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THE DIRICHLET AND FOCAL BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER QUASI-HALFLINEAR SINGULAR DIFFERENTIAL EQUATIONS

Dedicated to the blessed memory of my dear friend, Professor T. Chanturia

Abstract. For higher order quasi-halflinear singular differential equations, the Dirichlet and focal boundary value problems are considered. Analogues of the Fredholm first theorem are proved and on the basis of these results optimal in some sense sufficient conditions of solvability of the above-mentioned problems are found.

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1. Statement of the Problem and the Main Notation. In the interval $]a, b[$, we consider the differential equation

$$u^{(2m)} = \sum_{i=1}^k p_i(t) \left(\prod_{j=1}^m |u^{(j-1)}|^{\alpha_{ij}} \right) \operatorname{sgn} u + q(t, u, \dots, u^{(m-1)}) \quad (1)$$

with the Dirichlet boundary conditions

$$u^{(i-1)}(a) = 0, \quad u^{(i-1)}(b) = 0 \quad (i = 1, \dots, m) \quad (2)$$

and with the focal boundary conditions

$$u^{(i-1)}(a) = 0, \quad u^{(m+i-1)}(b) = 0 \quad (i = 1, \dots, m). \quad (3)$$

Here, m and k are arbitrary natural numbers, α_{ij} ($i = 1, \dots, k$; $j = 1, \dots, m$) are nonnegative constants such that

$$\alpha_{i1} > 0, \quad \sum_{j=1}^m \alpha_{ij} = 1 \quad (i = 1, \dots, m), \quad (4)$$

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$-\infty < a < b < +\infty$, the functions $p_i :]a, b[\rightarrow R$ ($i = 1, \dots, n$) are integrable on every compact interval contained in $]a, b[$, and $q :]a, b[\times R^m \rightarrow R$ is the function, satisfying the local Carathéodory conditions.

The equation (1) is said to be singular if at least one of the coefficients p_i ($i = 1, \dots, m$), or the function $q(\cdot, x_1, \dots, x_n)$, is not integrable on $[a, b]$, having singularity at one or at both boundary points of that interval. We say that the equation (1) has a strong singularity at the point a (at the point b) if for some $i \in \{1, \dots, k\}$ the condition

$$\int_a^t (s-a)^{2m-\sum_{j=1}^m j\alpha_{ij}} |p_i(s)| ds = +\infty$$

$$\left(\int_t^b (b-s)^{2m-\sum_{j=1}^m j\alpha_{ij}} |p_i(s)| ds = +\infty \right) \text{ for } a < t < b$$

is fulfilled.

In the case, when if (1) is a linear equation with strong singularities at the points a and b (at the point a), i.e., when $k = 1$, $\alpha_{11} = 1$, $\alpha_{1i} = 0$ for $i > 1$ and

$$\int_a^t (s-a)^{2m-1} |p_1(s)| ds = +\infty,$$

$$\int_t^b (b-s)^{2m-1} |p_1(s)| ds = +\infty \text{ for } a < t < b$$

$$\left(\int_a^t (s-a)^{2m-1} |p_1(s)| ds = +\infty, \int_t^b |p_1(s)| ds < +\infty \text{ for } a < t < b \right),$$

the problem (1), (2) (the problem (1), (3)) is thoroughly investigated in [1].

As for the case, when $\sum_{j=2}^m \alpha_{ij} = 1 - \alpha_{ij} > 0$ ($i = 1, \dots, k$) and the equation

(1) has strong singularities at the points a and b , the above-mentioned problems remain 4n studied. The present paper is devoted to fill up this gap.

Throughout the paper, we use the following notation.

$L([a, b])$ is the space of Lebesgue integrable functions $y : [a, b] \rightarrow R$.

$L_{loc}(]a, b[)$ ($L_{loc}([a, b])$) is the space of functions $y :]a, b[\rightarrow R$ which are integrable on $[a + \varepsilon, b - \varepsilon]$ (on $[a + \varepsilon, b]$) for arbitrarily small $\varepsilon > 0$.

$\tilde{C}_{loc}^{2m-1}(]a, b[)$ ($\tilde{C}_{loc}^{2m-1}([a, b])$) is the space of functions $u :]a, b[\rightarrow R$ which are absolutely continuous together with $u', \dots, u^{2(m-1)}$ on $[a + \varepsilon, b - \varepsilon]$ (on $[a + \varepsilon, b]$) for arbitrarily small $\varepsilon > 0$.

