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Tariel Kiguradze

ON SOLVABILITY OF ILL POSED INITIAL–BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER NONLINEAR HYPERBOLIC EQUATIONS

Abstract. The necessary and sufficient conditions for unique solvability of well posed initial-boundary value problems for higher order nonlinear hyperbolic equations are studied.

[illegible]

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Let $b > 0$, I be a compact interval containing zero, $\Omega = I \times [0, b]$, m and n be natural numbers, $m_0 \in \{0, \dots, m-1\}$, $p_{mk} \in C([0, b])$, $p_{jk} \in C(\Omega)$ $(j = m_0 + 1, \dots, m-1;\ k = 0, \dots, n)$ and $f : \Omega \times \mathbb{R}^{m_0+1} \times \mathbb{R}^{m_0+1 \times n} \to \mathbb{R}$ be a continuous function. In the rectangle Ω for the nonlinear hyperbolic equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(y) u^{(m,k)} + \sum_{j=m_0+1}^{m-1} \sum_{k=0}^{n} p_{jk}(x, y) u^{(j,k)} + \\ + f\Big(x, y, u^{(0,n)}, \dots, u^{(m_0,n)}, \mathcal{D}^{m_0,n-1}[u]\Big) \tag{1}$$

consider the initial–boundary problem

$$\begin{gathered} u^{(j,0)}(0, y) = \varphi_j(y) \qquad (j = 0, \dots, m_0), \\ h_k(u^{(m,0)}(x, \cdot))(x) = \psi_k(x) \qquad (k = 1, \dots, n). \end{gathered} \tag{2}$$

(If $m_0 = m - 1$, then there is no double sum in equation (1).) Here for any j and k

$$u^{(j,k)}(x, y) = \frac{\partial^{j+k} u(x, y)}{\partial x^j \partial y^k}, \quad \mathcal{D}^{m_0,n-1}[u](x, y) = \Big(u^{(j,k)}(x, y)\Big)_{0,0}^{m_0,n-1}$$

$\varphi_j \in C^n([0, b])$, $\psi_k \in C(I)$ and $h_k : C^{n-1}([0, b]) \to C(I)$ is a linear bounded operator.

Throughout the paper the following notations will be used.

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$\mathbb{R}$ is the set of real numbers; $\mathbb{R}^{m\times n}$ is the space of real $m\times n$ matrices

$$Z=(z_{ij})_{1,1}^{m,n}=\begin{pmatrix} z_{11} & \dots & z_{1n}\\ \cdot & \dots & \cdot\\ z_{m1} & \dots & z_{mn}\end{pmatrix}$$

with the norm $\|Z\|=\sum_{i=1}^m\sum_{j=1}^n|z_{ij}|$.

$C(I)$ and $C(\Omega)$, respectively, are the Banach spaces of continuous functions $z:I\to\mathbb{R}$ and $u:\Omega\to\mathbb{R}$, with the norms

$$\|z\|_{C(I)}=\max\{|z(x)|:x\in I\},\quad \|u\|_{C(\Omega)}=\max\{|u(x,y)|:(x,y)\in\Omega\}.$$

$C(I;\mathbb{R}^{m\times n})$ is the Banach space of continuous matrix functions $Z:I\to\mathbb{R}^{m\times n}$ with the norm $\|z\|_{C(I;\mathbb{R}^{m\times n})}=\max\{\|Z(x)\|:x\in I\}$.

$C^k(I)$ is the Banach space of k–times continuously differentiable functions $z:I\to\mathbb{R}$, with the norm

$$\|z\|_{C^k(I)}=\sum_{i=0}^k\|z^{(i)}\|_{C(I)}.$$

$C^{m,n}(\Omega)$ is the Banach space of functions $u:\Omega\to\mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j=0,\dots,m;\ k=0,\dots,n)$, with the norm

$$\|u\|_{C^{m,n}(\Omega)}=\sum_{j=0}^m\sum_{k=0}^n\|u^{(j,k)}\|_{C(\Omega)}.$$

Let $\zeta_k:\Omega\to\mathbb{R}$ $(k=1,\dots,n)$ be functions continuous and n–times continuously differentiable with respect to the second argument such that $\zeta_1(x,\cdot),\dots,\zeta_m(x,\cdot)$ is the fundamental set of solutions of the ordinary differential equation

$$z^{(n)}=\sum_{k=0}^{n-1}p_{mk}(y)z^{(k)} \tag{3}$$

for an arbitrarily given value of the parameter $x\in I$. Introduce the matrix function

