quad \text{for } a<t<b, \;\; c = \frac{a+b}{2},$$

$$h_2(p)(t) = (2m-1)\int_t^b [(-1)^m p(s)]_+ ds \quad \text{for } a<t<b,$$

$$(2m-1)!! = \prod_{i=1}^{2m}(2i-1), \;\; \mu_m = \Big(\frac{2^m}{(2m-1)!!}\Big)^2, \;\; \nu_m = 2\big((m-1)!(2m-1)\big)^{-\frac{1}{2}}.$$

Theorem 1. *Let $p \in L_{loc}(\,]a,b[)$ and let there exist a nonnegative constant ℓ such that*

$$(t-a)^{2m-1}h_1(p)(t) \le \ell \quad \textit{for } a<t<c$$

and

$$(b-t)^{2m-1}h_1(p)(t) \le \ell \quad \textit{for } c<t<b.$$

Let, moreover,

$$\mu_m\ell + \Big(\frac{b-a}{\pi}\Big)^{m-1}\nu_m\Bigg(\int_a^c (s-a)^{m-\frac{1}{2}}|\tau(s)-s|^{\frac{1}{2}}|p(s)|ds + \\ + \int_c^b (b-s)^{m-\frac{1}{2}}|\tau(s)-s|^{\frac{1}{2}}|p(s)|ds\Bigg) < 1. \tag{11}$$

Then for every $q \in L_{m-\frac{1}{2},m-\frac{1}{2}}(\,]a,b[)$ the problem (1), (2) *has one and only one solution.*

Corollary 1. *Let $p \in L_{loc}(\,]a,b[)$ and let there exist a nonnegative constant ℓ such that*

$$(-1)^m p(t) \le \ell(t-a)^{-2m} \quad \textit{for } a<t<c$$

and

$$(-1)^m p(t) \le \ell(b-t)^{-2m} \quad \textit{for } c<t<b.$$

If, moreover, the inequality (11) *holds, then for every $q \in L_{m-\frac{1}{2},m-\frac{1}{2}}(\,]a,b[)$ the problem* (1), (2) *has one and only one solution.*

Theorem 2. *Let $p \in L_{loc}(\,]a,b])$ and let there exist a nonnegative constant ℓ such that*

$$(t-a)^{2m-1}h_2(p)(t) \le \ell \quad \textit{for } a<t<b.$$

Let, moreover,

$$\mu_m \ell + 2^{m-1}\Big(\frac{b-a}{\pi}\Big)^{m-1}\nu_m \int\limits_a^b (s-a)^{m-\frac{1}{2}}|\tau(s)-s|^{\frac{1}{2}}|p(s)|ds < 1. \qquad (12)$$

Then for every $q \in L_{m-\frac{1}{2},0}(\,]a,b[)$ *the problem* $(1), (3)$ *has one and only one solution.*

Corollary 2. *Let* $p \in L_{loc}(\,]a,b])$ *and let there exist a nonnegative constant* ℓ *such that*

$$(-1)^m p(t) \le \ell (t-a)^{-2m} \quad \textit{for } \ a < t < b.$$

If, moreover, the inequality (12) *holds, then for every* $q \in L_{m-\frac{1}{2},0}(\,]a,b[)$ *the problem* $(1), (3)$ *has one and only one solution.*

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