$\tilde{C}^{2m-1,m}(]a, b[)$ ($\tilde{C}^{2m-1,m}(]a, b[)$) is the space of functions $u \in \tilde{C}_{loc}^{2m-1}(]a, b[)$ ($\tilde{C}_{loc}^{2m-1}(]a, b[)$) such that

$$\int_a^b |u^{(m)}(s)|^2 ds < +\infty.$$

$$\alpha_i = 2m + 1 - \sum_{j=1}^m j\alpha_{ij} \quad (i = 1, \dots, k); \quad (5)$$

$$\gamma_{1i} = \frac{2^m}{(2m-1)!!} \prod_{j=1}^m \left(\frac{2^{m-j+1}}{(2m-2j+1)!!} \right)^{\alpha_{ij}},$$

$$\gamma_{2i} = \frac{1}{(m-1)!\sqrt{2m-1}} \prod_{j=1}^m \left(\frac{1}{(m-j)!\sqrt{2m-2j+1}} \right)^{\alpha_{ij}} \quad (6)$$

$(i = 1, \dots, k);$

$$\begin{aligned} & \varphi_{1i}(t) = \\ & = ((t-a)^{-2m} + (b-t)^{-2m})^{\frac{1}{2}} \prod_{j=1}^m \left((t-a)^{2j-2m-2} + (b-t)^{2j-2m-2} \right)^{\frac{\alpha_{ij}}{2}}, \\ & \varphi_{2i}(t) = \\ & = ((t-a)^{1-2m} + (b-t)^{1-2m})^{\frac{1}{2}} \prod_{j=1}^m \left((t-a)^{2j-2m-1} + (b-t)^{2j-2m-1} \right)^{\frac{\alpha_{ij}}{2}} \end{aligned} \quad (7)$$

$(i = 1, \dots, k).$

As it has been said above, we assume that the function $q :]a, b[\times R^m \rightarrow R$ satisfies the local Carathéodory conditions, i.e., $q(t, \cdot, \dots, \cdot) : R^m \rightarrow R$ is continuous for almost all $t \in]a, b[$, $q(\cdot, x_1, \dots, x_m) :]a, b[\rightarrow R$ is measurable for any (x_1, \dots, x_m) , and

$$q^*(\cdot, y_1, \dots, y_m) \in L_{loc}(]a, b[) \quad \text{for } y_1 > 0, \dots, y_m > 0,$$

where

$$q^*(t, y_1, \dots, y_m) = \max \left\{ |q(t, x_1, \dots, x_m)| : |x_1| \leq y_1, \dots, |x_m| \leq y_m \right\}. \quad (8)$$

We investigate the problem (1), (2) in the case, when

$$\begin{aligned} & p_i \in L_{loc}(]a, b[) \quad (i = 1, \dots, k), \\ & \lim_{\rho \rightarrow \infty} \int_a^b \psi_1(t) \frac{q^*(t, \psi_1(t)\rho, \dots, \psi_m(t)\rho)}{\rho} dt = 0, \end{aligned} \quad (9)$$

where $\psi_j(t) = (t-a)^{m-j+\frac{1}{2}}(b-t)^{m-j+\frac{1}{2}}$ ($j = 1, \dots, m$), and the problem (1), (3) in the case, when

$$p_i \in L_{loc}(]a, b]) \quad (i = 1, \dots, k),$$

$$\lim_{\rho \rightarrow +\infty} \int_a^b (t-a)^{m-\frac{1}{2}} \frac{q^*(t, (t-a)^{m-\frac{1}{2}}\rho, \dots, (t-a)^{\frac{1}{2}}\rho)}{\rho} dt = 0. \quad (10)$$

In both cases the function q is sublinear with respect to the phase variables and, consequently, the equation (1) is quasi-halflinear.

In theorems on the existence of positive negative solutions of the problems (1), (2) and (1), (3), on the function q we impose either the restriction

$$\lim_{\rho > 0, \rho \rightarrow 0} \int_a^b (t-a)^m (b-t)^m \frac{q_*(t, \rho(t-a)^m(b-t)^m)}{\rho} dt = +\infty, \quad (11)$$

or the restriction

$$\lim_{\rho > 0, \rho \rightarrow 0} \int_a^b (t-a)^m \frac{q_*(t, \rho(t-a)^m)}{\rho} dt = +\infty, \quad (12)$$

where

$$q_*(t, y) = \inf \left\{ |q(t, x_1, \dots, x_m)| : (x_1, \dots, x_m) \in R^m, |x_1| \geq y \right\}. \quad (13)$$

A function $u \in \tilde{C}_{loc}^{2m-1}(]a, b])$ is said to be a solution of the equation (1) if it satisfies this equation almost everywhere on $]a, b[$. A solution of the equation (1) is said to be a solution of the problem (1), (2) (of the problem (1), (3)) if it satisfies the boundary conditions (2) (the boundary conditions (3)), where by $u^{(i-1)}(a)$ (by $u^{(j-1)}(b)$) it is understood the right (the left) limit of the function $u^{(i-1)}$ (of the function $u^{(j-1)}$) at the point a (at the point b).

