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**WIENER–HOPF AND WIENER–HOPF–HANKEL
OPERATORS WITH PIECEWISE-ALMOST
PERIODIC SYMBOLS ON WEIGHTED
LEBESGUE SPACES**

*Dedicated to the memory of
Professor Nikolai Ivanovich Muskhelishvili
on the occasion of his 120th birthday*

Abstract. We consider Wiener–Hopf, Wiener–Hopf plus Hankel, and Wiener–Hopf minus Hankel operators on weighted Lebesgue spaces and having piecewise almost periodic Fourier symbols. The main results concern conditions to ensure the Fredholm property and the lateral invertibility of these operators. In addition, under the Fredholm property, conclusions about the Fredholm index of those operators are also discussed.

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რეზიუმე. ჩვენ ვიხილავთ ვინერ–ჰოფის, ვინერ–ჰოფ პლუს ჰენკელისა და ვინერ–ჰოფ მინუს ჰენკელის ოპერატორებს უბან–უბან თითქმის პერიოდული ფურიეს სიმბოლოებით წონიან ლებეგის სივრცეებში. ძირითადი შედეგები ეხება პირობებს, რომლებიც უზრუნველყოფენ ამ ოპერატორების ფრედჰოლმობასა და შებრუნებადობას. დამატებით, ფრედჰოლმობასთან ერთად, განხილულია ან ოპერატორების ფრედჰოლმის ინდექსი.

1. INTRODUCTION

Wiener–Hopf and Hankel operators are known to be very important objects in the modeling of a great variety of applied problems. In fact, since their first appearance in the first half of the twentieth century, advances on the knowledge of their theory, consequent generalizations and their use have been continuously increasing. This circumstance is not indifferent of the interplay between these operators and singular integral operators – which can be identified in different monographs on the subject (cf., e.g., [3], [10], [17], [18]). Additionally, certain combinations of Wiener–Hopf and Hankel operators have also proved to be quite useful in the applications (and several examples of this can be seen e.g. in some wave diffraction problems when analysed by an operator theory approach [15], [16], [22]). A great part of the study in this kind of operators is concentrated in the description of their Fredholm and invertibility properties. In particular, for several classes of the so-called Fourier symbols of the operators, their Fredholm and invertibility properties are already characterized (see e.g. [1]–[5], [7]–[14], [19]–[21] and the references given there). Despite these advances, for some other classes of Fourier symbols and more general spaces, a complete description of the Fredholm and invertibility properties is still missing.

Within this scope, in the present paper we would like to consider Wiener–Hopf, Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators on weighted Lebesgue spaces and having piecewise-almost periodic Fourier symbols (i.e., a certain combination of piecewise continuous elements with almost periodic elements). The main efforts will be devoted to obtain invertibility and Fredholm descriptions of these operators. In view of stating the formal definitions of the operators under study, we will now introduce some preliminary notation.

Let E be a connected subspace of \mathbb{R} . A (Lebesgue) measurable function $w : E \rightarrow [0, \infty]$ is called a weight if $w^{-1}(\{0, \infty\})$ has (Lebesgue) measure zero. For $1 < p < \infty$, we denote by $L^p(\mathbb{R}, w)$ the usual Lebesgue space with the norm

$$\|f\|_{p,w} := \left(\int_{\mathbb{R}} |f(x)|^p w(x)^p dx \right)^{\frac{1}{p}}.$$

Additionally, $A_p(\mathbb{R})$ will denote the set of all weights w on \mathbb{R} for which the Cauchy singular integral operator $S_{\mathbb{R}}$ given by

$$(S_{\mathbb{R}}f)(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi i} \int_{\mathbb{R} \setminus (x-\epsilon, x+\epsilon)} \frac{f(t)}{t-x} dt, \quad x \in \mathbb{R},$$

is bounded on the space $L^p(\mathbb{R}, w)$. The weights $w \in A_p(\mathbb{R})$ are called *Muckenhoupt weights*.

Let \mathcal{F} denote the Fourier transformation. A function $\phi \in L^\infty(\mathbb{R})$ is a *Fourier multiplier* on $L^p(\mathbb{R}, w)$ if the map $f \mapsto \mathcal{F}^{-1}\phi \cdot \mathcal{F}f$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ into itself and extends to a bounded operator on $L^p(\mathbb{R}, w)$ (notice

that $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ is dense in $L^p(\mathbb{R}, w)$ whenever w belongs to $A_p(\mathbb{R})$. We let $\mathcal{M}_{p,w}$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R}, w)$. We will denote by $A_p^0(\mathbb{R})$ the set of all weights $w \in A_p(\mathbb{R})$ for which the functions $e_\lambda : x \mapsto e^{i\lambda x}$ belong to $\mathcal{M}_{p,w}$ for all $\lambda \in \mathbb{R}$. Let J be the reflection operator given by the rule $J\varphi(x) = \tilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$. We denote by $A_p^e(\mathbb{R})$ the subspace of all weights $w \in A_p(\mathbb{R})$ for which $Jw = w$. Additionally, let $A_p^{e,0}(\mathbb{R}) := A_p^0(\mathbb{R}) \cap A_p^e(\mathbb{R})$.

We shall use $L_+^p(\mathbb{R}, w)$ to denote the subspace of $L^p(\mathbb{R}, w)$ formed by all the functions supported in the closure of $\mathbb{R}_+ = (0, +\infty)$.

In what follows we will consider Wiener–Hopf operators defined by

$$W_\phi = r_+ \mathcal{F}^{-1} \phi \cdot \mathcal{F} : L_+^p(\mathbb{R}, w) \rightarrow L^p(\mathbb{R}_+, w),$$

and so-called Wiener–Hopf–Hankel operators [5], [14], [16], [22] (i.e., Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators) of the form

$$W_\phi \pm H_\phi : L_+^p(\mathbb{R}, w) \rightarrow L^p(\mathbb{R}_+, w) \quad (1.1)$$

with H_ϕ being the Hankel operator defined by

$$H_\phi = r_+ \mathcal{F}^{-1} \phi \cdot \mathcal{F} J.$$

Here, r_+ represents the operator of restriction from $L^p(\mathbb{R}, w)$ into $L^p(\mathbb{R}_+, w)$, $w \in A_p^e(\mathbb{R})$ and $\phi \in \mathcal{M}_{p,w}$ is the so-called Fourier symbol. For such Fourier symbol and weight, the operators in (1.1) are bounded.

2. AUXILIARY MATERIAL

2.1. The algebra of piecewise-almost periodic elements. In this subsection we will introduce the piecewise almost periodic elements (which will take the role of Fourier symbols of our main operators), and consider already some of their characteristics.

The smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all functions $e_\lambda := e^{i\lambda x}$ ($x \in \mathbb{R}$) is denoted by AP and called the algebra of almost periodic functions.

For $\phi \in AP$, there exists a number

$$M(\phi) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T \phi(x) dx$$

which is called the (Bohr) mean value of ϕ .

Let $\mathcal{G}B$ denote the group of all invertible elements of a Banach algebra B .

Theorem 2.1 (Bohr). *If $\phi \in \mathcal{G}AP$, then there exists a real number $k(\phi)$ and a function $\psi \in AP$ such that*

$$\phi(x) = e^{ik(\phi)x} e^{\psi(x)} \text{ for all } x \in \mathbb{R}.$$

The number $k(\phi)$ is uniquely determined and it is called the *mean motion* of ϕ . Considering $\phi \in \mathcal{GAP}$, the mean motion of ϕ can be obtained by

$$k(\phi) = \lim_{T \rightarrow \infty} \frac{(\arg \phi)(T) - (\arg \phi)(0)}{T}, \quad (2.2)$$

where $\arg \phi$ is any continuous argument of ϕ . The *geometric mean value* of the function ϕ is defined by $\mathbf{d}(\phi) = e^{M(\psi)}$.

