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GLOBAL AND BLOW-UP SOLUTIONS OF THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR SECOND ORDER NONLINEAR HYPERBOLIC EQUATIONS

Dedicated to Academician N. A. Izobov
on the occasion of his 70th birthday

Abstract. It is proved that the characteristic initial value problem for the second order hyperbolic equation

$$u_{xy} = f(x, y, u),$$

where $f : [0, a] \times [0, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, has at least one global, or local blow-up solution. Unimprovable in a sense conditions of existence and nonexistence of global and local blow-up solutions are established.

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$$u_{xy} = f(x, y, u),$$

სადაც $f : [0, a] \times [0, b] \times \mathbb{R} \to \mathbb{R}$ უწყვეტი ფუნქციაა, აქვს ერთი მაინც გლობალური ან ფეთქებადი ლოკალური ამონახსნი. ნაპოვნია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც, სათანადოდ, უზრუნველყოფენ გლობალური და ფეთქებადი ლოკალური ამონახსნების არსებობასა და არ არსებობას.

1. Formulation of the Main Results

Global solvability of initial and initial–boundary value problems for differential equations and blow-up phenomena of such problems have been attracting the attention of many mathematicians and are subjects of numerous studies (See [1–24] and the references cited therein). In the present paper we consider the characteristic initial value problem

$$u_{xy} = f(x, y, u), \tag{1.1}$$

$$u(x, 0) = c_1(x) \quad \text{for} \quad 0 \le x \le a, \qquad u(0, y) = c_2(y) \quad \text{for} \quad 0 \le y \le b \tag{1.2}$$

from that viewpoint. More precisely, we have proved a theorem on existence of either of global and local blow-up solutions of problem (1.1),(1.2), and obtained unimprovable in a sense conditions guaranteeing that problem (1.1),(1.2) : (*i*) has at least one global solution and no local blow-up solution; (*ii*) has at least one local blow-up solution and has no global solution.

Let

$$\Omega(a, b) = (0, a) \times (0, b), \quad \overline{\Omega}(a, b) = [0, a] \times [0, b].$$

For arbitrary $a_0 \in (0, a]$ and $b_0 \in (0, b]$ set

$$\Omega_0(a, b_0; a_0, b) = \Omega(a, b_0) \cup \Omega(a_0, b), \quad \overline{\Omega}_0(a, b_0; a_0, b) = \overline{\Omega}(a, b_0) \cup \overline{\Omega}(a_0, b).$$

If either $a_0 = a$ or $b_0 = b$, then it is clear that $\Omega_0(a, b_0; a_0, b) = \Omega(a, b)$.

Throughout the paper it is assumed that the function $f : \overline{\Omega}(a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous, and $c_1 : [0, a] \to \mathbb{R}$ and $c_2 : [0, b] \to \mathbb{R}$ are continuously differentiable functions satisfying the matching condition

$$c_1(0) = c_2(0). \tag{1.3}$$

We will use the following definitions:

Definition 1.1. Let D be a domain contained in $\Omega(a, b)$. A function $u : D \to \mathbb{R}$ will be called a solution of equation (1.1) in D, if it is continuous together with its partial derivatives u_x, u_y, u_{xy} and satisfies (1.1) at every point of D.

Definition 1.2. A function $u : \Omega(a, b) \to \mathbb{R}$ will be called a *global solution of problem* (1.1),(1.2), if it is a solution of equation (1.1) in the domain $\Omega(a, b)$, is uniformly continuous in $\Omega(a - \varepsilon, b - \varepsilon)$ for any sufficiently small $\varepsilon > 0$, and satisfies the initial conditions (1.2), where

$$u(x, 0) = \lim_{y \to 0} u(x, y), \quad u(0, y) = \lim_{x \to 0} u(x, y).$$

Definition 1.3. Let $a_0 \in (0, a)$ and $b_0 \in (0, b)$. A function

$$u : \Omega_0(a, b_0; a_0, b) \to \mathbb{R}$$

will be called a *local solution of problem* (1.1),(1.2), if it is a solution of equation (1.1), is uniformly continuous in the domain $\Omega_0(a, b_0 - \varepsilon; a_0 - \varepsilon, b)$ for any sufficiently small $\varepsilon > 0$, and satisfies the initial conditions (1.2).

Definition 1.4. A local solution u of problem (1.1),(1.2) defined in the domain $\Omega_0(a,b_0;a_0,b)$ will be called a *blow-up solution*, if

$$\sup\{|u(x,y)|:0<x<a\}\to+\infty \quad \text{as} \quad y\to b_0, \tag{1.4}$$

$$\sup\{|u(x,y)|:0<y<b\}\to+\infty \quad \text{as} \quad x\to a_0. \tag{1.5}$$

Theorem 1.1. *If $a_1\in(0,a)$ and $b_1\in(0,b)$ are sufficiently small, then problem* (1.1), (1.2) *has at least one uniformly continuous solution u in $\Omega(a,b_1;a_1,b)$. Moreover, for any such solution there exists either global, or a local blow-up solution of problem* (1.1), (1.2) *coinciding with u in $\Omega(a,b_1;a_1,b)$.*

Remark 1.1. In particular, Theorem 1.1 implies that if problem (1.1),(1.2) has no local blow-up solution (global solution), then it has a global (local blow-up) solution.