For the problems (1), (2) and (1), (3) we have proved the analogues of Fredholm first theorem (see Theorems 1–4) on the basis of which the sufficient conditions of solvability of these problems are established in the spaces $\tilde{C}^{2m-1, m}(]a, b])$ and $\tilde{C}^{2m-1, m}(]a, b[)$ (Theorems 5 and 6). The conditions are also found under which the problem (1), (2) (the problem (1), (3)) along with the trivial has a positive and negative on $]a, b[$ solutions (Theorems 7 and 8). All the above-mentioned theorems cover the case for which equation (1) has strong singularities at the points a and b . These theorems are new not only for a singular case, but also for a regular one, i.e for the case, where

$$p_i \in L([a, b]) \quad (i = 1, \dots, k),$$

$$q^*(\cdot, \rho_1, \dots, \rho_m) \in L([a, b]) \quad \text{for } \rho_1 > 0, \dots, \rho_m > 0$$

(see [2]–[11] and the references therein).

2. Fredholm Type Theorems. Along with (1), let us consider the half-linear homogeneous differential equation

$$u^{(n)} = \lambda \sum_{i=1}^k p_i(t) \left(\prod_{j=1}^m |u^{(j-1)}|^{\alpha_{ij}} \right) \operatorname{sgn} u, \quad (14)$$

depending on the parameter $\lambda \in [0, 1]$.

Theorem 1. *Let the condition (9) be fulfilled and almost everywhere on $]a, b[$ the inequalities*

$$(-1)^m p_i(t) \leq l_i \varphi_{1i}(t) + p_{0i}(t) \varphi_{2i}(t) \quad (i = 1, \dots, k) \quad (15)$$

be satisfied, where l_i ($i = 1, \dots, m$) are nonnegative constants, and $p_{0i} : [a, b] \rightarrow [0, +\infty[$ ($i = 1, \dots, k$) are integrable functions. If, moreover,

$$\sum_{i=1}^k \gamma_{1i} l_i < 1 \quad (16)$$

and for an arbitrary $\lambda \in [0, 1]$ the problem (14), (2) has only the trivial solution in the space $\tilde{C}^{2m-1, m}(]a, b[)$, then the problem (1), (2) has at least one solution in the same space.

Theorem 2. *Let the conditions of Theorem 1 and the condition (11) be fulfilled. If, moreover,*

$$(-1)^m p_i(t) \geq 0 \quad (i = 1, \dots, m), \quad (17)$$

$$(-1)^m q(t, x_1, \dots, x_n) x_1 \geq 0 \quad \text{for } t \in]a, b[, \quad (x_1, \dots, x_m) \in R^m,$$

then the problem (1), (2) in the space $\tilde{C}^{2m-1, m}(]a, b[)$ along with the trivial solution has a positive and a negative on $]a, b[$ solutions.

Theorem 3. *Let the condition (10) be fulfilled and almost everywhere on $]a, b[$ the inequalities*

$$(-1)^m p_i(t) \leq l_i (t-a)^{-\alpha_i} + p_{0i}(t) (t-a)^{1-\alpha_i} \quad (i = 1, \dots, k) \quad (18)$$

be satisfied, where $p_{0i} : [a, b] \rightarrow [0, +\infty[$ ($i = 1, \dots, m$) are integrable functions, and l_i ($i = 1, \dots, m$) are nonnegative constants, satisfying the inequality (16). If, moreover, for an arbitrary $\lambda \in [0, 1]$ the problem (14), (3) has only the trivial solution in the space $\tilde{C}^{2m-1, m}(]a, b[)$, then the problem (1), (3) in the same space has at least one solution.

