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ON LYAPUNOV STABILITY OF A CLASS OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND LINEAR IMPULSIVE SYSTEMS

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In this paper necessary and sufficient conditions are given for the stability in Lyapunov sense of solutions of a linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \quad \text{for} \quad t \in \mathbb{R}_+ \,, \tag{1}$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ ($\mathbb{R}_+ = [0, +\infty)$) are, respectively, matrix and vector-functions with bounded total variation components on every closed interval from $\mathbb{R}_+$, having defined properties analogously to the case of constant coefficients.

The results are realized for the linear impulsive system

$$\frac{dx}{dt} = Q(t)x + q(t) \quad \text{for} \quad t \in \mathbb{R}_+ \,, \tag{2}$$

$$x(t_k+) - x(t_k-) = G_k\, x(t_k-) + g_k \quad (k = 1, 2, \dots), \tag{3}$$

where $Q : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $q : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, matrix - and vector-functions with Lebesgue integrable components on every closed interval from $\mathbb{R}_+$, $G_k \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$, $g_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$, $0 < t_1 < t_2 < \cdots$, $\lim\limits_{k\to+\infty} t_k = +\infty$.

Analogous results are given, for example, in [1], [2] for systems of linear ordinary differential equations.

We use the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $[a, b]$ $(a, b \in \mathbb{R})$ is a closed interval from $\mathbb{R}$, $[t]$ is the integral part of $t \in \mathbb{R}$.

$\mathbb{C}$ is the space of all complex numbers.

$\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) is the set of all real (complex) $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$.

If $X \in \mathbb{C}^{n\times n}$, then X^{-1}, $\ln X$, $\det X$ and $r(X)$ are, respectively, the matrix, inverse to X, the logarithm (the principle value), the determinant and the spectral radius of X. $\operatorname{diag}(X_1, \dots, X_m)$, where $X_i \in \mathbb{C}^{n_i\times n_i}$ $(i = 1, \dots, m)$, $n_1 + \cdots + n_m = n$, is a quasi-diagonal $n \times n$-matrix; I_n is the identity $n \times n$-matrix; δ_{ij} is the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, 2, \dots)$; $Z_n = (\delta_{i+1\, j})_{i,j=1}^n$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{V}}(X)$ is the sum of total variations on $[a, b] \subset \mathbb{R}_+$ of its components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$.

$X(t-)$ and $X(t+)$ are the left and the right limits at the point $t \in \mathbb{R}_+$ ($X(0-) = X(0)$); $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$; $S_0(X)(t) = X(t) - \sum\limits_{0<\tau\le t} d_1x(t) - \sum\limits_{0\le\tau<t} d_2X(t)$ is the continuous part of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$.

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$BV_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ for which $\overset{b}{\underset{a}{V}}(X)<\infty$ for $a,b\in\mathbb{R}_+$.

$L_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ the components of which are the functions measurable and integrable in Lebesgue sense on every closed interval from $\mathbb{R}_+$.

$\widetilde{C}_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ the components of which are the functions absolutely continuous on every closed interval from $\mathbb{R}_+$;

$\widetilde{C}_{\text{loc}}(\mathbb{R}_+\backslash\{t_k\}_{k=1}^{\infty};\mathbb{R}^{n\times m})$, where $0<t_1<t_2<\dots$, is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ the restrictions of which on the arbitrary closed interval from $\mathbb{R}_+\backslash\{t_k\}_{k=1}^{\infty}$ are absolutely continuous functions.

If $g:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing function, $x:\mathbb{R}_+\to\mathbb{R}$ and $0\le s<t<\infty$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)+\sum_{s<\tau\le t}x(\tau)\,d_1\,g(\tau)+\sum_{s\le\tau<t}x(\tau)\,d_2\,g(\tau),$$

where $\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue-Stiltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_0(g)$ $\big($if $s=t$, then $\int_s^t x(\tau)\,dg(\tau)=0\big)$; if $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_s^t x(\tau)\,dg_1(\tau)-\int_s^t x(\tau)\,dg_2(\tau).$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in BV_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:\mathbb{R}_+\to\mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\Big(\sum_{k=1}^{n}\int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{l.m}\quad\text{for}\quad 0\le s\le t<\infty.$$

Under a solution of the system (1) we understand a vector-function $x\in BV_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ such that

$$x(t)-x(s)=\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\quad\text{for}\quad 0\le s\le t<\infty.$$

We assume that $A\in BV_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $A(0)=O_{n\times n}$, $f\in BV_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ and

$$\det\big(I_n+(-1)^j d_jA(t)\big)\neq 0\quad\text{for}\quad t\in\mathbb{R}_+\quad(j=1,2). \tag{4}$$

The stability in this or another sense of the solution of the system (1) is defined just in the same way as for systems of ordinary differential equations (see [1], [2]).