Theorem 1.2. *Let the inequality*

$$|f(x,y,z)|\le\varphi(|z|) \tag{1.6}$$

hold on $\Omega(a,b)\times\mathbb{R}$, where $\varphi:[0,+\infty)\to[0,+\infty)$ is a nondecreasing continuous function. If, moreover,

$$\int\limits_0^{+\infty}\frac{dz}{\Phi(z)}=+\infty, \tag{1.7}$$

where

$$\Phi(z)=1+\Big[\int\limits_0^z\varphi(s)\,ds\Big]^{\frac12} \quad for \quad z\ge0, \tag{1.8}$$

then problem (1.1), (1.2) *has at least one global solution and has no local blow-up solution. Moreover, its every global solution is uniformly continuous in $\Omega(a,b)$.*

Theorem 1.3. *Let the inequality*

$$f(x,y,z)\ge\varphi(z) \tag{1.9}$$

hold on $\Omega(a,b)\times[0,+\infty)$, where $\varphi:[0,+\infty)\to[0,+\infty)$ is a nondecreasing continuous function such that the function Φ, given by (1.8)*, satisfies the condition*

$$\int\limits_0^{+\infty}\frac{dz}{\Phi(z)}<+\infty. \tag{1.10}$$

Then there exists a positive number r such that if

$$c_1(x)+c_2(y)-c_1(0)>r \quad for \quad (x,y)\in\overline{\Omega}(a,b), \tag{1.11}$$

then problem (1.1), (1.2) *has no global solution and has at least one local blow-up solution.*

As an example consider the differential equation

$$u_{xy} = g(x,y) f_0(u), \tag{1.12}$$

where $g : \overline{\Omega}(a,b) \to (0,+\infty)$ and $f_0 : \mathbb{R} \to \mathbb{R}$ are continuous functions, and f_0 is nonnegative and nondecreasing on $[0,+\infty)$.

Set

$$F_0(z) = 1 + \Bigg[\int\limits_0^z f_0(s)\,ds\Bigg]^{\frac{1}{2}} \quad \text{for} \quad z \geq 0.$$

Theorems 1.2 and 1.3 imply

Corollary 1.1. *Problem* (1.12), (1.2) *is globally solvable for arbitrary continuously differentiable functions* $c_1 : [0,a] \to \mathbb{R}$ *and* $c_2 : [0,b] \to \mathbb{R}$ *satisfying the matching condition* (1.3) *if and only if*

$$\int\limits_0^{+\infty} \frac{dz}{F_0(z)} = +\infty.$$

2. Auxiliary Statements

2.1. **Lemma on existence of locally uniformly continuous solution of a characteristic initial value problem.** For equation (1.1) consider the characteristic initial value problem

$$u(x,y_0) = v_1(x) \quad \text{for} \quad x_0 \leq x \leq a, \qquad u(x_0,y) = v_2(y) \quad \text{for} \quad y_0 \leq y \leq b, \tag{2.1}$$

where $x_0 \in [0,a)$, $y_0 \in [0,b)$, and $v_1 : [x_0,a] \to \mathbb{R}$ and $v_2 : [y_0,b] \to$ are continuously differentiable functions such that

$$v_1(x_0) = v_2(y_0). \tag{2.2}$$

The function $f : [x_0,a] \times [y_0,b] \to \mathbb{R}$, as above, is assumed to be continuous.

Let $x_1 \in (x_0,a]$ and $y_1 \in (y_0,b]$. Set

$$\Omega_{11} = (x_0,a) \times (y_0,y_1), \quad \Omega_{12} = (x_0,x_1) \times (y_0,b), \quad \Omega_1 = \Omega_{11} \cup \Omega_{12},$$

$$M_{0i} = \sup\{|v_1(x) + v_2(y) - v_1(x_0)| : (x,y) \in \Omega_{1i}\} \quad (i=1,2), \tag{2.3}$$

$$M_i = \sup\{|f(x,y,z)| : (x,y) \in \Omega_{1i}, \ \ |z| \leq 1 + M_{0i}\} \quad (i=1,2). \tag{2.4}$$

By $\overline{\Omega}_1$ denote the closure of Ω_1, and by $C(\overline{\Omega_1})$ denote the Banach space of continuous functions $u : \overline{\Omega}_1 \to \mathbb{R}$.

Lemma 2.1. *If*

$$(a - x_0)(y_1 - y_0) M_1 \leq 1, \tag{2.5}$$

$$(x_1 - x_0)(b - y_0) M_2 \leq 1, \tag{2.6}$$

then problem (1.1), (2.1) *has at least one solution in* Ω_1. *Moreover, every such solution is uniformly continuous in* Ω_1 *and admits the estimates*

$$|u(x,y)| \le 1 + M_{01} \quad \textit{for} \quad (x,y) \in \Omega_{11}, \tag{2.7}$$

$$|u(x,y)| \le 1 + M_{02} \quad \textit{for} \quad (x,y) \in \Omega_{12}. \tag{2.8}$$

Proof. First assume that problem (1.1),(2.1) has a solution u in the domain Ω_1. Then the representation

$$u(x,y) = v_1(x) + v_2(y) - v_1(x_0) + \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} f(s,t,u(s,t))\,ds\,dt \tag{2.9}$$

is valid. On the other hand, in view of (2.1)–(2.3) it is clear that either u admits the estimate (2.7), or the inequality

$$|u(x^*,y^*)| > 1 + M_{01}$$

holds for some $x^* \in (x_0,a)$ and $y^* \in (y_0,y_1)$. In the latter case, according to (2.1)–(2.3), there exist $x_* \in (x_0,x^*)$ and $y_* \in (y_0,y^*)$ such that

$$|u(x,y)| \le 1 + M_{01} \quad \text{for} \quad x_0 \le x \le x_*, \quad y_0 \le y \le y_*$$

and

$$|u(x_*,y_*)| = 1 + M_{01}.$$

If along with this we take into account notations (2.3), (2.4) and inequality (2.5), then from (2.9) we get

$$\begin{aligned} 1 + M_{01} &\le M_{01} + \int\limits_{x_0}^{x_*}\int\limits_{y_0}^{y_*} |f(s,t,u(s,t))|\,ds\,dt \\ &\le M_{01} + M_1(x_*-x_0)(y_*-y_0) < M_{01}+1. \end{aligned}$$

The obtained contradiction proves the validity of estimate (2.7). The validity of estimate (2.8) can be proved similarly.

