Mem. Differential Equations Math. Phys. 53 (2011), 155–162

Short Communication

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THE PRINCIPLE OF A PRIORI BOUNDEDNESS FOR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. A general theorem (principle of a priori boundedness) on solvability of the boundary value problem

$$dx(t) = dA(t) \cdot f(t, x(t)), \quad h(x) = 0$$

is established, where $A : [a, b] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a nondecreasing matrix-function, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector-function belonging to the Carathéodory class corresponding to the matrix-function A, and $h : \mathrm{BV}_s([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous operator.

რეზიუმე. მოყვანილია ზოგადი თეორემა (აპრიორული შემოსაზღვრულობის პრინციპი)

$$dx(t) = dA(t) \cdot f(t, x(t)), \quad h(x) = 0$$

სასაზღვრო ამოცანის ამოხსნადობის შესახებ, სადაც $A : [a, b] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ არაკლებადი მატრიცული ფუნქციაა, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ არის A მატრიცის შესაბამისი კარათეოდორის კლასის ფუნქცია, ხოლო $h : \mathrm{BV}_s([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ კი უწყვეტი ოპერატორია.

2010 Mathematics Subject Classification. 34K10.
Key words and phrases. Systems of nonlinear generalized ordinary differential equations, the Lebesgue–Stiltjes integral, general boundary value problem, solvability, principle of a priori boundedness.

Let n be a natural number, $[a, b]$ be a closed interval of real axis, $A = (a_{ik})_{i,k=1}^{n} : [a, b] \to \mathbb{R}^{n\times m}$ be a nondecreasing matrix-function, f be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A, and let $h : \mathrm{BV}_s([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous operator satisfying the condition

$$\sup\Big\{\|h(x)\| : \ x \in \mathrm{BV}_s([a, b], \mathbb{R}^n), \ \|x\|_s \le \rho\Big\} < +\infty$$

for every $\rho \in]0, +\infty[$.

Reported on the Tbilisi Seminar on Qualitative Theory of Differential Equations on April 12, 2010.

Consider the nonlinear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot f(t, x(t)) \tag{1}$$

with the boundary condition

$$h(x) = 0. \tag{2}$$

The theorem on the existence of a solution of the problem (1), (2) which will be given below and be called the principle of a priori boundedness, generalizes the well-known Conti–Opial type theorems (see [8], [16]) and supplements earlier known criteria for the solvability of nonlinear boundary value problems for systems of generalized ordinary differential equations ([1], [2], [5], [6], [16]).

Analogous and related questions are investigated in [9]–[14] for the boundary value problems for the nonlinear systems of ordinary differential and functional differential equations. In the paper we use the methods of investigation given in [10] and [11].

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, i.e., [1]–[7], [15], [17] and the references therein).

Throughout the paper the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$, $]a, b[$, $[a, b[$ and $]a, b]$ $(a, b \in R)$ are, respectively, a closed, an open and semi-open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$$\mathbb{R}_+^{n\times m} = \Big\{ (x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m) \Big\}.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \leq b$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$

$$\|X\|_s = \sup \big\{ \|X(t)\| : \ t \in [a, b] \big\},$$

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded total variations $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$);

$\mathrm{BV}_s([a,b],\mathbb{R}^n)$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^n),\|\cdot\|_s)$;

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $I\subset\mathbb{R}$ is an interval, then $C(I,\mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X:I\to\mathbb{R}^{n\times m}$.

$s_j:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ $(j=0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$

$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau)\ \text{ and }\ s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau)\ \text{ for }\ a<t\le b,$$

and

$$s_0(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for }\ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then

$$\int_s^t x(\tau)\,dg(\tau)=$$

$$=\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)+\sum_{s<\tau\le t}x(\tau)d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$; if $a=b$, then we assume $\int\limits_a^b x(t)\,dg(t)=0$;

$L([a,b],R;g)$ is the space of all functions $x:[a,b]\to\mathbb{R}$ measurable and integrable with respect to the measures $\mu(g)$ with the norm

$$\|x\|_{L,g}=\int_a^b|x(t)|\,dg(t).$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D\subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to D$ such that $x_{kj}\in L([a,b],\mathbb{R};g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int_s^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}\ \text{ for }\ a\le s\le t\le b,$$

$$S_j(G)(t)\equiv\big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\ \ (j=0,1,2).$$

If $D_1 \subset R^n$, $D_2 \subset \mathbb{R}^{n\times m}$ and $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$, then $K([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(i) the function $f_{kj}(\cdot\,,x) : [a,b] \to \mathbb{R}$ is $\mu(g_{ik})$ measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t,\cdot) : D_1 \to \mathbb{R}$ is continuous, $\mu(g_{ik})$ almost for every $t \in [a,b]$, and

$$\sup\big\{|f_{kj}(.,x)| :\ x \in D_0\big\} \in L([a,b],\mathbb{R}; g_{ik})$$

for every compact $D_0 \subset D_1$.

