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A SURVEY OF RECENT RESULTS OF GEORGIAN MATHEMATICIANS ON BOUNDARY VALUE PROBLEMS FOR HOLOMORPHIC FUNCTIONS

Abstract. The present paper is a survey of the results obtained in recent years by the participants of the seminars held at A. Razmadze Mathematical Institute on boundary value problems for holomorphic and harmonic functions and on singular integral equations. These investigations take the source from to be the richest scientific heritage of N. Muskhelishvili.

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Introduction

Niko Muskhelishvili, an outstanding mathematician and mechanician would have been 110. His brilliant results in the theory of elasticity and in the problems of mathematical physics are widely known to scientists and specialists all over the world. In connection with the problems of applied character, N. Muskhelishvili elaborated methods of solution of the so-called piecewise continuous boundary value problems for holomorphic functions and of closely linked with them singular integral equations. His well-known monographs [65-67] contain fundamental results on the above-mentioned problems. They have played a decisive role in the formation of the now acknowledged Georgian school both in the plane theory of elasticity and in the theory of boundary value problems of mathematical physics.

In all that N. Muskhelishvili did in science and in organization of science as well, he set himself a high standard of excellence, and this was recognized by national and international honours of various kinds. His exceptional capacity for hard work enabled him to obtain effective solutions for quite a number of boundary value problems which were considered earlier as inaccessible.

In the present work we make an attempt to survey the results on linear boundary value problems for holomorphic functions obtained in recent years by the participants of the seminars on the above-mentioned and related with them problems of analysis which were held at A. Razmadze Mathematical Institute. The richest scientific heritage of N. Muskhelishvili was the impetus and the source of these investigations.

In the theory of holomorphic functions and singular integral equations the basic objects of his investigations are the following:

(a) The Riemann problem (linear conjugation problem): find a function ϕ from a given class of functions, holomorphic on the plane, cut along Γ – a curve or a finite family of nonintersecting curves, whose boundary values satisfy the conjugacy condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \tag{I}$$

where G and g are functions prescribed on Γ, and ϕ^+ and ϕ^- are boundary values of ϕ on Γ;

(b) The Riemann-Hilbert problem: in the domain bounded by a closed curve Γ, find a holomorphic function $\phi(z)$ such that its boundary values $\phi^+(t)$ satisfy the condition

$$\operatorname{Re}[G(t)\phi^+(t)] = f(t), \quad t \in \Gamma, \tag{II}$$

where G and g are functions given on Γ.

(c) The singular integral equation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau + \int_{\Gamma} k(t,\tau)\varphi(\tau)d\tau = f(t), \quad t \in \Gamma, \tag{III}$$

where a, b, f are functions from certain classes given on Γ, and φ is an unknown function.

The above-mentioned problems were for the first time formulated by B. Riemann [84]. Significant results on these problems have been obtained by D. Hilbert, Yu. Sokhotskiĭ, J. Plemelj, H. Poincaré, G.Bertrand, F. Noether, T. Carleman.

Depending on the assumptions imposed on the unknown functions, the boundary value problems are conventionally divided into three groups: (i) continuous problems with a continuous (up to the boundary) solution; (ii) piecewise continuous problems, when the conventionally is violated only at a finite number of boundary points; (iii) all other problems of discontinuous type.

For closed boundary curves, the solution of the problem (I) in the continuous statement has been given by F.D. Gakhov [21]. In solving a number of important problems of mechanics there arose a need to solve the problem for the case where Γ is a union of a finite number of simple smooth closed, mutually disjoint curves.

Early in the 40s of the last century, N. Muskhelishvili has constructed an elegent theory of solution of the above-cited problems in the piecewise continuous statement.

At the beginning of our exposition we present a brief survey of basic results obtained by N. Muskhelishvili for boundary value problems of the theory of functions and for singular integral equations. Further we will set forth the results on boundary value problems in the piecewise continuous statement which were obtained in recent years by his students and followers. The methods used in these works are penetrated with ideas of N. Muskhelishvili.

Here we give definitions of classes of functions appearing in the problems of linear conjugation in the piecewise continuous statement.