In view of (2.4),(2.7) and (2.8), from (2.9) we have

$$|u(x,y) - u(s,t)| \le M(|x-s| + |y-t|) \quad \text{for} \quad (x,y),\ (s,t) \in \Omega_1, \tag{2.10}$$

where

$$\begin{aligned} M = \max\{|v_1'(x)| + |v_2'(y)| : x_0 \le x \le a,\ \ y_0 \le y \le b\} \\ + (M_1+M_2)(a+b-x_0-y_0). \end{aligned} \tag{2.11}$$

Hence it follows that u is uniformly continuous in Ω_1 and, consequently, admits a continuous extension onto $\overline{\Omega}_1$.

Thus we have proved that the solvability of problem (1.1),(2.1) yields the solvability of the integral equation (2.9) in the space $C(\overline{\Omega}_1)$. On the other hand it is clear that if $u \in C(\overline{\Omega}_1)$ is a solution of (2.9), then its restriction on Ω_1 is a solution of problem (1.1),(1.2). Therefore, to complete the proof we need to show that the integral equation (2.9) has at least one solution in $C(\overline{\Omega}_1)$.

Introduce the operator

$$\mathcal{W}(u)(x,y) = v_1(x) + v_2(y) - v_1(x_0) + \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} f(s,t,u(s,t))\,ds\,dt.$$

The continuity of the functions $v_1 : [x_0,a] \to \mathbb{R}$, $v_2 : [y_0,b] \to \mathbb{R}$ and $f : \overline{\Omega}_1 \times \mathbb{R} \to \mathbb{R}$ implies that $\mathcal{W} : C(\overline{\Omega}_1) \to C(\overline{\Omega}_1)$ is a continuous operator.

Let $\mathbf{B}$ be the set of all functions $u \in C(\overline{\Omega})$ satisfying conditions (2.7),(2.8) and (2.10), where M is the number given by (2.11). It is clear that $\mathbf{B}$ is a convex and closed set. Moreover, by Arzella–Ascolli lemma, $\mathbf{B}$ is a compact.

By virtue of (2.3)–(2.8) for an arbitrary $u \in \mathbf{B}$ we have

$$|\mathcal{W}(u)(x,y)| \le M_{01} + (a-x_0)(y_1-y_0)M_1 \le M_{01}+1 \quad \text{for} \quad (x,y) \in \Omega_{11},$$
$$|\mathcal{W}(u)(x,y)| \le M_{02} + (x_1-x_0)(b-y_0)M_2 \le M_{02}+1 \quad \text{for} \quad (x,y) \in \Omega_{12},$$
$$|\mathcal{W}(u)(x,y) - \mathcal{W}(u)(s,t)| \le M(|x-s|+|y-t|) \quad \text{for} \quad (x,y),\ (s,t) \in \Omega_1.$$

Consequently, $\mathcal{W}$ is a continuous operator mapping the compact $\mathbf{B}$ into itself. By Schauder's theorem, $\mathcal{W}$ has a fixed point $u \in \mathbf{B}$, i.e. the integral equation (2.9) has a solution $u \in \mathbf{B}$. ☐

Remark 2.1. As it was noted above, if a solution u of problem (1.1),(1.2) is uniformly continuous in Ω_1, then it admits a continuous extension onto $\overline{\Omega}_1$. Moreover, the representation (2.9) implies that the extension of u has continuous partial derivatives u_x, u_y and u_{xy} on $\overline{\Omega}_1$ and satisfies equation (1.1) everywhere on $\overline{\Omega}_1$. Therefore the extension of u will be called a solution of problem (1.1),(1.2) in $\overline{\Omega}_1$.

2.2. **Some remarks on global and blow-up solutions of nonlinear autonomous ordinary differential equations of second order.** Consider the ordinary differential equation

$$w'' = \varphi(w) \tag{2.12}$$

with the initial and the boundary conditions

$$w(0) = \gamma_0, \qquad w'(0) = \gamma \tag{2.13}$$

and

$$w(0) = 0, \qquad \lim_{t \to t_0} w'(t) = +\infty, \tag{2.14}$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a continuous function,

$$\gamma_0 \ge 0, \quad \gamma > 0 \tag{2.15}$$

and $t_0 > 0$.