$$H(x)=\Big(h_j(\zeta_k(x,\cdot))(x)\Big)_{1,1}^{n,n}. \tag{4}$$

The linear case of problem (1),(2), that is, the equation

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(y)u^{(m,k)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n}p_{jk}(x,y)u^{(j,k)}+q(x,y) \tag{5}$$

with conditions (2) was studied in [2] and [3]. In [2] it was established that problem $(5),(2)$ is well–posed *if and only if*

$$\det H(x)\neq 0\quad\text{for}\quad x\in I, \tag{6}$$

i.e., for any $x\in I$ equation (3) does not have a nontrivial solution satisfying the boundary conditions

$$h_k(z)(x)=0\quad(k=1,\dots,n). \tag{7}$$

Criteria for so–called μ–well–posedness of problem $(5),(2)$ were proved in [2] for the case in which condition (6) fails but $\mu(x) \overset{\text{def}}{=} \det H(x) \not\equiv 0$.

In [5] it was proved that if (6) holds and f is Lipschitz continuous with respect to the phase variables, then problem $(1),(2)$ is locally well–posed.

In the present paper we study problem $(1),(2)$ in the ill–posed case $\det H(x) \equiv 0$. More precisely, we consider the case in which there exists $n_0 \in \{1,\dots,n\}$ such that for an arbitrary $x \in I$ problem (3),(7) has an n_0–dimensional space of solutions, i.e.,

$$\operatorname{rank} H(x) = n_1 \quad \text{for} \quad x \in I, \quad \text{where} \quad n_1 = n - n_0. \tag{8}$$

In ill–posed case problem $(5),(2)$ was studied in [3]. There was proved that without loss of generality (if necessary, considering an equivalent problem) one may assume that the matrix function $H(x)$ has the form

$$\text{either } H(x) \equiv \Theta_{n,n}; \ \text{ or } \ n_0 < n,$$

$$H(x) = \begin{pmatrix} \Theta_{n_0,n_0} & \Theta_{n_0,n_1} \\ \Theta_{n_1,n_0} & H_0(x) \end{pmatrix} \ \text{ and } \ \det H_0(x) \neq 0 \ \text{ for } \ x \in I,$$

where Θ_{n_i,n_k} is the zero $n_i \times n_k$ matrix, and E_{n_1,n_1} is the unit $n_1 \times n_1$ matrix.

It turns out that, unlike to well–posed case, in ill–posed case for solvability of problem $(1),(2)$ $(m-m_0)n_0$ compatibility conditions should be satisfied. Furthermore, additional regularity of the righthand side of equation (1) and the boundary data is also needed.

By $\zeta(\cdot,\cdot)$ denote the Cauchy function of equation (3) and set:

$$\Phi_{m_0}(y) = \Big(\varphi_{j-1}^{(k-1)}(y)\Big)_{1,1}^{m_0,n};$$

$$f_{jn}(x,y,z_0,\dots,z_{m_0},Z) = \frac{\partial f(x,y,z_0,\dots,z_{m_0},Z)}{\partial z_j} \qquad (j=0,\dots,m_0);$$

$$f_{jk}(x,y,z_0,\dots,z_{m_0},Z) = \frac{\partial f(x,y,z_0,\dots,z_{m_0},Z)}{\partial z_{jk}},$$

$$p_{jk}^0(y) = f_{jk}\Big(0,y,\varphi_0^{(n)}(y),\dots,\varphi_{m-1}^{(n)}(y),\Phi_{m_0}(y)\Big),$$

$$\rho_{jk}^0(y) = p_{jk}^0(y) + p_{jn}^0(y)p_{mk}(y) \quad (j=0,\dots,m_0;\ k=0,\dots,n-1);$$

$$\eta_{jk}^0(y) = \int_0^y \zeta(y,t)\sum_{l=0}^{n-1}\rho_{jl}^0(t)\zeta_k^{(0,l)}(0,t)\,dt \quad (j=0,\dots,m_0;\ k=1,\dots,n);$$

$$\lambda_{j_{ik}}^0 = h_i(\eta_{jk}^0)(0) \ \ (i,k=1,\dots,n), \qquad \Lambda_j^0 = \big(\lambda_{j_{ik}}^0\big)_{1,1}^{n_0,n_0} \quad (j=0,\dots,m_0).$$