Let Γ be a piecewise-smooth curve, i.e. the union of smooth arcs which may have a finite number of common points. The ends of one or several arcs are called knots. Points, different from knots, are called regular points.

Let $\varphi_k(t)$ be functions defined on closed arcs Γ_k forming Γ, and let φ be a function defined on Γ as

$$\varphi(t) = \varphi_k(t), \quad t \in \Gamma_k.$$

Thus the function $\varphi(t)$ is uniquely defined at all regular points of Γ; at the knots, however, where several arcs meet, we may leave this function undefined or ascribe to it one of the values. They say that φ belongs to the class H_0 on Γ if all the functions $\varphi_k(t)$ satisfy the Hölder condition (the condition H).

If the function $\varphi(t)$ prescribed on Γ satisfies the condition H on every closed segment of Γ not containing knots, while near each knots is representable in the form

$$\varphi(t) = \varphi^*(t)|t-c|^{-\alpha}, \quad 0 \le \alpha < 1, \tag{0.1}$$

where φ^* belongs to the class H_0 in the neighborhood of the point c, then we say that φ belongs to the class H^* on Γ.

A function, holomorphic in each finite domain on the complex plane not containing the points of the curve Γ, which is continuously extendable on Γ from the left and from the right except at the knots, while in a neighborhood of each knot satisfies the condition

$$|\phi(z)| < \mathrm{const}\,|z-c|^{-\alpha}, \quad 0 \le \alpha < 1,$$

is called the piecewise holomorphic function with the boundary curve Γ (or the jump curve Γ).

The problem of conjugation in a piecewise continuous statement is formulated as follows: find a piecewise holomorphic functions $\phi(z)$ having finite order at infinity and satisfying the boundary condition

$$\phi^+ = G(t)\phi^-(t) + g(t),$$

where $G(t)$ and $g(t)$ are the given on Γ functions of the class H_0, and $G(t) \neq 0$. ϕ^+ and ϕ^- denote boundary values of $\phi(z)$ at regular points, respectively, from the left and from the right;

A solution $X(z)$ of the homogeneous problem of conjugation is called canonical, if $X(z)$ and $1/X(z)$ are simultaneously piecewise holomorphic.

Next, under $\ln G(t)$ will be understood any definite value, continuously varying on each arc. Because of the fact that $G(t)$ belongs to the class H_0 and is distinct from zero on Γ, the function $\ln G(t)$ belongs to the class H_0 as well.

Let $c_1, \dots, c_n$ be the knots of the line Γ. If t approaches the knots c_k along one of the arcs Γ_k having the point c_k as its end point, the function $\ln G(t)$ tends to a definite limit which we denote by $\ln G_j(c_k)$. Thus, by definition,

$$\ln G_j(c_k) = \lim \ln G(t) \quad \text{for} \quad t \to c_k, \quad t \in \Gamma_k.$$

The function

$$X_0(z) = \exp \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\ln G(t)}{t-z} dt \tag{0.2}$$

satisfies the homogeneous boundary condition at regular points.

In the neighborhood of the knots c_k

$$X_0(z) = (z-c_k)^{\alpha_k + i\beta_k} H(z), \tag{0.3}$$

where $H(z)$ is a function holomorphic in each of the sectors in the neighborhood of c_k, tending to definite, different from zero, limits as $z \to c_k$ in any direction, not coming out of the given segment;

$$\alpha_k + i\beta_k = \frac{1}{2\pi i} \sum_j \pm \ln G_j(c_k), \tag{0.4}$$

where the sum extends to all j, numbers of arcs Γ_j having c_k as an end; note that the upper signs correspond to the outcoming arcs and the lower signs to the incoming ones.

Let now λ_k denote integers satisfying the conditions

$$-1 < \alpha_k + \lambda_k < 1, \quad k = 1, \ldots, n. \tag{0.5}$$

The function

$$X(z) = \prod_{k=1}^{n} (z - c_k)^{\lambda_k} X_0(z) \tag{0.6}$$

represents one of the canonical solutions.

The solution $X(z)$ is not, generally speaking, determined fully by the conditions (0.5). The point is that the number λ_k, corresponding to the knot c_k, is defined uniquely only in the case where α_k is an integer; in this case $\lambda_k = -\alpha_k$.