For $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we denote by PC or $PC(\dot{\mathbb{R}})$ the algebra of all functions $\varphi \in L^\infty(\mathbb{R})$ for which the one-sided limits $\varphi(x_0 - 0) = \lim_{x \rightarrow x_0 - 0} \varphi(x)$, $\varphi(x_0 + 0) = \lim_{x \rightarrow x_0 + 0} \varphi(x)$ exist for each $x_0 \in \dot{\mathbb{R}}$, and by $C(\dot{\mathbb{R}})$ the set of all (bounded and) continuous functions φ on the real line for which the two limits $\varphi(-\infty) := \lim_{x \rightarrow -\infty} \varphi(x)$, $\varphi(+\infty) := \lim_{x \rightarrow +\infty} \varphi(x)$ exist and coincide. Let $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC(\dot{\mathbb{R}})$ and $PC_0 := \{\varphi \in PC : \varphi(\pm\infty) = 0\}$. We denote by $C_{p,w}(\dot{\mathbb{R}})$ ($PC_{p,w}(\dot{\mathbb{R}})$) the closure in $\mathcal{M}_{p,w}$ of the set of all functions $\phi \in C(\dot{\mathbb{R}})$ (resp. $\phi \in PC(\dot{\mathbb{R}})$) with finite total variation.

We define $AP_{p,w}$ as the closure of the set of all almost periodic functions in $\mathcal{M}_{p,w}$. Let $SAP_{p,w}$ denote the smallest closed subalgebra of $\mathcal{M}_{p,w}$ that contains $C_{p,w}(\overline{\mathbb{R}})$ and $AP_{p,w}$, and denote by $PAP_{p,w}$ the smallest closed subalgebra of $\mathcal{M}_{p,w}$ that contains $PC_{p,w}$ and $AP_{p,w}$.

2.2. Operator relations. In order to relate operators and to transfer certain operator properties between different operators, we will be also using some known operator relations.

Definition 2.2. Consider two bounded linear operators $T : X_1 \rightarrow X_2$ and $S : Y_1 \rightarrow Y_2$ acting between Banach spaces. We say that T and S are equivalent, and denote this by $T \sim S$, if there are two boundedly invertible linear operators, $E : Y_2 \rightarrow X_2$ and $F : X_1 \rightarrow Y_1$, such that

$$T = E S F. \quad (2.3)$$

If two operators are equivalent, then they belong to the same *invertibility class*. More precisely, one of these operators is invertible, left-invertible, right-invertible or only generalized invertible, if and only if the other operator enjoys the same property.

Definition 2.3 ([6]). Let $T : X_1 \rightarrow X_2$ and $S : Y_1 \rightarrow Y_2$ be bounded linear operators. We say that T is Δ -related after extension to S if there is a bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \rightarrow X_{2\Delta}$ and invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F, \quad (2.4)$$

where Z is an additional Banach space and I_Z represents the identity operator in Z . In the particular case when $T_\Delta : X_{1\Delta} \rightarrow X_{2\Delta} = X_{1\Delta}$ is the identity operator, we say that the operators T and S are *equivalent after extension* and in such a case we will use the notation $T \overset{*}{\sim} S$.

In the following result, we describe a relation between Wiener–Hopf plus Hankel operators and Wiener–Hopf operators within the present framework. This result is well-known for non-weighted spaces (cf., e.g., [7, Theorem 2.1]) and the corresponding proof in the present case runs in a similar way. Anyway, we choose to present here a complete proof of it for the reader convenience.

Theorem 2.4. *Let $\phi \in \mathcal{GM}_{p,w}$ with $w \in A_p^e(\mathbb{R})$ and $1 < p < \infty$. The Wiener–Hopf plus Hankel operator*

$$W_\phi + H_\phi : L_+^p(\mathbb{R}, w) \rightarrow L^p(\mathbb{R}_+, w)$$

is Δ -related after extension to the Wiener–Hopf operator

$$W_{\phi\tilde{\phi}^{-1}} : L_+^p(\mathbb{R}, w) \rightarrow L^p(\mathbb{R}_+, w).$$

Proof. We shall use the characteristic functions χ_\pm to the positive/negative half-line.

Extending $W_\phi + H_\phi$ on the left by the zero extension operator, $\ell_0 : L^p(\mathbb{R}_+, w) \rightarrow L_+^p(\mathbb{R}, w)$, we obtain

$$W_\phi + H_\phi \sim \ell_0(W_\phi + H_\phi) : L_+^p(\mathbb{R}, w) \rightarrow L_+^p(\mathbb{R}, w).$$

After this we will extend

$$\ell_0(W_\phi + H_\phi) = \chi_+ \mathcal{F}^{-1}(\phi + \phi J) \mathcal{F}|_{\chi_+ L^p(\mathbb{R}, w)}$$

to the full $L^p(\mathbb{R}, w)$ space by using the identity in $L_-^p(\mathbb{R}, w)$. Next, we will extend the obtained operator to $[L^p(\mathbb{R}, w)]^2$ with the help of an auxiliary paired operator:

$$L_\phi = \mathcal{F}^{-1}(\phi - \phi J) \mathcal{F} \chi_+ + \chi_- : L^p(\mathbb{R}, w) \rightarrow L^p(\mathbb{R}, w).$$

Altogether, we have

$$\begin{bmatrix} \ell_0(W_\phi + H_\phi) & 0 & 0 \\ 0 & I_{\chi_- L^p(\mathbb{R}, w)} & 0 \\ 0 & 0 & L_\phi \end{bmatrix} = E_1 \mathcal{W}_\phi F_1$$

with

$$\begin{aligned} E_1 &= \frac{1}{2} \begin{bmatrix} I_{L^p(\mathbb{R}, w)} & J \\ I_{L^p(\mathbb{R}, w)} & -J \end{bmatrix}, \\ F_1 &= \begin{bmatrix} I_{L^p(\mathbb{R}, w)} & I_{L^p(\mathbb{R}, w)} \\ J & -J \end{bmatrix} \begin{bmatrix} I_{L^p(\mathbb{R}, w)} - \chi_- \mathcal{F}^{-1}(\phi - \phi J) \mathcal{F} \chi_+ & 0 \\ 0 & I_{L^p(\mathbb{R}, w)} \end{bmatrix}, \\ \mathcal{W}_\phi &= \begin{bmatrix} \mathcal{F}^{-1} \phi \mathcal{F} & 0 \\ \mathcal{F}^{-1} \tilde{\phi} \mathcal{F} & 1 \end{bmatrix} \chi_+ + \begin{bmatrix} 1 & \mathcal{F}^{-1} \phi \mathcal{F} \\ 0 & \mathcal{F}^{-1} \tilde{\phi} \mathcal{F} \end{bmatrix} \chi_- = \\ &= \begin{bmatrix} 1 & \mathcal{F}^{-1} \phi \mathcal{F} \\ 0 & \mathcal{F}^{-1} \tilde{\phi} \mathcal{F} \end{bmatrix} (\mathcal{F}^{-1} \Psi \mathcal{F} \chi_+ + \chi_-) = \\ &= \begin{bmatrix} 1 & \mathcal{F}^{-1} \phi \mathcal{F} \\ 0 & \mathcal{F}^{-1} \tilde{\phi} \mathcal{F} \end{bmatrix} (\chi_+ \mathcal{F}^{-1} \Psi \mathcal{F} \chi_+ + \chi_-) (I_{[L^p(\mathbb{R}, w)]^2} + \chi_- \mathcal{F}^{-1} \Psi \mathcal{F} \chi_+), \end{aligned}$$

where the operators χ_+ and χ_- are here defined on $[L^p(\mathbb{R}, w)]^2$ and

$$\Psi = \begin{bmatrix} 0 & -\widetilde{\phi\phi^{-1}} \\ 1 & \widetilde{\phi^{-1}} \end{bmatrix}.$$

The paired operator

$$I_{[L^p(\mathbb{R}, w)]^2} + \chi_- \mathcal{F}^{-1} \Psi \mathcal{F} \chi_+ : [L^p(\mathbb{R}, w)]^2 \rightarrow [L^p(\mathbb{R}, w)]^2$$

is an invertible operator with inverse given by

$$I_{[L^p(\mathbb{R}, w)]^2} - \chi_- \mathcal{F}^{-1} \Psi \mathcal{F} \chi_+ : [L^p(\mathbb{R}, w)]^2 \rightarrow [L^p(\mathbb{R}, w)]^2.$$

Thus, we have demonstrated that $W_\phi + H_\phi$ is Δ -related after extension with

$$W_\Psi = r_+ \mathcal{F}^{-1} \Psi \mathcal{F} : [L^p_+(\mathbb{R}, w)]^2 \rightarrow [L^p(\mathbb{R}_+, w)]^2.$$

Furthermore, we have

$$\begin{aligned} \begin{bmatrix} W_{\widetilde{\phi\phi^{-1}}} & 0 \\ 0 & I_{[L^p(\mathbb{R}_+, w)]} \end{bmatrix} &= \\ &= W_\Psi \ell_0 r_+ \mathcal{F}^{-1} \begin{bmatrix} \widetilde{\phi^{-1}} & 1 \\ -1 & 0 \end{bmatrix} \mathcal{F} \ell_0 : [L^p(\mathbb{R}_+, w)]^2 \rightarrow [L^p(\mathbb{R}_+, w)]^2 \end{aligned}$$

which shows an explicit equivalence after extension relation between $W_{\widetilde{\phi\phi^{-1}}}$ and W_Ψ . This, together with the Δ -relation after extension between $W_\phi + H_\phi$ and W_Ψ , concludes the proof. \square

Remark 2.5. From the proof of the last theorem we can also realize the last result as an equivalence after extension between the diagonal matrix operator $\text{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ and $W_{\widetilde{\phi\phi^{-1}}}$.

