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ON ESTIMATES OF SIGN-DEFINITE SOLUTIONS OF LINEAR EQUATIONS WITH DELAY

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Let $\mathbf{R} = (-\infty, \infty), \mathbf{R}_+ = [0, \infty), \Delta = \{(t, s) : t \geq s \geq 0\}$.

Consider the equation

$$(\mathcal{L}x)(t) = 0, \quad t \geq \tau \quad (\tau \in \mathbf{R}_+), \tag{1}$$

where

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) - ax(t) + \sum_{k=1}^{m} b_k x_{h_k}(t),$$

$$x_{h_k}(t) = \begin{cases} x_{h_k}(t), & \text{if } \ h_k(t) \geq \tau, \\ 0, & \text{if } \ h_k(t) < \tau; \end{cases}$$

$a \in \mathbf{R}$, $b_k \in \mathbf{R}_+$, $b = \sum\limits_{k=1}^{m} b_k$; $h_k : [\tau, \infty) \to \mathbf{R}$, $k = 1, \dots, m$, are given Lebesgue measurable on $\mathbf{R}_+$ functions such that $h_k(t) \leq t$ for almost all t, $0 < \omega_0 = \max\limits_k (vrai \sup\limits_t (t - h_k(t)) < \infty$.

By a solution of the equation (1) we mean a function $x : [\tau, \infty) \to R$, which is absolutely continuous on every finite interval $[\tau, T]$ and satisfies (1) almost everywhere.

As it is known [1], the Cauchy problem for the equation (1) is uniquely solvable and the solution can be expressed in the form

$$x(t) = C(t, \tau)x(\tau) \quad (t \geq \tau). \tag{2}$$

If the initial point τ in the equation (1) is an arbitrary one, then (1) becomes a family of equations with all the solutions described by the equality (2). Under this approach C becomes a function of two variables, t and τ, from Δ into $\mathbf{R}$; it is a common practice to call it the Cauchy function of the equation (1). The equality (2) implies that the Cauchy function as the function of the first argument satisfies the equation (1) with $C(\tau, \tau) = 1$. Let us point out another useful property of the Cauchy function: for every $(t, s) \in \Delta$, the estimate

$$|C(t, s)| \leq e^{(|a|+b)(t-s)} \tag{3}$$

holds.

The Cauchy function is the main subject of this paper, therefore it would be more convenient to express the definition of stability in terms of the properties of the Cauchy function.

Definition. We say that the equation (1) is a) *uniformly stable*, if there exists a $K > 0$ such that $|C(t, s)| \leq K$ for all $(t, s) \in \Delta$; and b) *exponentially stable*, if there exist constants $K > 0$ and $\gamma > 0$ such that the estimate

$$|C(t, s)| \leq Ke^{-\gamma(t-s)} \tag{4}$$

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holds for all $(t, s) \in \Delta$.

In the paper [2] some tests of the uniform and exponential stability for the equation (1) are obtained (Theorems 9, 10). Below we show that in the case of the positive Cauchy function these results can be refined by expressing the parameters K and γ in terms of the coefficients of the original equation.

Consider the Cauchy problem

$$\begin{cases} \dot{u}(t) = au(t) - bu(t - r(t)), & t \in R_+, \\ u(0) = 1, \end{cases} \tag{5}$$

where $a < b$. Let u_0 be a solution of the problem (5) with the delay $r = r_0(t)$ defined by the equality:

$$r_0(t) = \begin{cases} t, & \text{if } \ t \in [0, \omega], \\ \omega, & \text{if } \ t \in [\omega, \infty); \quad \omega = \text{const} > 0. \end{cases}$$

Let us denote $L = \sup\{t \in \mathbf{R}_+ : \dot{u}_0(s) < 0, \forall s < t\}$. Note that we allow L to be equal $+\infty$; in such case we deal with the class of equations (5) with the solutions steadily decreasing over R_+. To this particular situation, the most attention is given in the paper.

Let us construct the solution of (5) on $[0, \omega]$. It is easy to see that $\dot{u}_0(t) = a(1 - \frac{b}{a})$ $e^{at} < 0 \ \forall t \in [0, \omega]$. Hence $\omega < L \leq \infty$.