Let $u \in C^{m,n}(\Omega)$ be an arbitrary function satisfying the initial conditions

$$u^{(j,0)}(0,y) = \varphi_j(y) \qquad (j=0,\dots,m-1), \tag{9}$$

$f(x,y,z_0,\dots,z_{m_0},Z)$ be $m-m_0$–times continuously differentiable with respect to x, $z_0,\dots,z_{m_0}$ and Z, and let w be a solution of the ordinary differential equation

$$w^{(m)} = \sum_{j=m_0+1}^{m-1} p_{jn}(x,y)w^{(j)} + \\ + f(x,y,u^{(0,n)}(x,y),\dots,u^{(m_0,n)}(x,y),\mathcal{D}^{m_0,n-1}[u(x,y)]) \qquad (10)$$

satisfying the initial conditions

$$w^{(j)}(0) = \varphi_j^{(n)}(y) - \sum_{k=0}^{n-1} p_{mk}(y)\varphi_j^{(k)}(y) \qquad (j = 0,\dots,m-1). \qquad (11)$$

(If $m_0 = m-1$, then there is no sum in equation (10)). It is clear that $w \in C^{(2m-m_0,0)}(\Omega)$. Differentiating equation (10) $m-1-m_0$ times and taking into account (9) and (11), one can easily see that that for any $i \in \{0,\dots,m-1-m_0\}$ $w^{(m+i,0)}(0,y)$ can be expressed in terms of the functions $\varphi_0,\dots,\varphi_{m-1}$. More precisely,

$$w^{(m+i,0)}(0,y) = \mathcal{W}_i[\varphi_0,\dots,\varphi_{m-1}](y) \qquad (i = 0,\dots,m-1-m_0),$$

where $\mathcal{W}_i$ $(i = 0,\dots,m-1-m_0)$ continuous nonlinear operators.

If $h : C^{n-1}([0,b]) \to C^l(I)$, then for any $i \in \{0,\dots,l\}$ by $h^{(i)}$ denote the operator defined by the equality

$$h^{(i)}(z)(x) = \frac{d^i}{dx^i}\big[h(z)(x)\big].$$

Theorem 1. *Let there exist* $m_0 \in \{0,\dots,m-1\}$ *such that*

$$p_{jk}(x,y) + p_{jn}(x,y)p_{mk}(y) = 0 \\ (j = m_0+1,\dots,m-1;\ k = 0,\dots,n-1),^* \qquad (12)$$

$$\det \Lambda^0_{m_0} \neq 0. \qquad (13)$$

Furthermore, let $f(x,y,z_0,\dots,z_{m_0},Z)$ *be* $m-m_0$*–times continuously differentiable with respect to* x, $z_0,\dots,z_{m_0}$ *and* Z, $p_{jk} \in C^{m-m_0,0}(\Omega)$ $(j = m_0+1,\dots,m-1,\ k = 0,\dots,n)$, $\psi_k \in C^{m-m_0}(I)$ $(k = 1,\dots,n)$ *and* $h_k : C^{n-1}([0,b]) \to C^{m-m_0}(I)$ $(k = 1,\dots,n)$ *be linear bounded operators. Then problem* (1), (2) *has a unique local solution if and only if the following equalities hold*

$$\sum_{i=0}^{l} \frac{l!}{i!(l-i)!} h_k^{(l-i)}(\mathcal{W}_i[\varphi_0,\dots,\mathcal{W}_{m-1}])(0) = \psi_k^{(l)}(0) \\ (k = 1,\dots,n_0;\ l = 0,\dots,m-1-m_0). \qquad (14)$$

Remark 1. If $h_k : C^{n-1}([0,b]) \to \mathbb{R}$ $(k = 1,\dots,n)$ are bounded linear functionals, then (14) receives the form

$$h_k(\mathcal{W}_l[\varphi_0,\dots,\mathcal{W}_{m-1}]) = \psi_k^{(l)}(0) \quad (k = 1,\dots,n_0;\ l = 0,\dots,m-1-m_0).$$

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* If $m_0 = m-1$, then this condition is omitted.

If $m_0 = m-1$, then (14) has the form

$$h_k(\mathcal{W}_0[\varphi_0,\dots,\mathcal{W}_{m-1}])(0) = \psi_k(0) \quad (k=1,\dots,n_0),$$

where

$$\mathcal{W}_0(y) = \int_0^y \zeta(y,t)\, f\big(0,t,\varphi_0^{(n)}(t),\dots,\varphi_{m_0}^{(n)}(t),\Phi_{m_0}(t)\big)\,dt.$$