3. WIENER–HOPF OPERATORS ON WEIGHTED LEBESGUE SPACES

3.1. Fredholm theory for Wiener–Hopf operators with piecewise continuous symbols on weighted Lebesgue spaces. In the present subsection we will recall a Fredholm characterization of Wiener–Hopf operators with piecewise continuous Fourier symbols on weighted Lebesgue spaces (which we will use later on).

Let $\nu \in (0, 1)$. The set $\{e^{2\pi(x+i\nu)} : x \in \mathbb{R}\}$ is a ray starting at the origin and making the angle $2\pi\nu \in (0, 2\pi)$ with the positive real half-line. For $z_1, z_2 \in \mathbb{C}$, the Möbius transform

$$M_{z_1, z_2}(\zeta) := \frac{z_2 \zeta - z_1}{\zeta - 1}$$

maps 0 to z_1 and ∞ to z_2 . Thus,

$$\mathcal{A}(z_1, z_2; \nu) := \{M_{z_1, z_2}(e^{2\pi(x+i\nu)}) : x \in \mathbb{R}\} \cup \{z_1, z_2\}$$

is a circular arc between z_1 and z_2 (which contains its endpoints z_1, z_2). Finally, given $0 < \nu_1 \leq \nu_2 < 1$, we put

$$\mathcal{H}(z_1, z_2; \nu_1, \nu_2) := \bigcup_{\nu \in [\nu_1, \nu_2]} \mathcal{A}(z_1, z_2; \nu),$$

and refer to $\mathcal{H}(z_1, z_2; \nu_1, \nu_2)$ as the horn between z_1 and z_2 determined by ν_1 and ν_2 .

Let $1 < p < \infty$ and $w \in A_p(\mathbb{R})$. Then, each of the sets

$$\begin{aligned} I_x(p, w) &:= \{ \lambda \in \mathbb{R} : |(\xi - x)/(\xi + i)|^\lambda w(\xi) \in A_p(\mathbb{R}) \}, \quad x \in \mathbb{R}, \\ I_\infty(p, w) &:= \{ \lambda \in \mathbb{R} : |\xi + i|^{-\lambda} w(\xi) \in A_p(\mathbb{R}) \} \end{aligned} \quad (3.5)$$

is an open interval of length no greater than 1 which contains the origin:

$$I_x(p, w) = (-\nu_x^-(p, w), 1 - \nu_x^+(p, w)), \quad x \in \mathbb{R}, \quad (3.6)$$

with $0 < \nu_x^-(p, w) \leq \nu_x^+(p, w) < 1$.

Theorem 3.1 ([3, Theorem 17.7]). *Let $1 < p < \infty$, $w \in A_p(\mathbb{R})$, and let $\nu_x^\pm(p, w)$ be defined by (3.5)–(3.6). If $\psi \in PC_{p,w}$, then the operator W_ψ is Fredholm on the space $L^p(\mathbb{R}_+, w)$ if and only if*

$$\begin{aligned} 0 \notin \psi_{p,w}^\#(\mathbb{R}) &:= \left(\bigcup_{x \in \mathbb{R}} \mathcal{H}(\psi(x-0), \psi(x+0); \nu_\infty^-(p, w), \nu_\infty^+(p, w)) \right) \cup \\ &\quad \cup \mathcal{H}(\psi(+\infty), \psi(-\infty); \nu_0^-(p, w), \nu_0^+(p, w)). \end{aligned}$$

If W_ψ is Fredholm on the space $L^p(\mathbb{R}_+, w)$, then

$$\text{Ind } W_\psi = -\text{wind}_{p,w} \psi, \quad (3.7)$$

where $\text{wind}_{p,w} \psi$ is the winding number about the origin of the naturally oriented curve

$$\begin{aligned} \psi_{p,w}^0(\mathbb{R}) &:= \left(\bigcup_{x \in \mathbb{R}} \mathcal{A}(\psi(x-0), \psi(x+0); \nu_\infty^0(p, w)) \right) \cup \\ &\quad \cup \mathcal{A}(\psi(+\infty), \psi(-\infty); \nu_0^0(p, w)), \end{aligned}$$

with

$$\nu_x^0(p, w) := \frac{\nu_x^-(p, w) + \nu_x^+(p, w)}{2}. \quad (3.8)$$

Suppose that $\psi \in PC_{p,w}$ has only finitely many jumps at the points $\Lambda_\psi \subset \mathbb{R}$ and possibly at ∞ . If $0 \notin \psi_{p,w}^\#(\mathbb{R})$, then the *Cauchy index* $\text{ind}_{p,w} \psi$ of ψ with respect to p and w is defined by

$$\text{ind}_{p,w} \psi := \sum_l \text{ind}_l \psi + \sum_{x \in \Lambda_\psi} \left(-\nu_x^0(p, w) + \left\{ \nu_x^0(p, w) + \frac{1}{2\pi} \arg \frac{\psi(x+0)}{\psi(x-0)} \right\} \right),$$

where l ranges over the connected components of $\mathbb{R} \setminus \Lambda_\psi$, $\{c\}$ denotes the fractional part of the real number c and $\text{ind}_l \psi$ stands for the increment of

$\frac{1}{2\pi} \arg \psi$ on l , with $\arg \psi$ being any continuous argument of ψ on l . Additionally, we have that

$$\text{wind}_{p,w} \psi = \text{ind}_{p,w} \psi + \left(-\nu_{\infty}^0(p, w) + \left\{ \nu_{\infty}^0(p, w) + \frac{1}{2\pi} \arg \frac{\psi(-\infty)}{\psi(+\infty)} \right\} \right).$$

Thus, we can also write (3.7) in the form

$$\begin{aligned} \text{Ind} W_{\psi} &= - \sum_l \text{ind}_l \psi + \\ &+ \sum_{x \in \Lambda_{\psi} \cup \{\infty\}} \left(\nu_x^0(p, w) - \left\{ \nu_x^0(p, w) + \frac{1}{2\pi} \arg \frac{\psi(x+0)}{\psi(x-0)} \right\} \right), \end{aligned} \quad (3.9)$$

where $\psi(\infty \pm 0) := \psi(\mp \infty)$.

3.2. Wiener–Hopf operators with semi-almost periodic symbols on weighted Lebesgue spaces.

3.2.1. *Representation of semi-almost periodic functions.* The following theorem is an analogue of the corresponding classic Sarason’s result.

Theorem 3.2 ([13, Theorem 3.1.]). *Let $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$ and let u be a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$. Then, every function $\phi \in \text{SAP}_{p,w}$ can be uniquely represented in the form:*

$$\phi = (1 - u)\phi_{\ell} + u\phi_r + \phi_0,$$

where $\phi_{\ell}, \phi_r \in AP_{p,w}$, $\phi_0 \in C_{p,w}(\dot{\mathbb{R}})$ and $\phi_0(\infty) = 0$. The maps $\phi \mapsto \phi_{\ell}$ and $\phi \mapsto \phi_r$ are (continuous) Banach algebra homomorphisms of $\text{SAP}_{p,w}$ onto $AP_{p,w}$ of norm 1, where $\|\phi\|_{p,w} = \|\mathcal{F}^{-1}\phi \cdot \mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R}, w))}$.