If $G(t) \equiv \operatorname{diag}(t,\dots,t)$, then we omit G in the notation containing G.

The inequalities between the vectors and between the matrices are understood componentwise.

A vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is said to be a solution of the system (1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau)) \ \text{ for }\ a \le s \le t \le b.$$

Under the solution of the problem (1), (2) we mean solutions of the system (1) satisfying (2).

We assume that $g(t) \equiv \|A(t)\|$.

Definition 1. The pair (P,l) of a matrix-function $P \in K([a,b]\times\mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and a continuous operator $l : \mathrm{BV}_s([a,b],\mathbb{R}^n)\times \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is said to be consistent if:

(i) for any fixed $x \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ the operator $l(x,.) : \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ and for $\mu(g)$ almost all $t \in [a,b]$ the inequalities

$$\|P(t,z)\| \le \alpha(t,\|z\|), \ \ \|l(x,y)\| \le \alpha_0(\|x\|_s)\cdot\|y\|_s$$

are fulfilled, where $\alpha_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function, and $\alpha : [a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ is a function, measurable and integrable with respect to the measure $\mu(g)$ in the first argument and nondecreasing in the second one;

(iii) there exists a positive number β such that for any $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$, $q \in L([a,b],\mathbb{R}^n; A)$ and $c_0 \in \mathbb{R}^n$, for which the condition

$$\det\big(I_n + (-1)^j d_j A(t)\cdot P(t,y(t))\big) \ne 0 \ \text{ for }\ t \in [a,b] \ \ (j=1,2)$$

holds, an arbitrary solution y of the boundary value problem

$$dx(t) = dA(t)\cdot\big(P(t,y(t))\cdot x(t) + q(t)\big), \ \ l(x,y) = c_0$$

admits the estimate

$$\|y\|_s \le \beta(\|c_o\| + \|q\|_{L,g}).$$

Theorem 1. *Let there exist a positive number ρ and a consistent pair (P,l) of a matrix-function $P \in K([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l : \mathrm{BV}_s([a,b],\mathbb{R}^n)\times\mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ such that for any $\lambda\in]0,1[$ an arbitrary solution of the problem*

$$dx(t) = dA(t)\cdot\Big(P(t,x(t))\cdot x(t)+\lambda\big[f(t,x(t))-P(t,x(t))\cdot x(t)\big]\Big), \tag{3}$$

$$l(x,y)=\lambda[l(x,x)-h(x)] \tag{4}$$

admits the estimate

$$\|x\|_s\le\rho. \tag{5}$$

Then the problem (1), (2) *is solvable.*

Definition 2. Let $P\in K([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$. We say that a matrix-function $B_0\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{E}_{A,P}$ if the condition

$$\det\big(I_n+(-1)^jd_jB_0(t)\big)\ne 0 \ \text{ for } \ t\in[a,b] \ \ (j=1,2) \tag{6}$$

holds and there exists a sequence $x_k\in\mathrm{BV}([a,b],\mathbb{R}^n)$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int\limits_a^t dA(\tau)\cdot P(\tau,x_k(\tau))=B_0(t) \ \text{ uniformly on } \ [a,b]. \tag{7}$$

Definition 3. We say that the pair (P,l) of the matrix-function $P\in K([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and the continuous operator $l:\mathrm{BV}_s([a,b],\mathbb{R}^n)\times\mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ belongs to the Opial class $\mathcal{O}_0^A$ with respect to the matrix-function A if:

- (i) for any fixed $x\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$ the operator $l(x,.):\mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is linear;
- (ii) for any $z\in\mathbb{R}^n$, x and $y\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$ and for $\mu(g)$ almost all $t\in[a,b]$ the inequalities

$$\|P(t,z)\|\le\alpha(t), \tag{8}$$
$$\|l(x,y)\|\le\alpha_0\|y\|_s$$

  are fulfilled, where $\alpha_0\in\mathbb{R}_+$, and $\alpha:I\to\mathbb{R}_+$ is a function measurable and integrable with respect to the measure $\mu(g)$;
- (iii) for every matrix-function $B_0\in\mathcal{E}_{A,P}$ the following condition holds: if y is a solution of the system

$$dy(t)=dB_0(t)\cdot y(t),$$

  and, in addition,

$$\lim_{k\to+\infty}l(x_k,y)=0$$

  for some sequence $x_k\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$ $(k=1,2,\dots)$, then $y(t)\equiv 0$.