Remark 2. Let $\Omega_- = \{(x,y)\in\Omega : x\le 0\}$, $\Omega_+ = \{(x,y)\in\Omega : x\ge 0\}$, $m_1 = m - m_0$ and α_j $(j=0,\dots,m_1)$ are the natural numbers defined by the identity

$$(z+1)(z+2)\dots(z+m_1) = \sum_{j=0}^{m_1}\alpha_j z^j.$$

By Theorem 1 in [5], conditions (12),(13) ensure that that for an arbitrarily small $\varepsilon\neq 0$ the differential equation

$$\begin{aligned} u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(y)u^{(m,k)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)u^{(j,k)} + \\ + \frac{1}{m_1!}\sum_{j=1}^{m_1}\sum_{k=0}^{n-1}\alpha_j\varepsilon^j\rho_{m_0k}(x,y)u^{(m_0+j,k)} + \\ + f(x,y,u^{(0,n)},\dots,u^{(m_0,n)},\mathcal{D}^{m_0,n-1}[u]) \end{aligned} \tag{1_ε}$$

has a unique local solution u_ε satisfying the initial–boundary conditions (2). In fact we show that if along with the above mentioned conditions equalities (14) hold, then

$$\begin{aligned} &u_\varepsilon(x,y)\to u(x,y) \ \text{ uniformly on } \ \Omega_+ \ \text{ as } \ \varepsilon\downarrow 0,\\ &u_\varepsilon(x,y)\to u(x,y) \ \text{ uniformly on } \ \Omega_- \ \text{ as } \ \varepsilon\uparrow 0, \end{aligned}$$

where u is a solution of problem $(1),(2)$.

Set

$$p_{jk}[u](x,y) = f_{jk}\Big(x,y,u^{(0,n)}(x,y),\dots,u^{(m-1,n)}(x,y),\mathcal{D}^{m-1,n-1}[u](x,y)\Big),$$

$$\rho_{jk}[u](x,y) = p_{jk}[u](x,y) + p_{jn}[u](x,y)p_{mk}(y)$$

$$(j=0,\dots,m_0;\ k=0,\dots,n-1);$$

$$\eta_{jk}[u](x,y) = \int_0^y \zeta(y,t,x)\sum_{l=0}^{n-1}\rho_{jl}[u](x,t)\zeta_k^{(0,l)}(x,t)\,dt$$

$$(j=0,\dots,m_0;\ k=1,\dots,n);$$

$$\lambda_{j_{ik}}[u](x) = h_i(\eta_{jk}[u](x,\cdot))(x) \ \ (i,k=1,\dots,n),$$

$$\Lambda_j[u](x) = \big(\lambda_{j_{ik}}[u](x)\big)_{1,1}^{n_0,n_0} \quad (j=0,\dots,m_0).$$

Theorem 2. *Let all of the conditions of Theorem* 1 *hold and* $u_0 : I_0 \times [0,b] \to \mathbb{R}$ *be a non–continuable solution of problem* (1), (2) *such that*

$$\det \Lambda_{m_0}[u](x) \neq 0 \quad \textit{for} \quad x \in I_0. \tag{15}$$

Then I_0 *is an open set in* I. *Moreover, if* $a^* = \sup I_0 \notin I_0$, *then*

$$\lim_{x \to a^*} \sup \Big\{ \sum_{j=0}^{m_0} \|u_0^{(j,0)}(x,\cdot)\|_{C^n([0,b])} : y \in [0,b] \Big\} \to +\infty,$$

and if $a_* = \inf I_0 \notin I_0$, *then*

$$\lim_{x \to a_*} \sup \Big\{ \sum_{j=0}^{m_0} \|u_0^{(j,0)}(x,\cdot)\|_{C^n([0,b])} : y \in [0,b] \Big\} \to +\infty.$$

Remark 3. In Theorems 1 and 2 conditions (13) and (15) are sharp and cannot be weakened. Indeed in the rectangle $[0, m_0!] \times [0,b]$ consider the initial–periodic problem

$$u^{(m,n)} = |u|^{2m+1} u^{(m_0,0)} + u^{2m+1}; \tag{16}$$

$$\begin{gathered} u^{(j,0)}(0,y) = c_j \quad (j = 0, \ldots, m-1), \\ u^{(m,k-1)}(x,0) = u^{(m,k-1)}(x,\pi) \ \ (k = 1, \ldots, n), \end{gathered} \tag{17}$$

where $c_0 = 1$, $c_{m_0} = -1$ and $c_j = 0$ for $j \in \{1, \ldots, m-1\} \setminus \{m_0\}$. By Theorem 1, problem (16),(17) has a unique local solution u, which is independent of y (due to uniqueness). Therefore u is a solution to the initial value problem ordinary differential equation

$$z^{(m_0)} = -\operatorname{sgn}(z); \quad z^{(j)}(0) = c_j \quad (j = 0, \ldots, m_0 - 1). \tag{18}$$

But one can easily see that problem (18) has a unique non-continuable solution

$$z(x) = 1 - \frac{x^{m_0}}{m_0!}$$

defined on $[0, (m_0!)^{\frac{1}{m_0}}]$.