3.2.2. *Fredholm theory for Wiener–Hopf operators with semi-almost periodic symbols on weighted Lebesgue spaces.* Let us recall an analogue of *Duduchava–Saginashvili Theorem* for weighted Lebesgue spaces $L^p(\mathbb{R}_+, w)$ with Muckenhoupt weights $w \in \mathcal{A}_p^0(\mathbb{R})$.

Theorem 3.3 ([13, Proposition 4.7]). *Let $\phi \in \text{SAP}_{p,w} \setminus \{0\}$, with $1 < p < \infty$ and $w \in \mathcal{A}_p^0(\mathbb{R})$.*

- (a) *If $\phi \notin \mathcal{GSAP}$, then W_{ϕ} is not semi-Fredholm on $L_+^p(\mathbb{R}, w)$.*
- (b) *If $\phi \in \mathcal{GSAP}$ and $k(\phi_{\ell})k(\phi_r) < 0$, then W_{ϕ} is not semi-Fredholm on $L_+^p(\mathbb{R}, w)$.*
- (c) *If $\phi \in \mathcal{GSAP}$, $k(\phi_{\ell})k(\phi_r) \geq 0$ and $k(\phi_{\ell}) + k(\phi_r) > 0$, then W_{ϕ} is properly n -normal on $L_+^p(\mathbb{R}, w)$ and left-invertible.*
- (d) *If $\phi \in \mathcal{GSAP}$, $k(\phi_{\ell})k(\phi_r) \geq 0$ and $k(\phi_{\ell}) + k(\phi_r) < 0$, then W_{ϕ} is properly d -normal on $L_+^p(\mathbb{R}, w)$ and right-invertible.*
- (e) *If $\phi \in \mathcal{GSAP}$, $k(\phi_{\ell}) = k(\phi_r) = 0$ and*

$$0 \notin \mathcal{H}(\mathbf{d}(\phi_r), \mathbf{d}(\phi_{\ell}); \nu_0^-(p, w), \nu_0^+(p, w)),$$

then W_{ϕ} is Fredholm on $L_+^p(\mathbb{R}, w)$.

(f) If $\phi \in \mathcal{GSAP}$, $k(\phi_\ell) = k(\phi_r) = 0$ and

$$0 \in \mathcal{H}(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(p, w), \nu_0^+(p, w)),$$

then W_ϕ is not semi-Fredholm on $L_+^p(\mathbb{R}, w)$.

We would like to point out that although in [13, Proposition 4.7] do not appear the above left and right-invertibility conclusions (here added in propositions (c) and (d)), these lateral invertibility properties arise directly from the use of *Coburn-Simonenko Theorem* (since we are considering scalar Wiener–Hopf operators).

Lemma 3.4 ([3, Lemma 3.12]). *Let $A \subset (0, \infty)$ be an unbounded set and consider $\{I_\alpha\}_{\alpha \in A} := \{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ to be a family of intervals such that $x_\alpha \geq 0$ and $|I_\alpha| = y_\alpha - x_\alpha \rightarrow \infty$ as $\alpha \rightarrow \infty$. If $\phi \in \mathcal{GSAP}$ is such that $k(\phi_\ell) = k(\phi_r) = 0$ and $\arg \phi$ is any continuous argument of ϕ , then the limit*

$$\frac{1}{2\pi} \lim_{\alpha \rightarrow \infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} ((\arg \phi)(x) - (\arg \phi)(-x)) dx \quad (3.10)$$

exists, is finite and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ and $\arg \phi$.

For $\phi \in \mathcal{GSAP}$ such that $k(\phi_\ell) = k(\phi_r) = 0$, the value (3.10) is denoted by $\text{ind } \phi$ and called the *Cauchy index* of ϕ . Following [19, Section 4.3] we can generalize this notion of Cauchy index for *SAP* functions with $k(\phi_\ell) + k(\phi_r) = 0$.

The following theorem provides a formula for the Fredholm index of Wiener–Hopf operators with semi-almost periodic symbols on $L^p(\mathbb{R}_+, w)$.

Theorem 3.5 ([13, Theorem 4.8]). *If $\phi \in \mathcal{GSAP}_{p,w}$, $k(\phi_\ell) = k(\phi_r) = 0$ and*

$$0 \notin \mathcal{H}(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(p, w), \nu_0^+(p, w)),$$

then the operator W_ϕ is Fredholm and

$$\text{Ind } W_\phi = -\text{ind } \phi + \nu_0^0(p, w) - \left\{ \nu_0^0(p, w) + \frac{1}{2\pi} \arg \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)} \right\}, \quad (3.11)$$

where

$$\nu_0^0(p, w) := \frac{\nu_0^-(p, w) + \nu_0^+(p, w)}{2}.$$

3.3. Wiener–Hopf operators with piecewise-almost periodic symbols on weighted Lebesgue spaces. Motivated by the material in the previous subsections, the main purpose of the present subsection will be to establish an analogue invertibility and Fredholm description for Wiener–Hopf operators acting between L^p spaces ($1 < p < \infty$) with Muckenhoupt weights $w \in A_p^0(\mathbb{R})$, and with $PAP_{p,w}$ Fourier symbols.

3.3.1. Representation of $PAP_{p,w}$ piecewise-almost periodic functions.

Theorem 3.6. *Let $w \in A_p^0(\mathbb{R})$, and let u be a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$.*

- (i) *If $\phi \in PAP_{p,w}$, then there are uniquely determined functions φ_r , $\varphi_\ell \in AP_{p,w}$ and $\phi_0 \in PC_{p,w}^0$ such that*

$$\phi = (1 - u)\varphi_\ell + u\varphi_r + \phi_0. \quad (3.12)$$

- (ii) *If $\phi \in \mathcal{G}PAP_{p,w}$, then there exists $\varphi \in \mathcal{G}SAP_{p,w}$ and $\psi \in \mathcal{G}PC_{p,w}$ satisfying $\psi(-\infty) = \psi(+\infty) = 1$, such that $\phi = \varphi\psi$ and*

$$W_\phi = W_\varphi W_\psi + K_1 = W_\psi W_\varphi + K_2, \quad (3.13)$$

with compact operators K_1 and K_2 .

- (iii) *In addition, the φ_ℓ and φ_r elements used in (i) coincide with the local representatives of $\varphi \in \mathcal{G}SAP_{p,w}$ used in (ii) and their unique existence is ensured by Theorem 3.2.*

Proof. Part (i) is an immediate consequence of Theorem 3.2.

To prove part (ii), suppose that ϕ is in $\mathcal{G}PAP_{p,w}$ and put $f := (1 - u)\varphi_\ell + u\varphi_r$ where the elements u , φ_ℓ and φ_r have the properties described in (3.12). Then, $\phi = f + \phi_0$ (with $\phi_0 \in PC_{p,w}^0$). From the hypothesis there is a constant $C \in (0, \infty)$ such that $|f(x)|$ is bounded away from zero for $|x| > C$, and therefore, we can find a function $f_0 \in C_{p,w}^0(\mathbb{R})$ such that $\varphi := f + f_0 \in \mathcal{G}SAP_{p,w}$. Consequently, we have

$$\phi = \varphi + \phi_0 - f_0 = \varphi(1 + \varphi^{-1}(\phi_0 - f_0)) =: \varphi\psi,$$

and it is clear that $\psi = \varphi^{-1}\phi \in \mathcal{G}PC_{p,w}$ and $\psi(-\infty) = \psi(+\infty) = 1$. Since ϕ is continuous on \mathbb{R} and ψ is continuous at ∞ , we deduce that (3.13) holds with compact operators K_1 and K_2 .

The proposition (iii) follows immediately from the construction performed for (ii). \square

3.3.2. Fredholm theory of Wiener–Hopf operators with piecewise-almost periodic functions on weighted Lebesgue spaces. We are now in condition to derive a Fredholm characterization for Wiener–Hopf operators with $PAP_{p,w}$ Fourier symbols on weighted Lebesgue spaces.

Theorem 3.7. *Consider $w \in A_p^0(\mathbb{R})$ and $\phi \in PAP_{p,w}$ such that ϕ is not identically zero.*

- (a) *If $\phi \in \mathcal{G}PAP_{p,w}$, $k(\phi_\ell) = k(\phi_r) = 0$ and*

$$0 \notin \phi_{p,w}^\#(\mathbb{R}) \cup \mathcal{H}(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(p, w), \nu_0^+(p, w)),$$

then W_ϕ is Fredholm on $L^p(\mathbb{R}_+, w)$.