Lemma 2.2. *If condition* (1.7) *holds, then problem* (2.12), (2.13) *has a unique solution w defined in the interval* $[0,+\infty)$ *and satisfying the inequalities*

$$w(t) > \gamma_0, \quad w'(t) > 0 \quad \textit{for} \quad t > 0. \tag{2.16}$$

Proof. First assume that problem (2.12),(2.13) has a solution w defined in the interval $[0,+\infty)$. Then, in view of nonnegativity of φ and condition (2.15), inequalities (2.16) hold. Multiplying (2.12) by w', integrating from 0 to t, and taking into account (2.13), we get

$$w'^2(t) = \Phi^2_{\gamma_0,\gamma}(w(t)) \quad \text{for} \quad t \ge 0,$$

where

$$\Phi_{\gamma_0,\gamma}(z) = \Big[\gamma^2 + 2\int\limits_{\gamma_0}^{z} \varphi(s)\,ds\Big]^{\frac{1}{2}} \quad \text{for} \quad z \ge \gamma_0. \tag{2.17}$$

Hence according to (2.16) we have

$$\frac{w'(t)}{\Phi_{\gamma_0,\gamma}(w(t))} = 1 \quad \text{for} \quad t \ge 0.$$

Therefore

$$\Psi_{\gamma_0,\gamma}(w(t)) = t \quad \text{for} \quad t \ge 0, \tag{2.18}$$

where

$$\Psi_{\gamma_0,\gamma}(z) = \int\limits_{\gamma_0}^{z} \frac{ds}{\Phi_{\gamma_0,\gamma}(s)}.$$

In view of (1.8) and (2.17) it is clear that

$$\Phi_{\gamma_0,\gamma}(z) < (\gamma+2)\Phi(z) \quad \text{for} \quad z \ge \gamma_0.$$

Hence, in view of (1.7), if follows that

$$\lim_{z\to+\infty} \Psi_{\gamma_0,\gamma}(z) \ge \frac{1}{\gamma+2} \lim_{z\to+\infty} \int\limits_{\gamma_0}^{z} \frac{ds}{\Phi(s)} = +\infty.$$

Consequently, the function $\Psi_{\gamma_0,\gamma} : [\gamma_0,+\infty) \to [0,+\infty)$ has the inverse $\Psi^{-1}_{\gamma_0,\gamma} : [0,+\infty) \to [\gamma_0,+\infty)$. Therefore (2.18) implies that

$$w(t) = \Psi^{-1}_{\gamma_0,\gamma}(t) \quad \text{for} \quad t \ge 0. \tag{2.19}$$

Thus we have proved that if problem (2.12),(2.13) has a solution defined on $[0,+\infty)$, then it is unique and admits the representation (2.19). On the other hand, from the definition of $\Psi^{-1}_{\gamma_0,\gamma}$ it follows that the function given by (2.19) is indeed a solution of problem (2.12),(2.13) satisfying inequalities (2.16). □

Lemma 2.3. *If condition* (1.10) *holds and*

$$\limsup_{z\to 0} \frac{\varphi(z)}{z} < +\infty, \tag{2.20}$$

then problem (2.12), (2.14) *has a unique solution* w *and*

$$w(t) > 0, \quad w'(t) > 0 \quad \textit{for} \quad 0 < t < t_0. \tag{2.21}$$

Proof. For an arbitrary $\gamma > 0$ set

$$\Phi_\gamma(z) = \Big[\gamma^2 + 2\int_0^z \varphi(s)\,ds\Big]^{\frac{1}{2}}, \quad \Psi_\gamma(z) = \int_0^z \frac{ds}{\Phi_\gamma(s)} \quad \text{for} \quad z \geq 0.$$

Then according to (1.8) and (1.10) we have

$$\lim_{z\to+\infty} \int_0^z \varphi(s)\,ds = +\infty, \quad \lim_{z\to+\infty} \frac{\Phi_\gamma(z)}{\Phi(z)} = \sqrt{2}$$

and

$$T(\gamma) = \int_0^{+\infty} \frac{dz}{\Phi_\gamma(z)} < +\infty. \tag{2.22}$$

Hence it follows that for an arbitrary $\gamma > 0$ in the interval $[0, T(\gamma))$ the differential equation (2.12) has a unique solution $w_\gamma(t)$ satisfying the initial conditions

$$w_\gamma(0) = 0, \quad w'_\gamma(0) = \gamma.$$

Besides,

$$w_\gamma(t) > 0, \quad w'_\gamma(t) > 0 \quad \text{for} \quad 0 < t < T(\gamma), \qquad \lim_{t\to T(\gamma)} w_\gamma(t) = +\infty.$$

On the other hand, (1.10),(2.20) and (2.22) imply that $T : (0, +\infty) \to (0, +\infty)$ is a continuous decreasing function such that

$$\lim_{\gamma\to 0} T(\gamma) = +\infty, \qquad \lim_{\gamma\to+\infty} T(\gamma) = 0.$$

Consequently, the inverse function T^{-1} maps $(0, +\infty)$ onto $(0, +\infty)$.

From the above said it is clear that if $\gamma_0 = T^{-1}(t_0)$, then the function $w(t) = w_{\gamma_0}(t)$ is the unique solution of problem (2.12),(2.14) satisfying the inequalities (2.21). □

2.3. **Lemmas on differential inequalities.** Along with the differential equation (1.1) consider the differential inequalities

$$|u_{xy}| \leq \varphi(|u|) \tag{2.23}$$

and

$$u_{xy} \geq \varphi(|u|) \tag{2.24}$$

with the initial conditions (1.2), where $\varphi : [0, +\infty) \times [0, +\infty)$ is a continuous nondecreasing function. As before, the functions $c_1 : [0, a] \to \mathbb{R}$ and $c_2 : [0, b] \to \mathbb{R}$ are assumed to be continuously differentiable and satisfying the matching condition (1.3).

Global, local and blow-up solutions of problem (2.23),(1.2) (problem (2.24),(1.2)) are defined similarly to the definitions for problem (1.1),(1.2). More precisely, in Definitions 1.1–1.4 equation (1.1) should be replaced by inequality (2.23) (inequality (2.24)).

Lemma 2.4. *If condition* (1.7) *holds, then problem* (2.23), (1.2) *has no local blow-up solution and its arbitrary global solution is uniformly continuous on* $\Omega(a,b)$.