Define the function r^* as follows: $r^*(t) = r_0(t), \forall t \in R_+$ for $L = \infty$; $r^*(t) = r_0(t), \forall t \in [0, L]$ for $L < \infty$; and $r^*(t + L) = r^*(t)$.

The equation (5) with the delay $r = r^*(t)$ will be called the test-equation. Denote by u the solution of the test-equation, with the initial condition $u(0) = 1$, and by $U(t, s)$ the Cauchy function for the test-equation.

We outline some properties of the test-equation.

Lemma 1. *If $L < \infty$, then $u(t + L) = u(t)u(L) \ \forall t \in R_+$; in particular, $u(nL) = u^n(L)$.*

Proof. It is clear that the function $z(t) = u(t + L)$ satisfies the test-equation. Due to the representation (2) we have

$$u(t + L) = z(t) = u(t)z(0) = u(t)u(L). \ \square$$

Lemma 2. *If $L < \infty$, then $u(L) < 0$.*

Proof. Assume the contrary: $u(L) = u_0(L) \geq 0$. In this case, it follows from the equation (5) that $u_0(L - \omega) \leq \frac{a}{b} u_0(L) < u_0(L)$, which contradicts the definition of L. $\square$

Lemma 3. *The following statements are equivalent:*

a) *$U(t, s) > 0 \ \forall (t, s) \in \Delta$;*

b) *$L = \infty$;*

c) *$b\omega \leq e^{a\omega - 1}$.*

Proof. *a)* $\Rightarrow$ *b)* holds by Lemma 2. The implications *b)* $\Rightarrow$ *c)* and *c)* $\Rightarrow$ *a)* are established in the works [3, 4].

Introduce the notation

$$D_0 = \left\{(a\omega, b\omega) : a\omega < b\omega \leq e^{a\omega - 1}, a\omega < 1\right\}, \quad D_1 = \left\{(a\omega, b\omega) : b\omega > e^{a\omega - 1}, a\omega < 1\right\}.$$

Lemma 4. *Let $(a\omega_0, b\omega_0) \in D_0$. Then there exists $\omega > \omega_0$ such that $(a\omega, b\omega) \in D_1$.*

Proof. Consider the ray $P = \{\alpha = a\tau, \beta = b\tau, \tau \geq \omega_0\}$. It is clear that for $a < b$ the intersection point of P and the curve $\beta = e^{\alpha - 1}$ lies in the domain where $\alpha < 1$, and after this point the ray lies above this curve. Hence there exists a point $(a\omega, b\omega) \in P$ such that $a\omega < 1, b\omega > e^{a\omega - 1}$, i.e. $(a\omega, b\omega) \in D_1$. $\square$

Theorem 1. *If* $(a\omega_0, b\omega_0) \in D_0$*, then the test-equation is exponentially stable and its Cauchy function is positive.*

Proof. The function $U(t,s)$ is positive by Lemma 3. The exponential stability is provided by Lemmas 1 and 4, the properties of $U(t,s)$ [5], and the theorem on continuous dependence of solution to (5) on delay [6]. □

Lemma 5 ([2, 3])**.** *Assume that* $L < \infty$ *and* $s \in \mathbf{R}_+, 0 \le t - h_k(t) \le \omega$ *for almost all* $t \in \mathbf{R}_+$ *and all* $k = \overline{1,m}$*. If, for some* $t_0 \ge s + n(L+\omega), n \ge 2$*, the inequalities* $C(t_0,s) > 0, C_t(t_0,s) \ge 0$ *are fulfilled, then there exists a point* $t_1 \ge s + (n-1)(L+\omega)$ *such that*

$$C(t_0,s) = C(t_1,s)u(L). \tag{6}$$

Next, we investigate the equation (1). As it will be shown in the sequel, the properties of the test-equation define the behavior of the solution of (1) in many respects. For instance, the assumptions of Theorem 1 provide positiveness of the Cauchy function as well as exponential stability of the equation (1).

Theorem 2. *If the Cauchy function of* (1) *is positive for all* $(t,s) \in \Delta$*, then there exists an* $l > 0$ *such that the function* $C(\cdot,s)$ *is steadily decreasing at all* $t \ge s + l$*.*

Proof. Lemma 4 and the definition of ω_0 imply the existence of an $\omega > \omega_0$ such that $t - h_k(t) \le \omega$, $(a\omega, b\omega) \in D_1$. Due to Lemma 3 $L < \infty$. Suppose that, for some $t_0 \ge s + 2(L+\omega)$, the inequality $C_t(t_0,s) \ge 0$ holds. Then due to Lemma 5 there exists a point $t_1 \ge s + L + \omega$ such that the equality (6), where according to Lemma 2, $u(L) < 0$, is satisfied. Hence $C(t_1,s) < 0$, and we have a contradiction with the theorem condition.