- (b) *If $\phi \in \mathcal{G}PAP_{p,w}$, $k(\phi_\ell)k(\phi_r) \geq 0$, $k(\phi_\ell) + k(\phi_r) > 0$ and $0 \notin \phi_{p,w}^\#(\mathbb{R})$, then W_ϕ is properly n -normal and left-invertible.*
- (c) *If $\phi \in \mathcal{G}PAP_{p,w}$, $k(\phi_\ell)k(\phi_r) \geq 0$, $k(\phi_\ell) + k(\phi_r) < 0$ and $0 \notin \phi_{p,w}^\#(\mathbb{R})$, then W_ϕ is properly d -normal and right-invertible.*

(d) In all other cases, W_ϕ is not normally solvable.

Proof. If $\phi \notin \mathcal{GPAP}_{p,w}$, we see from [3, Corollary 2.8], that W_ϕ is not normally solvable.

So, let us now assume that $\phi \in \mathcal{GPAP}_{p,w}$. Then, we can write $\phi = \varphi\psi$, $\varphi \in \mathcal{GSAP}_{p,w}$ and $\psi \in \mathcal{GPC}_{p,w}$ (satisfying $\psi(-\infty) = \psi(+\infty) = 1$). Taking into account (3.13), we see that W_ϕ is Fredholm if and only if both operators W_φ and W_ψ are Fredholm –which by Theorem 3.1 and Theorem 3.3 happens if conditions stated in part (a) are satisfied.

Having in mind (3.13) and since W_ψ is Fredholm under the conditions of parts (b) and (c) (cf. Theorem 3.1), we deduce that W_ϕ is properly n -normal (resp. properly d -normal) if and only if so is W_φ . Therefore, we obtain part (b) (resp. part (c)) from Theorem 3.3 and *Coburn–Simonenko Theorem*.

To complete the proof, we use the following fact: considering linear and bounded operators A and B acting between Banach spaces (such that AB can be computed), if AB is n -normal (resp. d -normal) then B is n -normal (resp. A is d -normal). This, [3, Theorem 2.2] and (3.13) show that W_ϕ is n -normal (resp. d -normal) if and only if so are both W_φ and W_ψ , and hence we get part (d) for $\phi \in \mathcal{GPAP}$ as a consequence of *Coburn–Simonenko Theorem* and Theorems 3.1 and 3.3. \square

Corollary 3.8. *Let $\phi \in \mathcal{GPAP}_{p,w}$. If W_ϕ is a Fredholm operator, then*

$$\begin{aligned} \text{Ind}W_\phi &= \text{Ind}W_\varphi + \text{Ind}W_\psi = - \sum_l \text{ind}_l \psi - \text{ind} \varphi + \\ &+ \sum_{x \in \Lambda_\psi \cup \{\infty\}} \left(\nu_x^0(p, w) - \left\{ \nu_x^0(p, w) - \frac{1}{2\pi} \arg \frac{\psi(x-0)}{\psi(x+0)} \right\} \right) + \\ &+ \nu_0^0(p, w) - \left\{ \nu_0^0(p, w) + \frac{1}{2\pi} \arg \frac{\mathbf{d}(\varphi_\ell)}{\mathbf{d}(\varphi_r)} \right\}, \end{aligned} \quad (3.14)$$

where $\phi = \psi\varphi$ is a corresponding factorization in the sense of Theorem 3.6 (ii).

Proof. This is obtained by jointing together Theorem 3.6(ii) and formulas (3.9) and (3.11). \square

3.3.3. Example of an invertible Wiener–Hopf operator with a piecewise-almost periodic Fourier symbol on weighted Lebesgue spaces. Let $p = 2$ and choose the weight function $w(x) = |x|^{\frac{1}{4}}$. We will consider the function ϕ (see Figure 1), given by

$$\phi(x) = (1 - u(x))3e^{e^{ix}}g(x) + u(x)e^{e^{-2ix}}g(x) + \frac{g(x)}{x^2 + 1}, \quad (3.15)$$

where u is the real-valued function

$$u(x) = \frac{1}{2} + \frac{1}{\pi} \tanh(x)$$

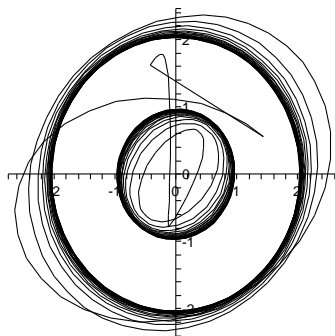


FIGURE 1. The range of $\phi(x)$ for x between -100 and 100 .

$$\text{and } g(x) = \begin{cases} e^x + 1, & \text{if } x < 0 \\ e^{\frac{2i}{x-i}}, & \text{if } x \geq 0 \end{cases}.$$

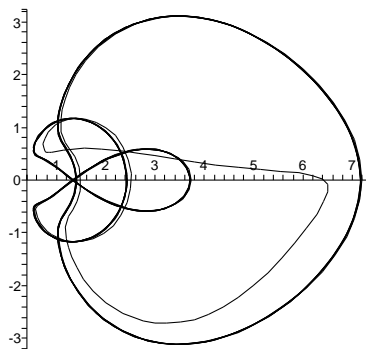


FIGURE 2. The range of $\varphi(x)$ for x between -100 and 100 .

It is clear that ϕ admits a factorization $\phi = \varphi\psi$ in the sense of Theorem 3.6 (ii) with

$$\varphi(x) = (1 - u(x))3e^{ix} + u(x)e^{e^{-2ix}} + \frac{1}{x^2 + 1}, \quad (3.16)$$

and $\psi(x) = g(x)$.

The function φ (cf. Figure 2) is invertible and we have $\varphi \in \mathcal{GSAP}_{2,w}$.

The element ψ is also an invertible function (see Figure 3). Moreover, $\psi(-\infty) = \psi(+\infty) = 1$. Observing that φ and ψ are invertible, one obtains that ϕ is invertible.

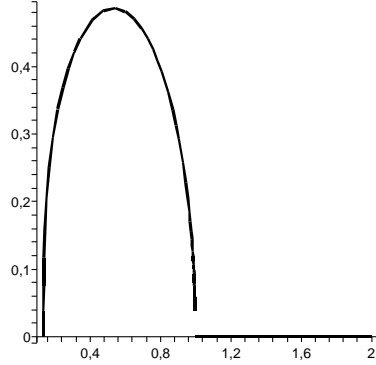


FIGURE 3. The range of $\psi(x)$ for x between -100 and 100.

From Theorem 3.6 (iii), we have that the almost periodic representatives of $\phi \in \mathcal{GPAP}_{2,w}$ coincide with the almost periodic representatives of $\varphi \in \mathcal{GSAP}_{2,w}$. From the definition of $k(\phi)$, it results that $k(\phi_\ell) = k(\phi_r) = 0$.