Proof. Let

$$\gamma_0 = 1 + \max\{|c_1(x) + c_2(y) - c_1(0)| : 0 \le x \le a,\ 0 \le y \le b\}, \tag{2.25}$$

γ be an arbitrarily fixed positive number, and w be a solution of problem (2.12),(2.13). By Lemma 2.2, w is defined on $[0,+\infty)$ and satisfies inequalities (2.16).

The function $(x,y) \to w(x+y)$ is a solution of the differential equation

$$w_{xy} = \varphi(w).$$

Therefore the representation

$$w(x+y) = w(x) + w(y) - w(0) + \int_0^x \int_0^y \varphi(w(s+t))\,ds\,dt \tag{2.26}$$

is valid. On the other hand, (2.16) and (2.25) imply that

$$|c_1(x) + c_2(y) - c_1(0)| < \gamma_0 < w(x) + w(y) - w(0) \quad \text{for} \quad (x,y) \in \Omega(a,b). \tag{2.27}$$

First prove that if u is a local solution of problem (2.23),(1.2) in some domain $\Omega_0(a,b_0;a_0,b)$, then

$$|u(x,y)| < w(x+y) \le w(a+b) \quad \text{for} \quad (x,y) \in \Omega_0(a,b_0;a_0,b). \tag{2.28}$$

Assume the contrary that (2.28) is violated, i.e. the inequality

$$|u(x_0,y_0)| \ge w(x_0+y_0) \tag{2.29}$$

holds for some $(x_0,y_0) \in \Omega_0(a,b_0;a_0,b)$. Without loss of generality one can assume that $(x_0,y_0) \in \Omega(a,b_0)$, since the case $(x_0,y_0) \in \Omega(a_0,b)$ can be considered similarly.

Setting

$$u(x,0) = \lim_{y\to 0} u(x,y) \quad \text{for} \quad 0 \le x \le x_0,$$
$$u(0,y) = \lim_{x\to 0} u(x,y) \quad \text{for} \quad 0 \le y \le y_0,$$

the function u becomes continuous in $\overline{\Omega}(x_0,y_0)$. Let

$$v(x,y) = w(x+y) - |u(x,y)| \quad \text{for} \quad 0 \le x \le x_0,\ \ 0 \le y \le y_0.$$

Then in view of (1.2),(2.27) and (2.29) we have

$$v(x,0) > 0 \quad \text{for} \quad 0 \le x \le x_0, \quad v(0,y) > 0 \quad \text{for} \quad 0 \le y \le y_0$$

and

$$v(x_0,y_0) \le 0.$$

Hence, by continuity of v on $\overline{\Omega}(x_0, y_0)$, there exist $x_1 \in (0, x_0]$ and $y_1 \in (0, y_0]$ such that

$$v(x,y) > 0 \quad \text{for} \quad 0 \le x < x_1, \quad 0 \le y \le y_1 \tag{2.30}$$

and

$$v(x_1, y_1) = 0. \tag{2.31}$$

In view of (2.23),(2.27) and (2.30), the representation

$$u(x,y) = c_1(x) + c_2(y) - c_1(0) + \int_0^x \int_0^y u_{st}(s,t)\, ds\, dt \tag{2.32}$$

implies that

$$|u(x_1,y_1)| \le |c_1(x_1) + c_2(y_1) - c_1(0)| + \int_0^{x_1} \int_0^{y_1} \varphi(u(s,t))\, ds\, dt$$
$$< w(x_1) + w(y_1) - w(0) + \int_0^{x_1} \int_0^{y_1} \varphi(w(s+t))\, ds\, dt.$$

Hence, by virtue of (2.26), we find

$$v(x_1,y_1) = w(x_1+y_1) - |u(x_1,y_1)| > 0,$$

which contradicts to the equality (2.31). The obtained contradiction proves the validity of the estimate (2.28).

Similarly we can prove that if u is a global solution of problem (2.23),(1.2), then

$$|u(x,y)| < w(x+y) \le w(a+b) \quad \text{for} \quad (x,y) \in \Omega(a,b). \tag{2.33}$$

In view of the estimate (2.28) (the estimate (2.33)), (2.23) implies that

$$|u_{xy}(x,y)| \le r \quad \text{for} \quad (x,y) \in \Omega_0(a,b_0;a_0,b) \ \ ((x,y) \in \Omega(a,b)),$$

where $r = \max\{\varphi(z) : 0 \le z \le w(a+b)\}$. By virtue of the representation (2.32), the latter inequality ensures the uniform continuity of u in the domain $\Omega_0(a,b_0;a_0,b)$ (in the domain $\Omega(a,b)$). Consequently, problem (2.23),(1.2) has no local blow-up solution and its arbitrary local and global solutions are uniformly continuous. $\square$