Definition 1. The system (1) is called stable in this or another sense if every its solution is stable in the same sense.

It is evident that the system (1) is stable if and only if the zero solution of its corresponding homogeneous system

$$dx(t)=dA(t)\cdot x(t)\quad\text{for}\quad t\in\mathbb{R}_+ \tag{1_0}$$

is stable in the same sense.

Therefore, the stability is not the property of some solution of the system (1). It is the common property of all solutions and the vector-function f does not affect this property. Hence it is the property only of the matrix-function A. Thus the following definition is natural.

Definition 2. The matrix-function A is called stable in this or another sense if the system (1_0) is stable in the same sense.

Theorem 1. *Let the matrix-function $A \in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_0(A)(t) = \sum_{l=1}^{m} s_0(\alpha_l)(t)\cdot B_l \quad \text{for} \quad t\in\mathbb{R}_+$$

and

$$I_n + (-1)^j d_j A(t) = \exp\left((-1)^j \sum_{l=1}^{m} d_j\alpha_l(t)\cdot B_l\right) \ \text{for} \ t\in\mathbb{R}_+ \quad (j=1,2),$$

where $B_l \in \mathbb{R}^{n\times n}$ $(l=1,\dots,m)$ are pairwise permutable constant matrices, and $\alpha_l \in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(l=1,\dots,m)$ are such that

$$\lim_{t\to+\infty} \alpha_l(t) = +\infty \quad (l=1,\dots,m). \tag{5}$$

Then: a) *the matrix-function A is stable if and only if every eigenvalue of the matrices B_l $(l=1,\dots,m)$ has the nonpositive real part and, in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;* b) *the matrix-function A is asymptotically stable if and only if every eigenvalue of the matrices B_l $(l=1,\dots,m)$ has the negative real part.*

If the matrix-function $A \in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ has at most a finite number of discontinuity points in $[0,t]$ for every $t>0$, then by $\nu_1(t)$ and $\nu_2(t)$ we denote, respectively, the number of points $\tau \in]0,t]$ for which $\|d_1A(t)\| \neq 0$ and the number of points $\tau\in[0,t[$ for which $\|d_2A(t)\|\neq 0$.

Corollary 1. *Let $A \in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that $S_0(A)(t)\equiv\alpha(t)A_0$ and $d_jA(t) = A_j$ if $\|d_jA(t)\|\neq 0$ $(t\in\mathbb{R}_+;\ j=1,2)$, where $\alpha\in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}_+)$ is a continuous function such that*

$$\lim_{t\to+\infty}\alpha(t) = +\infty,$$

and $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$ are pairwise permutable constant matrices. Let, moreover, there exist numbers $\beta_1,\beta_2\in\mathbb{R}_+$ such that

$$\lim_{t\to+\infty}\sup\left|\nu_j(t) - \beta_j\,\alpha(t)\right| < +\infty \quad (j=1,2).$$

Then: a) *the matrix-function A is stable if and only if every eigenvalue of the matrix $P = A_0 - \beta_1\ln(I_n - A_1) + \beta_2\ln(I_n + A_2)$ has the nonpositive real part and, in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;* b) *the matrix-function A is asymptotically stable if and only if every eigenvalue of the matrix P has the negative real part.*

Corollary 2. *Let the matrix-function $A\in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_0(A)(t) = C\,\mathrm{diag}\left(S_0(G_1)(t),\dots,S_0(G_m)(t)\right)C^{-1} \quad \text{for} \quad t\in\mathbb{R}_+$$

and

$$I_n + (-1)^j d_jA(t) = C\,\mathrm{diag}\left(\exp((-1)^j d_jG_1(t)),\dots,\exp((-1)^j d_jG_m(t))\right)C^{-1} \quad \text{for} \quad t\in\mathbb{R}_+ \quad (j=1,2),$$

where $C\in\mathbb{C}^{n\times n}$ is a nonsingular constant matrix, $G_l(t)\equiv\sum\limits_{i=0}^{n_l-1}\alpha_{li}(t)Z^i_{n_l}$ $(l=1,\dots,m)$, $\sum\limits_{l=1}^{m} n_l = n$, $\alpha_{li}\in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(l=1,\dots,m;\ i=1,\dots,n_l-1)$, and α_{l0} is a complex-valued function such that $\mathrm{Re}\,\alpha_{l0}$ and $\mathrm{Im}\,\alpha_{l0}\in BV_{\rm loc}(\mathbb{R}_+;\mathbb{R})$. Then: a) *the matrix-function A is stable if and only if*

$$\sup\left\{\exp(\mathrm{Re}\,\alpha_{l0}(t))\prod_{i=1}^{n_l-1}(1+\alpha_{li}(t))^{\left[\frac{n_l-1}{i}\right]} : t\in\mathbb{R}_+\right\} < +\infty \ \ (l=1,\dots,m);$$

b) *the matrix-function A is asymptotically stable if and only if*

$$\lim_{t\to+\infty}\exp(\operatorname{Re}\alpha_{l_0}(t))\prod_{i=1}^{n_l-1}(1+\alpha_{li}(t))^{\left[\frac{n_l-1}{i}\right]}=0\quad(l=1,\dots,m).$$