We have that

$$\begin{aligned} I_0(2, |x|^{\frac{1}{4}}) &= \left\{ \mu \in \mathbb{R} : \left| \frac{\xi}{\xi + i} \right|^\mu |\xi|^{\frac{1}{4}} \in A_2(\mathbb{R}) \right\} = \\ &= \left\{ \mu \in \mathbb{R} : -\frac{1}{2} < \mu + \frac{1}{4} < \frac{1}{2} \right\} = \\ &= \left(-\frac{3}{4}, 1 - \frac{3}{4} \right), \end{aligned}$$

whence $\nu_0^-(2, |x|^{\frac{1}{4}}) = \nu_0^+(2, |x|^{\frac{1}{4}}) = \frac{3}{4}$. In the same way, we obtain

$$\nu_\infty^-(2, |x|^{\frac{1}{4}}) = \nu_\infty^+(2, |x|^{\frac{1}{4}}) = \frac{1}{4}.$$

Consequently, and observing that the only discontinuity point of ϕ is 0, we have

$$\begin{aligned} \phi_{p,w}^\#(\mathbb{R}) &= \mathcal{H}\left(\phi(0-0), \phi(0+0); \nu_\infty^-(2, |x|^{\frac{1}{4}}), \nu_\infty^+(2, |x|^{\frac{1}{4}})\right) \cup \\ &\quad \cup \mathcal{H}\left(\phi(+\infty), \phi(-\infty); \nu_0^-(2, |x|^{\frac{1}{4}}), \nu_0^+(2, |x|^{\frac{1}{4}})\right) = \\ &= \mathcal{H}\left(2e + 1, 2e^3 + e^2; \frac{1}{4}, \frac{1}{4}\right) \cup \mathcal{H}\left(1, 3; \frac{3}{4}, \frac{3}{4}\right) = \\ &= \mathcal{A}\left(2e + 1, 2e^3 + e^2; \frac{1}{4}\right) \cup \mathcal{A}\left(1, 3; \frac{3}{4}\right). \end{aligned}$$

Since $\mathbf{d}(\phi_r) = 1$ and $\mathbf{d}(\phi_\ell) = 3$, it also results that

$$\mathcal{H}\left(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(2, |x|^{\frac{1}{4}}), \nu_0^+(2, |x|^{\frac{1}{4}})\right) = \mathcal{H}\left(1, 3; \frac{3}{4}, \frac{3}{4}\right) = \mathcal{A}\left(1, 3; \frac{3}{4}\right).$$

Therefore, we have to consider the arcs $\mathcal{A}(2e+1, 2e^3+e^2; \frac{1}{4})$ and $\mathcal{A}(1, 3; \frac{3}{4})$ (see Figure 4).

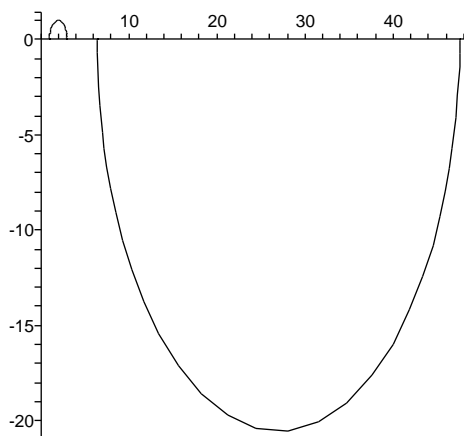


FIGURE 4. The arcs $\mathcal{A}(2e+1, 2e^3+e^2; \frac{1}{4})$ and $\mathcal{A}(1, 3; \frac{3}{4})$.

Since these arcs do not contain the origin, the operator

$$W_\phi : L_+^2(\mathbb{R}, |x|^{\frac{1}{4}}) \rightarrow L^2(\mathbb{R}_+, |x|^{\frac{1}{4}})$$

is a Fredholm operator (cf. Theorem 3.7 (a)).

Let us now compute the Fredholm index of this operator.

From the definition of $\psi(x)$, we have that if $x < 0$ then $\arg \psi = 0$, and if $x \geq 0$ then $\arg \psi = \frac{2x}{x^2+1}$. Thus, $\text{ind}_l \psi = 0$ and consequently $\sum_l \text{ind}_l \psi = 0$.

The only point of discontinuity of ψ is zero and

$$\nu_0^0(2, |x|^{\frac{1}{4}}) = \frac{\nu_0^-(2, |x|^{\frac{1}{4}}) + \nu_0^+(2, |x|^{\frac{1}{4}})}{2} = \frac{3}{4}.$$

Additionally, $\arg \frac{\psi(x+0)}{\psi(x-0)} = \arg \frac{e^2}{2} = 0$.

On the other hand, we have that $\text{ind} \varphi = 0$ and

$$\arg \frac{\mathbf{d}(\varphi_\ell)}{\mathbf{d}(\varphi_r)} = \arg(3) = 0.$$

Using these results and substituting on formula (3.14), we obtain that

$$\text{Ind} W_\phi = 0.$$

Consequently, putting together this information with *Coburn–Simonenko Theorem*, we conclude that the Wiener–Hopf operator of this example is invertible.

4. FREDHOLM AND LATERAL INVERTIBILITY OF
WIENER–HOPF–HANKEL OPERATORS WITH PIECEWISE-ALMOST
PERIODIC FUNCTIONS ON WEIGHTED LEBESGUE SPACES

We will turn now to Wiener–Hopf–Hankel operators with piecewise-almost periodic symbols on Lebesgue spaces with Muckenhoupt weights $w \in A_p^{0,e}(\mathbb{R})$. Here, we are also looking for corresponding possible invertibility and Fredholm properties. In fact, we will be able to identify conditions under which the Wiener–Hopf plus/minus Hankel operators are left or right-invertible (and not Fredholm) or have the Fredholm property.

4.1. Fredholm theory of Wiener–Hopf–Hankel operators with piecewise-almost periodic functions on $L^p(\mathbb{R}_+, w)$. We will now identify conditions to ensure the Fredholm and lateral invertibility of our Wiener–Hopf plus/minus Hankel operators.

Theorem 4.1. *Let $w \in A_p^{e,0}(\mathbb{R})$ and $\phi \in \mathcal{GPAP}_{p,w}$ ($1 < p < \infty$).*

(a) *If $k(\phi_\ell) + k(\phi_r) = 0$ and*

$$0 \notin (\phi\widetilde{\phi^{-1}})_{p,w}^\#(\mathbb{R}) \cup \mathcal{H}\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p, w), \nu_0^+(p, w)\right)$$

then $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are Fredholm operators.

(b) *If $k(\phi_\ell) + k(\phi_r) > 0$ and $0 \notin (\phi\widetilde{\phi^{-1}})_{p,w}^\#(\mathbb{R})$, then both operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are left-invertible (and at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly n -normal).*

(c) *If $k(\phi_\ell) + k(\phi_r) < 0$ and $0 \notin (\phi\widetilde{\phi^{-1}})_{p,w}^\#(\mathbb{R})$, then both operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are right-invertible (and at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly d -normal).*

(d) *If $k(\phi_\ell) + k(\phi_r) = 0$ and*

$$0 \in (\phi\widetilde{\phi^{-1}})_{p,w}^\#(\mathbb{R}) \cup \mathcal{H}\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p, w), \nu_0^+(p, w)\right),$$

then at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is not normally solvable on $L_+^p(\mathbb{R}, w)$.

Proof. From the definition of $PAP_{p,w}$, we have the following representation of ϕ :

$$\phi = (1 - u)\phi_\ell + u\phi_r + \phi_0,$$

where $\phi_\ell, \phi_r \in AP_{p,w}$, $\phi_0 \in PC_{p,w}(\mathbb{R})$, $\phi_0(\infty) = 0$ and u is a monotonically increasing real-valued function in $C(\mathbb{R})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$.

Taking into consideration Bohr's theorem and the definition of the geometric mean value, it follows that

$$\begin{aligned}\phi_\ell &= e_{k(\phi_\ell)} \mathbf{d}(\phi_\ell) e^{w_\ell}, \\ \phi_r &= e_{k(\phi_r)} \mathbf{d}(\phi_r) e^{w_r},\end{aligned}$$

with $w_\ell, w_r \in AP_{p,w}$, $M(w_\ell) = M(w_r) = 0$ (and $\mathbf{d}(\phi_\ell)\mathbf{d}(\phi_r) \neq 0$). Thus,

$$\phi = (1 - u)\mathbf{d}(\phi_\ell)e_{k(\phi_\ell)}e^{w_\ell} + u\mathbf{d}(\phi_r)e_{k(\phi_r)}e^{w_r} + \phi_0. \quad (4.17)$$

Due to the transfer of regularity properties from the Wiener–Hopf operator $W_{\phi\widetilde{\phi}^{-1}}$ to the Wiener–Hopf plus and minus Hankel operators $W_\phi \pm H_\phi$, we will study the regularity properties of the Wiener–Hopf operator $W_{\phi\widetilde{\phi}^{-1}} : L_+^p(\mathbb{R}, w) \rightarrow L^p(\mathbb{R}_+, w)$. In view of this, we obtain

$$\widetilde{\phi\phi^{-1}} = \frac{(1 - u)\mathbf{d}(\phi_\ell)e_{k(\phi_\ell)}e^{w_\ell} + u\mathbf{d}(\phi_r)e_{k(\phi_r)}e^{w_r} + \phi_0}{(1 - \widetilde{u})\mathbf{d}(\phi_\ell)e_{-k(\phi_\ell)}e^{\widetilde{w}_\ell} + \widetilde{u}\mathbf{d}(\phi_r)e_{-k(\phi_r)}e^{\widetilde{w}_r} + \widetilde{\phi}_0} \quad (4.18)$$

being the almost periodic representatives of $\widetilde{\phi\phi^{-1}}$ given by

$$\begin{aligned} (\widetilde{\phi\phi^{-1}})_\ell &= \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)} e_{k(\phi_\ell)+k(\phi_r)}e^{w_\ell-\widetilde{w}_r}, \\ (\widetilde{\phi\phi^{-1}})_r &= \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)} e_{k(\phi_\ell)+k(\phi_r)}e^{w_r-\widetilde{w}_\ell}. \end{aligned}$$