Lemma 2.5. *Let*

$$\max\{a,b\} < t_0 < a+b, \tag{2.34}$$

and problem (2.12), (2.14) *have a solution* w *satisfying inequalities* (2.21). *If, moreover, condition* (1.11) *holds, where*

$$r = w(a) + w(b), \tag{2.35}$$

then problem (2.24), (1.2) *has no global solution.*

Proof. Assume the contrary that problem (2.24),(1.2) has a global solution u. Then in view of inequalities (1.11) and (2.24), the representation (2.32) yields

$$u(x,y) > r > 0 \quad \text{for} \quad (x,y) \in \Omega(a,b). \tag{2.36}$$

According to (2.34) there exist $a_0 \in (0,a)$ and $b_0 \in (0,b_0)$ such that $a_0+b_0 = t_0$. If along with this we take into account conditions (2.14),(2.36) and continuity of u at point (a_0,b_0), then it becomes clear that

$$\lim_{(x,y)\to(a_0,b_0)} \frac{w(x+y)}{u(x,y)} = +\infty.$$

Therefore for some $a_1 \in (0,a_0)$ and $b_1 \in (0,b_0)$ we have

$$w(a_1+b_1) > u(a_1,b_1). \tag{2.37}$$

Due to the uniform continuity in $\Omega(a_1,b_1)$, the function u admits a continuous extension onto $\overline{\Omega}(a_1,b_1)$. Set

$$v(x,y) = u(x,y) - w(x+y).$$

Then in view of (1.2),(1.11),(2.21) and (2.35) we have

$$v(x,0) = c_1(x) - w(x) > w(a) + w(b) - w(x) > 0 \quad \text{for} \quad 0 \le x \le a_1,$$
$$v(0,y) = c_2(y) - w(y) > w(a) + w(b) - w(y) > 0 \quad \text{for} \quad 0 \le y \le b_1.$$

On the other hand it follows from (2.37) that $v(a_1,b_1) < 0$. Therefore there exist $x_0 \in (0,a_1]$ and $(0,b_1]$ such that

$$v(x,y) > 0 \quad \text{for} \quad 0 \le x < x_0, \;\; 0 \le y \le y_0 \tag{2.38}$$

and

$$v(x_0,y_0) = 0. \tag{2.39}$$

Taking into account inequalities (1.11),(2.24) and (2.38), from (2.26) and (2.32) we find

$$\begin{aligned} v(x_0,y_0) &= c_1(x_0) + c_2(y_0) - c_1(0) - w(x_0) - w(y_0) \\ &\quad + \int_0^{x_0}\int_0^{y_0} \big(u_{st}(s,t) - \varphi(w(s+t))\big)\,ds\,dt \\ &> r - w(x) - w(y) + \int_0^{x_0}\int_0^{y_0} \big(\varphi(u(s,t)) - \varphi(w(s+t))\big)\,ds\,dt \\ &\ge r - w(x_0) - w(y_0). \end{aligned}$$

Hence, in view of conditions (2.21) and (2.35), it follows that

$$v(x_0,y_0) > r - w(a) - w(b) = 0.$$

But this contradicts to the equality (2.37). The obtained contradiction proves that problem (2.24),(1.2) has no global solution. □

3. Proofs of the Main Results

Proof of Theorem 1.1. Let

$$M^0 = \max\{|c_1(x) - c_1(0)| : 0 \le x \le a\} + \max\{|c_2(y)| : 0 \le y \le b\},$$
$$M = 1 + \max\{|f(x, y, z)| : (x, y) \in \overline{\Omega}(a, b),\ |z| \le 1 + M^0\},$$

and a_1 and b_1 be arbitrary numbers satisfying the inequalities

$$0 < a_1 \le \min\Big\{\frac{1}{Mb},\ \frac{a}{2}\Big\}, \quad 0 < b_1 \le \min\Big\{\frac{1}{Ma},\ \frac{b}{2}\Big\}.$$

Then by Lemma 2.1, problem (1.1),(1.2) has a uniformly continuous solution u_1 in the domain $\Omega_1 = \Omega_0(a, b_1; a_1, b)$. Our goal is to prove that u_1 is a restriction on the set Ω_1 of some either global, or local blow-up solution of problem (1.1),(1.2).

By u_1 we will understand its continuous extension onto $\overline{\Omega}_1$. Set

$$\begin{aligned}
M_{11}^0(t) &= \max\{|u_1(x, b_1) - u_1(a_1, b_1)| : a_1 \le x \le a\} \\
&\quad + \max\{|u_1(a_1, y)| : b_1 \le y \le t\}, \\
M_{11}(t) &= 1 + \max\{|f(x, y, z)| : (x, y) \in \overline{\Omega}(a, b),\ |z| \le 1 + M_{11}^0(t)\} \\
&\qquad\qquad \text{for} \quad b_1 \le t \le b, \\
M_{12}^0(s) &= \max\{|u_1(a_1, y) - u_1(a_1, b_1)| : b_1 \le y \le b\} \\
&\quad + \max\{|u_1(x, b_1)| : a_1 \le x \le s\}, \\
M_{12}(s) &= 1 + \max\{|f(x, y, z)| : (x, y) \in \overline{\Omega}(a, b),\ |z| \le 1 + M_{12}^0(s)\} \\
&\qquad\qquad \text{for} \quad a_1 \le s \le a.
\end{aligned}$$

It is clear that $M_{11} : [b_1, b] \to (0, +\infty)$ and $M_{12} : [a_1, a] \to (0, +\infty)$ are continuous nondecreasing functions. If

$$aM_{11}\Big(\frac{2b}{3}\Big)\Big(\frac{2b}{3} - b_1\Big) \le 1 \quad \Big(bM_{12}\Big(\frac{2a}{3}\Big)\Big(\frac{2a}{3} - a_1\Big) \le 1\Big),$$

then set

$$b_2 = \frac{2b}{3} \quad \Big(a_2 = \frac{2a}{3}\Big),$$

and if

$$aM_{11}\Big(\frac{2b}{3}\Big)\Big(\frac{2b}{3} - b_1\Big) > 1 \quad \Big(bM_{12}\Big(\frac{2a}{3}\Big)\Big(\frac{2a}{3} - a_1\Big) > 1\Big),$$

then then there exist $b_2 \in \big(b_1, \frac{2b}{3}\big)$ $\big(a_2 \in \big(a_1, \frac{2a}{3}\big)\big)$ such that

$$aM_{11}(b_2)(b_2 - b_1) = 1 \quad \big(bM_{12}(a_2)(a_2 - a_1) = 1\big).$$

Consequently, in all of the considered cases we have

$$b_2 = b_1 + \min\Big\{\frac{1}{aM_{11}(b_2)}, \frac{2b}{3} - b_1\Big\}, \quad a_2 = a_1 + \min\Big\{\frac{1}{bM_{12}(a_2)}, \frac{2a}{3} - a_1\Big\}.$$