Theorem 2. *Let $\alpha_{il}\in\mathbb{R}$ $(i,l=1,\dots,n)$, and $\mu_i:\mathbb{R}_+\to\mathbb{R}$ $(i=1,\dots,n)$ be nondecreasing functions such that $s_0(\mu_i)\in\widetilde{C}_{\rm loc}(\mathbb{R}_+;\mathbb{R})$ $(i=1,\dots,n)$,*

$$\lim_{t\to+\infty}a_0(t)=+\infty,\quad \sigma_i=\lim_{t\to+\infty}\inf(\alpha_{ii}\,d_2\,\mu_i(t))>-1\quad(i=1,\dots,n)$$

and

$$\alpha_{ii}<0\quad(i=1,\dots,n),\quad r(H)<1, \tag{6}$$

where $a_0(t)\equiv\int\limits_0^t\eta_0(s)\,ds+\sum\limits_{0<s\le t}\ln|1-\eta_1(s)|-\sum\limits_{0\le s<t}\ln|1+\eta_2(s)|$, $\eta_0(t)\equiv\min\big\{|\alpha_{ii}|(s_0(\mu_i)(t))'\ :\ i=1,\dots,n\big\}$, $\eta_j(t)\equiv\max\big\{\alpha_{ii}d_j\mu_i(t)\ :\ i=1,\dots,n\big\}$ $(j=1,2)$, $H=((1-\delta_{il})(1+|\sigma_i|)^{-1}|\alpha_{il}|\,|\alpha_{ii}|^{-1})_{i,l=1}^n$. *Then the matrix-function $A(t)=(\alpha_{il}\mu_i(t))_{i,l=1}^n$ is asymptotically stable. Conversely, if this matrix-function is asymptotically stable,*

$$\alpha_{il}\ge0\quad(i\ne l;\ \ i,l=1,\dots,n) \tag{7}$$

and

$$\sum_{l=1,\,l\ne i}^n\alpha_{il}d_1\mu_i(t)<\min\big\{1-\alpha_{ii}d_1\mu_i(t),|1+\alpha_{ii}d_1\mu_i(t)|\big\}\ \ \textit{for}\ \ t\in\mathbb{R}_+\ \ (i=1,\dots,n),$$

then the condition (6) *holds.*

Impulsive systems. Under a solution of the impulsive system (2), (3) we understand a continuous from the left vector-function $x\in\widetilde{C}_{\rm loc}(\mathbb{R}_+\backslash\{t_k\}_{k=1}^{+\infty};\mathbb{R}^n)$ satisfying both the system (2) almost everywhere on $]t_k,t_{k+1}[$ and the relation (3) at the point t_k for every $k=\{1,2,\dots\}$.

The stability in this or another sense of the solutions of the system (2), (3) is defined as above as well as to the stability of this system.

Besides, the homogeneous system, corresponding to the impulsive system (2), (3), is defined by the pair $(Q,\{G_k\}_{k=1}^\infty)$. Therefore, here we speak on the stability of this pair.

We assume that

$$\det(I_n+G_k)\ne0\quad(k=1,2,\dots). \tag{8}$$

By $\nu(t)$ $(t>0)$ we denote the number of the points t_k $(k=1,2,\dots)$ belonging to $[0,t[$.

It is easy to show that the vector-function x is a solution of the impulsive system (2), (3) if and only if it is a solution of the system (1), where

$$A(t)\equiv\int\limits_0^tQ(\tau)d\tau+\sum_{0\le t_k<t}G_k,\quad f(t)\equiv\int\limits_0^tq(\tau)d\tau+\sum_{0\le t_k<t}g_k.$$

In addition, the condition (4) is equivalent to the condition (8). Thus the impulsive system (2), (3) is a particular case of the system (1) and the following results immediately follow from the analogous results given above for the system (1).