From this, taking into account that $w_\ell, w_r \in AP_{p,w}$ are such that $M(w_\ell) = M(w_r) = 0$ (which additionally implies that $M(\widetilde{w}_\ell) = M(\widetilde{w}_r) = 0$), we have

$$k((\widetilde{\phi\phi^{-1}})_\ell) = k((\widetilde{\phi\phi^{-1}})_r) = k(\phi_\ell) + k(\phi_r), \quad (4.19)$$

$$\mathbf{d}((\widetilde{\phi\phi^{-1}})_\ell) = \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}, \quad \mathbf{d}((\widetilde{\phi\phi^{-1}})_r) = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}. \quad (4.20)$$

Applying now Theorem 3.7 to the Wiener–Hopf operator $W_{\phi\widetilde{\phi}^{-1}}$ and having in mind (4.19)–(4.20), it follows that:

- (a) If $k(\phi_\ell) + k(\phi_r) < 0$ and $0 \notin (\phi\phi^{-1})_{p,w}^\#(\mathbb{R})$, then $W_{\phi\widetilde{\phi}^{-1}}$ is properly d -normal and right-invertible on $L_+^p(\mathbb{R}, w)$;
- (b) If $k(\phi_\ell) + k(\phi_r) > 0$ and $0 \notin (\phi\phi^{-1})_{p,w}^\#(\mathbb{R})$, then $W_{\phi\widetilde{\phi}^{-1}}$ is properly n -normal and left-invertible on $L_+^p(\mathbb{R}, w)$;
- (c) If $k(\phi_\ell) + k(\phi_r) = 0$ and

$$0 \notin (\phi\phi^{-1})_{p,w}^\#(\mathbb{R}) \cup \mathcal{H}\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p, w), \nu_0^+(p, w)\right), \quad (4.21)$$

then $W_{\phi\widetilde{\phi}^{-1}}$ is a Fredholm operator on $L_+^p(\mathbb{R}, w)$;

- (d) If $k(\phi_\ell) + k(\phi_r) = 0$ and

$$0 \in (\phi\phi^{-1})_{p,w}^\#(\mathbb{R}) \cup \mathcal{H}\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p, w), \nu_0^+(p, w)\right),$$

then $W_{\phi\widetilde{\phi}^{-1}}$ is not normally solvable on $L_+^p(\mathbb{R}, w)$.

To arrive at the final assertion, we can interpret the Δ -relation after extension between the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf operator $W_{\phi\widetilde{\phi}^{-1}}$ as an equivalence after extension between $\text{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ and $W_{\phi\widetilde{\phi}^{-1}}$ (cf. Remark 2.5).

In this way, we get in cases (a) and (b) that $\text{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ is properly d -normal and right-invertible or properly n -normal and left-invertible, respectively. This implies that – in the case (a) – at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly d -normal and both are right-invertible; in the case (b), at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly n -normal and both operators are right-invertible.

The case (c) leads to the Fredholm property for both $W_\phi \pm H_\phi$.

In case (d), we have that $\text{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ is not normally solvable, which implies that at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is not normally solvable. \square

4.2. A formula for the sum of the indices of Fredholm Wiener–Hopf plus and minus Hankel operators.

Theorem 4.2. *Let $\phi \in \mathcal{GPAP}_{p,w}$. If $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are both Fredholm operators, then*

$$\begin{aligned} \text{Ind}[W_\phi + H_\phi] + \text{Ind}[W_\phi - H_\phi] &= \text{Ind}W_\theta + \text{Ind}W_\zeta = - \sum_l \text{ind}_l \zeta - \text{ind}\theta + \\ &+ \sum_{x \in \Lambda_\zeta \cup \{\infty\}} \left(\nu_x^0(p, w) - \left\{ \nu_x^0(p, w) - \frac{1}{2\pi} \arg \frac{\zeta(x-0)}{\zeta(x+0)} \right\} \right) + \\ &+ \nu_0^0(p, w) - \left\{ \nu_0^0(p, w) + \frac{1}{2\pi} \arg \frac{\mathbf{d}(\theta_\ell)}{\mathbf{d}(\theta_r)} \right\}, \end{aligned} \quad (4.22)$$

where $\widetilde{\phi\phi^{-1}} = \zeta\theta$ is a corresponding factorization in the sense of Theorem 3.6 (ii).

Proof. Let $\phi \in \mathcal{GPAP}_{p,w}$ such that $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are both Fredholm.

Recalling that $\text{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ is equivalent after extension with $W_{\widetilde{\phi\phi^{-1}}}$ (cf. Remark 2.5), it holds that

$$\text{Ind}W_{\widetilde{\phi\phi^{-1}}} = \text{Ind}(W_\phi + H_\phi) + \text{Ind}(W_\phi - H_\phi). \quad (4.23)$$

From the Fredholm index formula for the Wiener–Hopf operators with $PAP_{p,w}$ Fourier symbols presented in Corollary 3.8, we have

$$\text{Ind}W_{\widetilde{\phi\phi^{-1}}} = \text{Ind}W_\theta + \text{Ind}W_\zeta, \quad (4.24)$$

where $\widetilde{\phi\phi^{-1}} = \zeta\theta$. Thus, combining (4.23) and (4.24), it follows

$$\begin{aligned} \text{Ind}[W_\phi + H_\phi] + \text{Ind}[W_\phi - H_\phi] &= \text{Ind}W_\theta + \text{Ind}W_\zeta = - \sum_l \text{ind}_l \zeta - \text{ind}\theta + \\ &+ \sum_{x \in \Lambda_\zeta \cup \{\infty\}} \left(\nu_x^0(p, w) - \left\{ \nu_x^0(p, w) - \frac{1}{2\pi} \arg \frac{\zeta(x-0)}{\zeta(x+0)} \right\} \right) + \\ &+ \nu_0^0(p, w) - \left\{ \nu_0^0(p, w) + \frac{1}{2\pi} \arg \frac{\mathbf{d}(\theta_\ell)}{\mathbf{d}(\theta_r)} \right\}. \quad \square \end{aligned}$$

We would like to remark that due to the method here used we are not able to separate the Fredholm indices of both Wiener–Hopf plus and minus Hankel operators. In view of this, we have the above dependence of both symbols by means of the sum of the corresponding Fredholm indices.

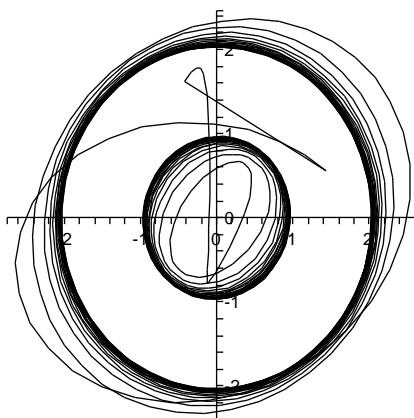


FIGURE 5. The range of $\phi(x)$ defined in (4.25) (for x between -50 and 50).