By Lemma 2.1[1], in the closed domain

$$\overline{\Omega}_2 = \big([a_1, a] \times [b_1, b_2]\big) \cup \big([a_1, a_2] \times [b_1, b]\big)$$

equation (1.1) has a solution u_2 satisfying the initial conditions

$$u_2(x, b_1) = u_1(x, b_1) \quad \text{for} \quad a_1 \le x \le a,$$
$$u_2(a_1, y) = u_1(a_1, y) \quad \text{for} \quad b_1 \le y \le b.$$

Repeating this process on and on, we get the numerical and functional sequences $\big(a_k\big)_{k=1}^{+\infty}$, $\big(b_k\big)_{k=1}^{+\infty}$, $\big(M^0_{k1}(b_{k+1})\big)_{k=1}^{+\infty}$, $\big(M_{k1}(b_{k+1})\big)_{k=1}^{+\infty}$, $\big(M^0_{k2}(a_{k+1})\big)_{k=1}^{+\infty}$, $\big(M_{k2}(a_{k+1})\big)_{k=1}^{+\infty}$, and $\big(u_k\big)_{k=1}^{+\infty}$ such that: for any $k \ge 1$ the function u_{k+1} is a solution of equation (1.1) defined on the set

$$\overline{\Omega}_{k+1} = \big([a_k, a] \times [b_k, b_{k+1}]\big) \cup \big([a_k, a_{k+1}] \times [b_k, b]\big)$$

and satisfying the initial conditions

$$u_{k+1}(x, b_k) = u_k(x, b_k) \quad \text{for} \quad a_k \le x \le a,$$
$$u_{k+1}(a_k, y) = u_k(a_k, y) \quad \text{for} \quad b_k \le y \le b;$$

$$M^0_{k1}(b_{k+1}) = \max\{|u_k(x, b_k) - u_k(a_k, b_k)| : a_k \le x \le a\} + \max\{|u_k(a_k, y)| : b_k \le y \le b_{k+1}\},$$

$$M_{k1}(b_{k+1}) = 1 + \max\{|f(x, y, z)| : (x, y) \in \overline{\Omega}(a, b), \tag{3.1}$$
$$|z| \le 1 + M^0_{k1}(b_{k+1})\}; \tag{3.2}$$

$$M^0_{k2}(a_{k+1}) = \max\{|u_k(a_k, y) - u_1(a_k, b_k)| : b_k \le y \le b\} + \max\{|u_k(x, b_k)| : a_k \le x \le a_{k+1}\}, \tag{3.3}$$

$$M_{k2}(a_{k+1}) = 1 + \max\{|f(x, y, z)| : (x, y) \in \overline{\Omega}(a, b), |z| \le 1 + M^0_{k2}(a_{k+1})\}; \tag{3.4}$$

$$b_{k+1} = b_k + \min\Big\{\frac{1}{aM_{k1}(b_{k+1})}, \frac{(k+1)b}{k+2} - b_k\Big\}, \tag{3.5}$$

$$a_{k+1} = a_k + \min\Big\{\frac{1}{bM_{k2}(a_{k+1})}, \frac{(k+1)a}{k+2} - a_k\Big\}. \tag{3.6}$$

It is clear that $\big(a_k\big)_{k=1}^{+\infty}$ and $\big(b_k\big)_{k=1}^{+\infty}$ are increasing sequences satisfying the inequalities

$$0 < a_k < \frac{k}{k+1}a, \quad 0 < b_k < \frac{k}{k+1}b \quad (k = 1, 2, \dots).$$

Set

$$\lim_{k\to+\infty} a_k = a_0, \quad \lim_{k\to+\infty} b_k = b_0, \tag{3.7}$$

$$u(x, y) = \begin{cases} u_1(x, y) & \text{for} \quad (x, y) \in \Omega_1, \\ u_k(x, y) & \text{for} \quad (x, y) \in \overline{\Omega}_k \ \ (k = 2, 3, \dots). \end{cases} \tag{3.8}$$

[1]See Remark 2.1.

If either $a_0 = a$, or $b_0 = b$, then

$$\Omega_1 \cup \Big(\bigcup_{k=2}^{+\infty} \overline{\Omega}_k \Big) = \Omega(a, b),$$

and if

$$a_0 < a, \quad b_0 < b, \tag{3.9}$$

then

$$\Omega_1 \cup \Big(\bigcup_{k=2}^{+\infty} \overline{\Omega}_k \Big) = \Omega_0(a, b_0; a_0, b).$$

In the first (the second) case the function u given by (3.8) is a global solution of problem (1.1),(1.2) (local solution of problem (1.1),(1.2) in the domain $\Omega_0(a, b_0; a_0, b)$) and matches with u_1 in the domain $\Omega_0(a, b_1; a_1; b)$.

To complete the proof of the theorem it remains to show that if inequalities (3.9) hold, then the local solution u a is blow-up solution, i.e. it satisfies conditions (1.4) and (1.5).