Theorem 3. *Let $Q\in L_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G_k\in\mathbb{R}^{n\times n}$ $(k=1,2,\dots)$ be such that*

$$\int\limits_0^tQ(\tau)d\tau=\sum_{l=1}^m\alpha_{0l}(t)B_l\quad\textit{for}\quad t\in\mathbb{R}_+$$

and

$$G_k=\exp\bigg(\sum_{l=1}^m\alpha_{kl}B_l\bigg)-I_n\quad(k=1,2,\dots),$$

where $B_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m)$ *are pairwise permutable constant matrices,* $\alpha_{0l} \in BV_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ $(l = 1,\dots,m)$ *are continuous functions, and* $\alpha_{kl} \in \mathbb{R}$ $(l = 1,\dots,m;$ $k = 1,2,\dots)$ *are numbers such that the functions*

$$\alpha_l(t) = \alpha_{0l}(t) + \sum_{0\le t_k<t} \alpha_{kl} \quad (l = 1,\dots,m)$$

are nonnegative and satisfy the condition (5). *Then:* a) *the pair* $(Q,\{G_k\}_{k=1}^{\infty})$ *is stable if and only if every eigenvalue of the matrices* B_l $(l = 1,\dots,m)$ *has the nonpositive real part and, in addition, every eigenvalue with the zero real part is simple;* b) *the pair* $(Q,\{G_k\}_{k=1}^{\infty})$ *is asymptotically stable if and only if every eigenvalue of the matrices* B_l $(l = 1,\dots,m)$ *has the negative real part.*

Corollary 3. *Let*

$$Q(t) \equiv \alpha(t)Q_0, \quad G_k = G_0 \quad (k = 1,2,\dots)$$

and there exist $\beta \in \mathbb{R}_+$ *such that*

$$\lim_{t\to+\infty} \sup|\nu(t) - \beta t| < +\infty,$$

where Q_0 *and* G_0 *are permutable constant* $n\times n$*-matrices, and* $\alpha \in L_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ *is such that*

$$\int_0^{+\infty} \alpha(t)\,dt = +\infty.$$

Then: a) *the pair* $(Q,\{G_k\}_{k=1}^{\infty})$ *is stable if and only if every eigenvalue of the matrix* $P = Q_0 + \beta\ln(I + G_0)$ *has the nonpositive real part and, in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;* b) *the pair* $(Q,\{G_k\}_{k=1}^{\infty})$ *is asymptotically stable if and only if every eigenvalue of the matrix* P *has the negative real part.*

Corollary 4 ([3])**.** *Let* $Q(t) \equiv Q_0$, $G_k = G_0$ $(k = 1,2,\dots)$ *and* $t_{k+1} - t_k = \eta = \mathrm{const}$ $(k = 1,2,\dots)$, *where* Q_0 *and* G_0 *are permutable constant* $n\times n$*-matrices. Then the conclusion of Corollary* 3 *is true, where* $P = Q_0 + \eta^{-1}\ln(I + G_0)$.

Theorem 4. *Let* $\alpha_{il} \in \mathbb{R}$, $\nu_{ik} \in \mathbb{R}_+$ $(i,l = 1,\dots,n;\ k = 1,2,\dots)$ *and the functions* $\nu_i \in L_{\mathrm{loc}}(\mathbb{R}_+ : \mathbb{R}_+)$ $(i = 1,\dots,n)$ *be such that the conditions* (6),

$$\int_0^{\infty} \eta(s)ds + \sum_{0\le t_k<\infty} \ln|1 + \eta_k| = -\infty$$

and

$$\sigma_i = \lim_{k=+\infty} \inf(\alpha_{ii}\nu_{ki}) > -1 \ \ (i = 1,\dots,n)$$

hold, where $\eta(t) \equiv \min\{|\alpha_{ii}|\nu_i(t) : i = 1,\dots,n\}$, $\eta_k = \max\{\alpha_{ii}\nu_{ki} : i = 1,\dots,n\}$ $(k = 1,2,\dots)$, $H = ((1-\delta_{il})(1+|\sigma_i|)^{-1}|\alpha_{il}||\alpha_{ii}|^{-1})_{i,l=1}^{n}$. *Then the pair* $(Q,\{G_k\}_{k=1}^{\infty})$, *where* $Q(t) \equiv (\alpha_{il}\nu_i(t))_{i,l=1}^{n}$ *and* $G_k = (\alpha_{il}\nu_{ki})_{i,l=1}^{n}$ $(k = 1,2,\dots)$, *is asymptotically stable. Conversely, if this pair is asymptotically stable and the condition* (7) *holds, then the condition* (6) *holds as well.*

Remark 1. Some results on the stability and the asymptotic stability of the linear system of ordinary differential equations

$$\frac{dx}{dt} = Q(t)x + q(t)$$

follow from Theorems 1, 2 and Corollaries 2–4 if there we assume $G_k = O_{n\times n}$, $\alpha_{kl} = 0$, $\nu_{ki} = 0$ $(l = 1,\dots,m;\ i = 1,\dots,n;\ k = 1,2,\dots)$ and $\beta = 0$.

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