Theorem 4. *If along with the conditions of Theorem 3 the conditions (12) and (17) are fulfilled, then the problem (1), (3) in the space $\tilde{C}^{2m-1, m}(]a, b[)$ along with trivial solution has a positive and a negative on $]a, b[$ solutions.*

Remark 1. The condition (16) in Theorems 1 and 3 is unimprovable and it cannot be replaced by the condition

$$\sum_{i=1}^k \gamma_{1i} l_i \leq 1. \quad (16')$$

Indeed, if

$$\begin{aligned} k = 1, \quad \alpha_{11} = 1, \quad \alpha_{1j} = 0 \text{ for } j \neq 1, \quad l_1 = (-1)^m 4^{-m} ((2m-1)!!)^2, \\ p_1(t) \equiv l_1(t-a)^{-m}, \\ q(t, x_1, \dots, x_m) \equiv \left(\prod_{i=1}^{2m} (\nu - i + 1) - l_1 \right) (t-a)^{\nu-2m}, \quad \nu > m, \end{aligned}$$

then all the conditions of Theorems 1 and 3 are fulfilled, except (16), instead of which the condition (16') is fulfilled, but nevertheless, as is shown in [1], the problem (1), (2) (the problem (1), (3)) in the case under consideration has no solution in the space $\tilde{C}^{2m-1,m}(]a, b[)$ (in the space $\tilde{C}^{2m-1,m}(]a, b])$).

3. Existence Theorems. On the basis of Theorems 1 and 3 we prove Theorems 5 and 6 below which contain effective conditions for solvability of the problems (1), (2) and (1), (3).

Theorem 5. *Let the condition (9) be fulfilled and almost everywhere on $]a, b[$ the inequalities (15) be satisfied, where l_i ($i = 1, \dots, m$) and $p_{0i} :]a, b[\rightarrow [0, +\infty[$ ($i = 1, \dots, m$) are, respectively, nonnegative numbers and integrable functions, satisfying the inequality*

$$\sum_{i=1}^k \left(\gamma_{1i} l_i + \gamma_{2i} \int_a^b p_{0i}(t) dt \right) < 1. \quad (19)$$

Then the problem (1), (2) in the space $\tilde{C}^{2m-1,m}(]a, b[)$ has at least one solution.

Theorem 6. *Let the condition (10) be fulfilled and almost everywhere on $]a, b[$ the inequalities (18) be satisfied, where l_i ($i = 1, \dots, m$) and $p_{0i} :]a, b[\rightarrow [0, +\infty[$ ($i = 1, \dots, m$) are, respectively, nonnegative numbers and integrable functions, satisfying the inequality (19). Then the problem (1), (3) in the space $\tilde{C}^{2m-1,m}(]a, b[)$ has at least one solution.*

Remark 2. If we use the example given in Remark 1, then it will become clear that the strict inequality (19) in Theorems 4 and 5 cannot be replaced by the nonstrict inequality

$$\sum_{i=1}^k \left(\gamma_{1i} l_i + \gamma_{2i} \int_a^b p_{0i}(t) dt \right) \leq 1.$$

4. Theorems on the Non-Unique Solvability of the Problems (1), (2) and (1), (3).

Theorem 7. *If along with the conditions of Theorem 5 the conditions (11) and (17) are fulfilled, then the problem (1), (2) in the space $\tilde{C}^{2m-1,m}(]a, b[)$ along with the trivial solution has a positive and a negative solutions on $]a, b[$.*

Theorem 8. *If along with the conditions of Theorem 6 are fulfilled the conditions (12) and (17), then the problem (1), (3) in the space $\tilde{C}^{2m-1,m}(\]a, b[)$ along with a trivial solution has a positive and a negative on $\]a, b[$ solutions.*

As examples, we consider the differential equations

$$u^{(2m)} = (-1)^m \prod_{i=1}^k l_i \varphi_{1i}(t) \left(\prod_{j=1}^m |u^{(j-1)}|^{\alpha_{ij}} \right) \operatorname{sgn} u + \\ + (-1)^m (t-a)^{-\mu} (b-t)^{-\mu} q_0(t) |u|^\lambda \operatorname{sgn} u \quad (20)$$

and

$$u^{(2m)} = (-1)^m \prod_{i=1}^k l_i (t-a)^{-\alpha_i} \left(\prod_{j=1}^m |u^{(j-1)}|^{\alpha_{ij}} \right) \operatorname{sgn} u + \\ + (-1)^m (t-a)^{-\mu} q_0(t) |u|^\lambda \operatorname{sgn} u, \quad (21)$$

where $q_0 : \]a, b[\rightarrow]0, +\infty[$ is an integrable function,

$$0 < \lambda < 1, \quad \mu = \left(m - \frac{1}{2} \right) (\lambda + 1),$$

and l_i ($i = 1, \dots, m$) are the nonnegative constants, satisfying the inequality (16). According to Theorem 7 (Theorem 8), the problem (20), (2) (the problem (21), (2)) in the space $\tilde{C}^{2m-1,m}(\]a, b[)$ (in the space $\tilde{C}^{2m-1,m}(\]a, b[)$) along with the trivial solution has a positive and a negative on $\]a, b[$ solutions.

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