By (3.7) and (3.9), we have

$$\lim_{k\to+\infty} \big(b_{k+1} - b_k\big) = 0, \quad \lim_{k\to+\infty} \Big(\frac{(k+1)b}{k+2} - b_k \Big) = b - b_0,$$
$$\lim_{k\to+\infty} \big(a_{k+1} - a_k\big) = 0, \quad \lim_{k\to+\infty} \Big(\frac{(k+1)a}{k+2} - a_k \Big) = a - a_0.$$

Therefore (3.5) and (3.6) imply

$$\lim_{k\to+\infty} M_{k1}(b_{k+1}) = +\infty, \quad \lim_{k\to+\infty} M_{k2}(a_{k+1}) = +\infty.$$

Taking into account these equalities from (3.2) and (3.4) we conclude that

$$\lim_{k\to+\infty} M^0_{k1}(b_{k+1}) = +\infty, \quad \lim_{k\to+\infty} M^0_{k2}(a_{k+1}) = +\infty.$$

However, in view of (3.1),(3.2) and (3.8) the latter equalities imply that

$$\max\{|u_k(x, b_k) - u_k(a_k, b_k)| : a_k \le x \le a\}$$
$$+ \max\{|u_k(a_k, y)| : b_k \le y \le b_{k+1}\} \to +\infty \quad \text{as} \quad k \to +\infty \tag{3.10}$$

and

$$\max\{|u_k(a_k, y) - u_1(a_k, b_k)| : b_k \le y \le b\}$$
$$+ \max\{|u_k(x, b_k)| : a_k \le x \le a_{k+1}\} \to +\infty \quad \text{as} \quad k \to +\infty. \tag{3.11}$$

First show that condition (1.4) holds. Assume the contrary. Then there exists a positive number r_0 such that

$$\limsup_{y\to b_0} r(y) < r_0, \tag{3.12}$$

where $r(y) = \sup\{|u(x, y)| : 0 < x < a\}$. Set

$$M_0 = 2r_0 + \max\{|c_2(y)| : 0 \le y \le b\} \tag{3.13}$$

and

$$M = \max\{|f(x, y, z)| : (x, y) \in \overline{\Omega}(a, b), \ |z| \le 1 + M_0\}.$$

Choose $a_{01} \in (0, a_0)$ such that

$$a_{01}\, b\, M \le 1. \tag{3.14}$$

According to (3.12) there exists $b_{01} \in (0, b_0)$ such that

$$(b_0 - b_{01}) a M \le 1 \tag{3.15}$$

and

$$r(b_{01}) < r_0. \tag{3.16}$$

The restriction of u on the domain

$$\Omega_{01} = \Big((0, a) \times (b_{01}, b_0) \Big) \cup \Big((0, a_{01}) \times (b_{01}, b) \Big)$$

is a solution of equation (1.1) subject to the initial conditions

$$u(x, b_{01}) = v_1(x) \quad \text{for} \quad 0 \le x \le a, \quad u(a_{01}, y) = v_2(y) \quad \text{for} \quad b_{01} \le y \le b,$$

where

$$v_1(x) = u(x, b_{01}) \quad \text{for} \quad 0 \le x \le a, \quad v_2(y) = c_2(y) \quad \text{for} \quad b_{01} \le y \le b.$$

Besides, v_1 and v_2 are continuously differentiable and satisfy the matching condition

$$v_1(0) = v_2(b_{01}).$$

On the other hand, (3.13) and (3.16) imply that

$$|v_1(x) + v_2(y) - v_1(0)| < M_0 \quad \text{for} \quad 0 \le x \le a, \;\; b_{01} \le y \le b. \tag{3.17}$$

By Lemma 2.1, inequalities (3.14),(3.15) and(3.17) guarantee the validity of the estimate

$$|u(x, y)| < 1 + M_0 \quad \text{for} \quad 0 < x < a, \;\; b_{01} < y < b_0.$$

But this estimate contradicts to the condition (3.10). The obtained contradiction proves the validity of the condition (1.4).

The validity of (1.5) can be proved similarly. □

Proof of Theorem 1.2. According to (1.6) an arbitrary global (local) solution of problem (1.1),(1.2) is a global (local) solution of problem (2.23),(1.2). On the other hand, by Lemma 2.4, problem (2.23),(1.2) has no local blow-up solution and its arbitrary global solution is uniformly continuous in $\Omega(a, b)$. Now if we apply Theorem 1.1, then the validity of Theorem 1.2 will become obvious. □

Proof of Theorem 1.3. Without loss of generality we may assume that the function φ satisfies condition (2.20), since otherwise we could replace it by the following one:

$$\varphi_0(z) = \begin{cases} \varphi(z) & \text{for} \quad z > 1, \\ z\varphi(z) & \text{for} \quad 0 \le z \le 1. \end{cases}$$

Let t_0 be an arbitrarily fixed number satisfying condition (2.34). By Lemma 2.3, problem (2.12),(2.14) has a unique solution w satisfying inequalities (2.21). Set $r = w(a) + w(b)$, and show that if inequality (1.11) holds, then problem (1.1),(1.2) has no global solution.

Assume the contrary that problem (1.1),(1.2) has a global solution. Then (1.9) and (1.11) imply that

$$u(x, y) > 0 \quad \text{for} \quad (x, y) \in \Omega(a, b)$$

and u is global solution of problem (2.24),(1.2). However, in this case, by Lemma 2.5, problem (2.24),(1.2) has no global solution. The obtained contradiction proves that if inequality (1.11) holds, then problem (1.1),(1.2) has no global solution. But then, by Theorem 1.1, problem (1.1),(1.2) has at least one blow-up solution. □

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