Remark 1. By equalities (7) and (8) the condition

$$\|d_jA(t)\|\cdot\alpha(t)<1 \ \text{ for } \ t\in[a,b] \ \ (j=1,2)$$

guarantees the condition (6).

Corollary 1. *Let there exist a positive number ρ and a pair $(P,l) \in \mathcal{O}_0^A$ such that for any $\lambda \in]0,1[$ an arbitrary solution of the problem* (3), (4) *admits the estimate* (5)*. Then the problem* (1), (2) *is solvable.*

The following result belongs to Z. Opial (see, [9], [16]).

Corollary 2. *Let the pair $(P,l) \in \mathcal{O}_0^A$ be such that*

$$\big|f(t,x) - P(t,x)x\big| \le \alpha(t,\|x\|) \ \ \textit{for} \ \ t \in [a,b], \ \ x \in \mathbb{R}^n, \tag{9}$$

$$|h(x) - l(x)| \le l_0(|x|) + l_1(\|x\|_s) \ \ \textit{for} \ \ x \in \mathrm{BV}_s([a,b],\mathbb{R}^n), \tag{10}$$

where $\alpha \in K([a,b]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ is a nondecreasing in second variable vector-function, $l_0 : \mathrm{BV}_s([a,b],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, $l_1 \in C(\mathbb{R}_+,\mathbb{R}_+^n)$. Let, moreover,

$$\lim_{k\to+\infty} \frac{1}{\rho} \int_a^b dV(A)(\tau)\cdot\alpha(\tau,\rho) = \lim_{\rho\to+\infty} \frac{\|l_1(\rho)\|}{\rho} = 0.$$

Then the problem (1), (2) *is solvable.*

By $Y_P(x)$ we denote the fundamental matrix of the system

$$dy(t) = dA(t)\cdot P(t,x(t))y(t))$$

for every $x \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$, satisfying the condition $Y_P(x)(a) = I_n$.

Corollary 3. *Let conditions* (9) *and* (10) *hold, where P and l are, respectively, matrix-function and operator, satisfying the conditions* (i) *and* (ii) *of Definition* 3*; $l_0 : \mathrm{BV}_s([a,b],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and a nondecreasing in second variable vector-function $\alpha \in K([a,b]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ and a vector-function $l_1 \in C(\mathbb{R}_+,\mathbb{R}_+^n)$ are such that the condition*

$$\inf\Big\{\big|\det\big(l(x,Y_P(x))\big)\big| : \ x \in \mathrm{BV}_s([a,b],\mathbb{R}^n)\Big\} > 0$$

holds. Then the problem (1), (2) *is solvable.*

Corollary 4. *Let $P(t,x) \equiv P_0(t)$ and $l(x,y) \equiv l_0(y)$, where $P_0 \in L([a,b],\mathbb{R}^{n\times n};A)$, and $l_0 : \mathrm{BV}_s([a,b],R^n) \to \mathbb{R}^n$ is a bounded linear operator such that*

$$\det\big(I_n + (-1)^j d_jA(t)\cdot P_0(t)\big) \ne 0 \ \ \textit{for} \ \ t \in [a,b] \ \ (j=1,2)$$

and the problem

$$dy(t) = dA(t)\cdot P_0(t)y(t), \ \ l_0(y) = 0$$

has only the trivial solution. Let, moreover, there exist a positive number ρ such that for every $\lambda \in]0,1[$ an arbitrary solution of the problem

$$dx(t) = dA(t)\cdot\big(P_0(t)\cdot x(t) + \lambda[f(t,x(t)) - P_0(t)\cdot x(t)]\big),$$
$$l_0(x) = \lambda[l_0(x) - h(x)]$$

admits the estimate (5)*.Then the problem* (1), (2) *is solvable.*

Acknowledgement

This work is supported by the Georgian National Science Foundation (Project # GNSF/ST09_175_3-101).

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(Received 30.06.2010)

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