4.3. An example within the Wiener–Hopf–Hankel framework. Let $p = 2$, $w(x) = |x|^{\frac{1}{2}}$ and consider the function ϕ (see Figure 5) given by

$$\phi(x) = (1 - u(x))g(x)e^{-i\pi x} + u(x)2ig(x)e^{i\pi x}, \quad (4.25)$$

where

$$u(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x) \quad \text{and} \quad g(x) = \begin{cases} 1 + \frac{1}{x+i}, & x \geq 0 \\ 1 + \frac{1}{x-i}, & x < 0 \end{cases}.$$

It is clear that ϕ admits a factorization $\phi = \varphi\psi$ in sense of Theorem 3.6 (ii), with

$$\begin{aligned} \varphi(x) &= (1 - u(x))e^{-i\pi x} + u(x)2ie^{i\pi x}, \\ \psi(x) &= g(x). \end{aligned}$$

We observe that φ is an invertible function ($\varphi \in \mathcal{GSAP}_{2,w}$), cf. Figure 6, and it is clear that ψ is also an invertible function ($\psi \in \mathcal{GPC}_{2,w}$); see Figure 7. Moreover, $\psi(\pm\infty) = 1$. It therefore follows that ϕ is invertible.

From the definition of mean motion, we have that $k(\phi_\ell) + k(\phi_r) = 0$.

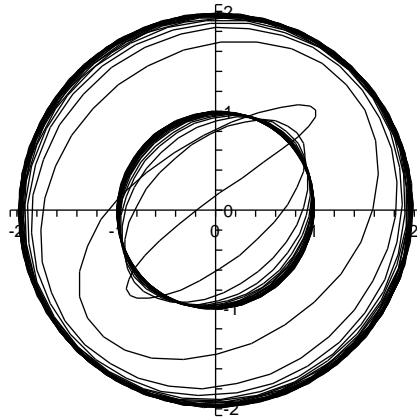


FIGURE 6. The range of $\varphi(x)$ (for x between -50 and 50).

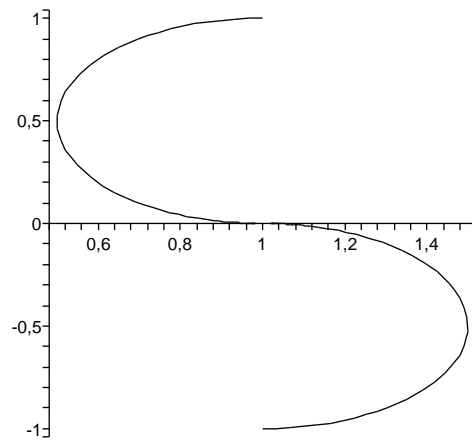


FIGURE 7. The range of $\psi(x)$ (for x between -50 and 50).

Since $\phi = \varphi\psi$, it results that $\phi\widetilde{\phi}^{-1} = \varphi\widetilde{\varphi}^{-1}\psi\widetilde{\psi}^{-1}$, with

$$\psi\widetilde{\psi}^{-1}(x) = \begin{cases} \frac{x^2 + 2i}{x^2 - 2x + 2}, & x < 0 \\ 1, & x = 0 \\ \frac{x^2 - 2i}{x^2 - 2x + 2}, & x > 0 \end{cases}.$$

Recalling that $p = 2$ and $w(x) = |x|^{\frac{1}{5}}$, we have

$$\begin{aligned} I_0(x) &= \left\{ \mu \in \mathbb{R} : \left| \frac{\xi}{\xi + i} \right|^\mu |\xi|^{\frac{1}{5}} \in A_2(\mathbb{R}) \right\} = \\ &= \left\{ \mu \in \mathbb{R} : -\frac{1}{2} < \mu + \frac{1}{5} < \frac{1}{2} \right\} = \\ &= \left\{ \mu \in \mathbb{R} : -\frac{7}{10} < \mu < 1 - \frac{7}{10} \right\}. \end{aligned}$$

Thus, $\nu_0^-(2, |x|^{\frac{1}{5}}) = \nu_0^+(2, |x|^{\frac{1}{5}}) = \frac{7}{10}$. In the same way,

$$\nu_\infty^-(2, |x|^{\frac{1}{5}}) = \nu_\infty^+(2, |x|^{\frac{1}{5}}) = \frac{3}{10}.$$

The only discontinuity point of ϕ and $\phi\widetilde{\phi}^{-1}$ is 0. Then, we have

$$\begin{aligned} (\phi\widetilde{\phi}^{-1})_{p,w}^\#(\mathbb{R}) &:= \mathcal{H}\left(\phi\widetilde{\phi}^{-1}(0-0), \phi\widetilde{\phi}^{-1}(0+0); \nu_\infty^-(2, |x|^{\frac{1}{5}}), \nu_\infty^+(2, |x|^{\frac{1}{5}})\right) \cup \\ &\cup \mathcal{H}\left(\phi\widetilde{\phi}^{-1}(+\infty), \phi\widetilde{\phi}^{-1}(-\infty); \nu_0^-(2, |x|^{\frac{1}{5}}), \nu_0^+(2, |x|^{\frac{1}{5}})\right) = \\ &= \mathcal{H}\left(i, -i; \frac{3}{10}, \frac{3}{10}\right) \cup \mathcal{H}\left(2i, -\frac{1}{2}i; \frac{7}{10}, \frac{7}{10}\right) = \\ &= \mathcal{A}\left(i, -i; \frac{3}{10}\right) \cup \mathcal{A}\left(2i, -\frac{1}{2}i; \frac{7}{10}\right). \end{aligned}$$

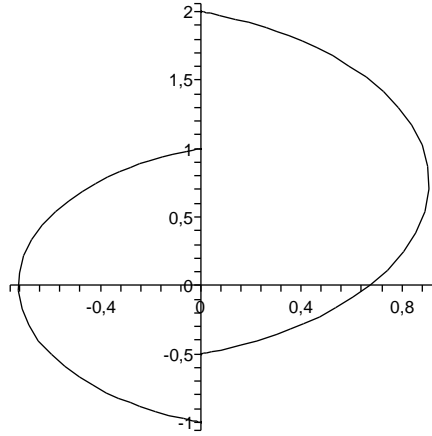


FIGURE 8. The arcs $\mathcal{A}(i, -i; \frac{3}{10})$ and $\mathcal{A}(2i, -\frac{1}{2}i; \frac{7}{10})$.

Additionally, since $\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)} = 2i$ and $\frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)} = -\frac{1}{2}i$, we obtain

$$\begin{aligned} \mathcal{H}\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(2, |x|^{\frac{1}{5}}), \nu_0^+(2, |x|^{\frac{1}{5}})\right) &= \\ &= \mathcal{H}\left(2i, -\frac{1}{2}i; \frac{7}{10}, \frac{7}{10}\right) = \mathcal{A}\left(2i, -\frac{1}{2}i; \frac{7}{10}\right). \end{aligned}$$

Figure 8 shows the arcs $\mathcal{A}(i, -i; \frac{7}{10})$ and $\mathcal{A}(2i, -\frac{1}{2}i; \frac{7}{10})$. Since these arcs do not contain the origin, the operators

$$W_\phi \pm H_\phi : L_+^2(\mathbb{R}, |x|^{\frac{1}{5}}) \rightarrow L^2(\mathbb{R}_+, |x|^{\frac{1}{5}})$$

have the Fredholm property.

Let us calculate their Fredholm index sum.

If $x < 0$, we have $\arg(\psi\widetilde{\psi^{-1}}) = \arctan(\frac{2}{x^2})$, if $x > 0$, then

$$\arg(\psi\widetilde{\psi^{-1}}) = \arctan\left(-\frac{2}{x^2}\right) = -\arctan\left(\frac{2}{x^2}\right)$$

and for $x = 0$, $\arg(\psi\widetilde{\psi^{-1}}) = 0$. Therefore,

$$\sum_{\ell} \text{ind}_{\ell} \psi\widetilde{\psi^{-1}} = 0.$$

Additionally, $\arg \frac{\psi\widetilde{\psi^{-1}(0-0)}}{\psi\widetilde{\psi^{-1}(0+0)}} = \arg \frac{i}{-i} = 0$. On the other hand, we have $\text{ind} \varphi\widetilde{\varphi^{-1}} = 0$ and

$$\arg \frac{\mathbf{d}((\varphi\widetilde{\varphi^{-1}})_{\ell})}{\mathbf{d}((\varphi\widetilde{\varphi^{-1}})_{r})} = \arg \left(\left(\frac{\mathbf{d}(\varphi_{\ell})}{\mathbf{d}(\varphi_r)} \right)^2 \right) = \arg \left(\frac{1}{4} \right) = 0.$$

Finally, using this data in the formula (4.22), we obtain

$$\text{Ind}[W_\phi + H_\phi] + \text{Ind}[W_\phi - H_\phi] = 0.$$

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