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ON OSCILLATORY SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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The proposed note deals with the asymptotic properties of solutions of the differential equation

$$\big(r(t)u''\big)' = p_0(t)u + p_1(t)u' + q(t), \tag{1}$$

where $r : [0,+\infty[\to]0,+\infty[$, $p_i : [0,+\infty[\to R$ $(i = 1,2)$ and $q : [0,+\infty[\to R$ are continuous functions.

The following theorem is valid.

Theorem. *Let*

$$\begin{gathered}\int\limits_0^{+\infty} \frac{t}{r(t)}\,dt < +\infty,\\ \int\limits_0^{+\infty} |p_i(t)|\,dt < +\infty \ \ (i = 0,1), \quad \int\limits_0^{+\infty} |q(t)|\,dt < +\infty.\end{gathered} \tag{2}$$

Then an arbitrary oscillatory solution of the equation (1) *satisfies the conditions*

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} u'(t) = \lim_{t\to+\infty} r(t)u''(t) = 0. \tag{3}$$

Proof. Let the sequence $(t_k)_{k=1}^{+\infty}$ be such that

$$u(t_k) = 0, \quad 1 < t_k < t_{k+1} \quad (k = 1,2,\dots).$$

Then for each natural k there exists $\widetilde{t}_k \in]t_k, t_{k+2}[$ such that

$$u''(\widetilde{t}_k) = 0.$$

Hence (1) implies

$$u''(t) = \frac{1}{r(t)} \int\limits_{\widetilde{t}_k}^{t} \big[p_0(s)u(s) + p_1(s)u'(s) + q(s)\big]\,ds.$$

If now we set

$$\begin{aligned}\rho_{ik} &= \max\big\{|u^{(i)}(t)| :\ t_k \le t \le t_{k+2}\big\} \ \ (i = 0,1),\\ \rho_{2k} &= \max\big\{r(t)|u''(t)| :\ t_k \le t \le t_{k+2}\big\}\end{aligned}$$

and

$$\varepsilon_k = \int\limits_{t_k}^{t_{k+2}} \Big(\frac{t}{r(t)} + |p_0(t)| + |p_1(t)| + |q(t)|\Big)\,dt,$$

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then from the latter identity we find that

$$|u''(t)| \le \frac{1}{r(t)}\,\varepsilon_k(\rho_{0k}+\rho_{1k}+1) \ \text{ for } \ t_k \le t \le t_{k+2} \tag{4}$$

and

$$\rho_{2k} \le \varepsilon_k(\rho_{0k}+\rho_{1k}+1) \ \text{ for } \ t_k \le t \le t_{k+2}. \tag{5}$$

On the other hand, by virtue of conditions (2) we have

$$\lim_{k\to+\infty} \varepsilon_k = 0. \tag{6}$$

Therefore without loss of generality it can be assumed that

$$\varepsilon_k < \frac{1}{2} \quad (k=1,2,\dots). \tag{7}$$

By the Green formula, for each natural k we have

$$u(t) = \int_{t_k}^{t_{k+2}} G_k(t,s)u''(s)\,ds, \quad u'(t) = \int_{t_k}^{t_{k+2}} \frac{\partial G_k(t,s)}{\partial s}\,u'(s)\,ds \tag{8}$$

$$\text{for } \ t_k \le t \le t_{k+2},$$

where

$$G_k(t,s) = \begin{cases} \dfrac{(t-t_k)(s-t_{k+2})}{t_{k+2}-t_k} & \text{for } \ t < s, \\ \dfrac{(t-t_{k+2})(s-t_k)}{t_{k+2}-t_k} & \text{for } \ t > s. \end{cases}$$

Moreover,

$$|G_k(t,s)| \le s - t_k < s, \quad \Big|\frac{\partial G_k(t,s)}{\partial t}\Big| \le 1 \ \text{ for } \ t_k \le t,s \le t_{k+2}.$$

By virtue of these estimates and inequalities (4), from (8) we find that

$$\rho_{0k} \le \varepsilon_k(\rho_{0k}+\rho_{1k}+1)\int_{t_k}^{t_{k+2}} \frac{s}{r(s)}\,ds \le \varepsilon_k^2(\rho_{0k}+\rho_{1k}+1),$$

$$\rho_{1k} \le \varepsilon_k(\rho_{0k}+\rho_{1k}+1)\int_{t_k}^{t_{k+2}} \frac{ds}{r(s)} \le$$

$$\le \varepsilon_k(\rho_{0k}+\rho_{1k}+1)\int_{t_k}^{t_{k+2}} \frac{s\,ds}{r(s)} \le \varepsilon_k^2(\rho_{0k}+\rho_{1k}+1).$$

Therefore

$$\rho_{0k}+\rho_{1k} \le 2\varepsilon_k^2(\rho_{0k}+\rho_{1k}) + 2\varepsilon_k^2.$$

Hence by (5)–(7) it follows that

$$\rho_{0k}+\rho_{1k} \le 4\varepsilon_k^2, \quad \rho_{2k} \le 2\varepsilon_k \quad (k=1,2,\dots)$$

and

$$\lim_{k\to+\infty} \rho_{ik} = 0 \ \ (i=0,1,2).$$

Therefore equalities (3) are fulfilled. □

The proven theorem completes the previously known results on asymptotic behavior of solutions of liner differential equations of third order (see [1]–[9] and the references cited therein).

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