For the practicality of Theorem 2 it is desirable to obtain conditions of positiveness of the Cauchy function of the equation (1) in terms of the coefficients of the original problem. It is easy with the following result of [4]. □

Lemma 6 ([4])**.** *The Cauchy function of the equation* (1) *is positive on* Δ*, if and only if there exists a function* v*, absolutely continuous on every finite segment* $[0,T]$*, such that* $v(t) > 0$ *and* $(\mathcal{L}v)(t) \le 0 \quad \forall t \in R_+$*.*

Theorem 3. *If* $(a\omega_0, b\omega_0) \in D_0$*. Then the following estimates of the Cauchy function of* (1) *hold:*

$$0 < C(t,s) \le Ke^{(a-b)(t-s)} \quad \forall (t,s) \in \Delta.$$

Proof. We use Lemma 6 to prove the positiveness of $C(t,s)$. Let us put $v(t) = e^{-\alpha t} > 0$, where $\alpha = -\omega_0^{-1} \ln b\omega_0 > 0$. Due to the definitions of ω_0, the set D_0, and the choice of α we have: $(\mathcal{L}v)(t) = -e^{-\alpha t}(\alpha + a - \sum_{k=1}^{m} b_k e^{\alpha(t-h_k(t))}) \le \omega_0^{-1} e^{-\alpha t}(1 - a\omega_0 - \ln b\omega_0) \le 0.$

Therefore, the Cauchy function is positive on Δ, and due to Theorem 2 it is steadily decreasing in t for $t \ge s + l$.

Denote $l_0 = l + \omega_0$.

As it was mentioned before, the function $C(\cdot,s)$ satisfies the equation

$$C_t(t,s) = aC(t,s) - bC(t,s) + \sum_{k=1}^{m} b_k(C(t,s) - C(h_k(t),s)),$$

which, due to the Cauchy formula [1], for $t \ge s + l_0$ is equivalent to the integral equation

$$C(t,s) = e^{(a-b)(t-s-l_0)}C(s+l_0,s) +$$

$$+\int_{s+l_0}^{t} e^{(a-b)(t-\tau)} \sum_{k=1}^{m} b_k (C(\tau,s) - C(h_k(\tau),s)) d\tau.$$

Since $\tau \geq h_k(\tau)$, the inequality $C(\tau, s) \geq C(h_k(\tau), s)$ holds for all $k = \overline{1,m}$. Hence, with the estimate (3) we obtain:

$$C(t,s) \leq e^{(a-b)(t-s-l_0)} C(s+l_0,s) \leq e^{(|a|-a+2b)l_0} e^{(a-b)(t-s)}. \quad \square$$

For completeness let us investigate the behavior of the Cauchy function of the equation (1) on the boundary of D_0 (in case $a\omega_0 = b\omega_0$).

Theorem 4. *If* $a\omega_0 = b\omega_0 < 1$, *then the estimates for the Cauchy function of the equation* (1) *hold:*

$$0 \leq C(t,s) \leq K \quad \forall (t,s) \in \Delta.$$

Proof. By Theorem 3 and continuous dependence of the solution of (1) on the coefficients [6] the Cauchy function $C(t,s)$ is nonnegative for all $(t,s) \in \Delta$. The boundedness of $C(t,s)$ in (t,s) is established in [7]. $\square$

Remark. The estimate $a\omega_0 < 1$ in Theorems 3, 4 is the best possible. The following example shows it.

Example.

$$\begin{gathered} \dot{x}(t) = x(t) - x(t-1), \quad t \in R_+, \\ (x(\tau) = \tau, \quad \tau \in (-1, 0]). \end{gathered} \tag{7}$$

It is easy to check by the direct substitution that the solution of (7) is the function $x(t) = t$, which is not bounded on R_+; hence (see [8], Lemma 6.6.2) the equation (7) can be neither exponentially nor even uniformly stable.

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