Corollary 1. *Let all of the conditions of Theorem* 1 *hold, and let there exist* $\delta > 0$ *such that*

$$|\det \Lambda_{m_0}[v](x)| \geq \delta \quad \textit{for} \quad x \in I$$

for any $v \in C^{m_0,n}(\Omega)$ *and*

$$|f(x,y,z_0,\ldots,z_{m_0},Z)| \leq \delta^{-1}\Big(1 + \sum_{i=0}^{m_0} |z_i| + \|Z\|\Big).$$

Then problem (1), (2) *has a unique solution in* Ω *if and only if* (14) *holds.*

Finally for the equation

$$u^{(m,2)} = -u^{(m,0)} + f(x, y, u^{(0,n)}, \dots, u^{(m-1,n)}, \mathcal{D}^{m_0,n-1}[u]), \tag{19}$$

consider the initial–Dirichlet and initial–periodic problems

$$\begin{aligned} &u^{(j,0)}(0, y) = \varphi_j(y) \quad (j = 0, \dots, m-1), \\ &u^{(m,0)}(x, 0) = 0, \ \ u^{(m,0)}(x, \pi) = 0 \end{aligned} \tag{20}$$

and

$$\begin{aligned} &u^{(j,0)}(0, y) = \varphi_j(y) \quad (j = 0, \dots, m-1), \\ &u^{(m,k)}(x, 0) = u^{(m,k)}(x, 2\pi) \ \ (k = 0, 1). \end{aligned} \tag{21}$$

Corollary 2. *Let $f(x, y, z_0, \dots, z_{m_0}, Z)$ be continuously differentiable with respect to x, $z_0, \dots, z_{m-1}$ and Z, and let*

$$\int_0^\pi \big(\big(p^0_{m-1\,0}(t) - p^0_{m-1\,2}(t)\big)\sin^2 t + p^0_{m-1\,1}(t)\cos t \sin t\big)\, dt \neq 0.$$

Then problem (19), (20) *is locally uniquely solvable if and only if*

$$\int_0^\pi f(0, t, \varphi_0^{(n)}(t), \dots, \varphi_{m-1}^{(n)}(t), \Phi_{m-1}(t)) \sin t \, dt = 0.$$

Corollary 3. *Let $f(x, y, z_0, \dots, z_{m_0}, Z)$ be continuously differentiable with respect to x, $z_0, \dots, z_{m-1}$ and Z, and let*

$$\det \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \neq 0,$$

where

$$\lambda_{11} = \int_0^{2\pi} \big(\big(p^0_{m-1\,2}(t) - p^0_{m-1\,2}(t)\big)\sin^2 t + p^0_{m-1\,1}(t)\cos t \sin t\big)\, dt,$$

$$\lambda_{12} = \int_0^{2\pi} \big(\big(p^0_{m-1\,2}(t) - p^0_{m-1\,2}(t)\big)\cos t \sin t - p^0_{m-1\,1}(t)\sin^2 t\big)\, dt,$$

$$\lambda_{21} = \int_0^{2\pi} \big(\big(p^0_{m-1\,2}(t) - p^0_{m-1\,2}(t)\big)\cos t \sin t + p^0_{m-1\,1}(t)\cos^2 t\big)\, dt,$$

$$\lambda_{22} = \int_0^{2\pi} \big(\big(p^0_{m-1\,2}(t) - p^0_{m-1\,2}(t)\big)\cos^2 t - p^0_{m-1\,1}(t)\cos t \sin t\big)\, dt.$$

Then problem (19), (21) *is locally uniquely solvable if and only if*

$$\int_0^{2\pi} f(0,t,\varphi_0^{(n)}(t),\dots,\varphi_{m-1}^{(n)}(t),\Phi_{m-1}(t))\sin t\,dt=0,$$

$$\int_0^{2\pi} f(0,t,\varphi_0^{(n)}(t),\dots,\varphi_{m-1}^{(n)}(t),\Phi_{m-1}(t))\cos t\,dt=0.$$

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Author's address:

Florida Institute of Technology
Department of Mathematical Sciences
150 W. University Blvd.
Melbourne, Fl 32901
USA
E-mail: tkigurad@fit.edu