The knots c_k, for which α_k are integers, are called singular and the remaining knots are called nonsingular.

Sometimes it is advisable to require that an unknown solution should be bounded in the neighborhood of some preassigned nonsingular knots $c_1, \ldots, c_p$.

Solutions satisfying this condition, are called solutions of the class $h(c_1, \ldots, c_p)$. Classes h have been introduced in the work of [68].

To every class of solutions we put (to within a constant multiplier) into correspondence a canonical solution.

Indeed, a canonical solution of the class $h(c_1, \ldots, c_p)$, where $c_1, \ldots, c_p$ are the given nonsingular knots, will be called a solution $X(z)$ defined by the formula (0.6), where the integers λ_k are chosen such that $\alpha_k + \lambda_k > 0$ for the knots $c_1, \ldots, c_p$, and $\alpha_k + \lambda_k < 0$ for all the rest knots.

The integer number $\varkappa$ defined by the formula

$$\varkappa = -\sum_{k=1}^{n} \lambda_k \tag{0.7}$$

is called the index of G in the given class $h(c_1, \ldots, c_p)$.

From (0.6) it follows that $X(z)$ is at infinity of the order $(-\varkappa)$ and

$$\lim_{z \to \infty} z^{\varkappa} X(z) = 1.$$

The general solution of the class $h(c_1, \ldots, c_n)$ of problem (I) has the form

$$\phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(t)dt}{X^+(t)(t-z)} + X(z)p(z), \tag{0.8}$$

where $p(z)$ is an arbitrary polynomial.

Of special interest, from the point of view of their applications, are solutions of the inhomogeneous problem (I) vanishing at infinity.

Due to the fact that the order of the function $X(z)$ is at infinity equal exactly to $-\varkappa$, i.e., the index in the class $h(c_1, \ldots, c_p)$ is taken with the opposite sign, we come to the following conclusions:

For $\varkappa \geq 0$, the solutions of the given class, vanishing at infinity, are given by the formula

$$\phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(t)dt}{X^+(t)(t-z)} + X(z)p_{\varkappa-1}(z), \tag{0.9}$$

where $p_{\varkappa-1}$ is an arbitrary polynomial of order not higher than $\varkappa{-}1$ ($p_{\varkappa-1} = 0$ for $\varkappa = 0$).

For $\varkappa < 0$, solutions of the given class, vanishing at infinity, exist if and only if

$$\int_\Gamma \frac{t^k g(t)}{X^+(t)} dt = 0, \quad k = 0, \ldots, -\varkappa - 1. \tag{0.10}$$

If these conditions are satisfied, the solution is unique and given by the formula (0.9) with $p_{\varkappa-1} \equiv 0$.

As is seen from the above-said, the main tool for the investigation of the problem of conjugation (as well as of related singular integral equations) is the Cauchy type integral

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)}{t-z} dt, \quad z \notin \Gamma$$

and the Cauchy type singular integral

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau, \quad t \in \Gamma.$$

N. I. Muskhelishvili has made considerable contribution to the development of the theory of the above-mentioned integral operators. He proved that if $\varphi \in H^*(\Gamma)$, where Γ is a general piecewise smooth curve, then the corresponding Cauchy type integral is a piecewise holomorphic function. He obtained the well-known asymptotic formulas describing the behaviour of the Cauchy type integral at the knots. The other important fact established by N. I. Muskhelishvili is the invariance of the class $H^*(\Gamma)$ in the case of general piecewise smooth curves.

The solutions of piecewise continuous problems were applied by N. Muskhelishvili for the construction of the theory of singular integral equations. He studied completely equations of the type

$$k\varphi \equiv A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t_0} + \int\limits_\Gamma k(t_0,\tau)\varphi(\tau)d\tau = f(t_0),$$

$$k'\psi \equiv A(t)\psi(t) - \frac{1}{\pi i}\int\limits_\Gamma \frac{B(\tau)\psi(\tau)}{\tau - t}d\tau + \int\limits_\Gamma k(\tau,t)\psi(\tau)d\tau = g(t),$$

where A, B, k are given functions of the class H_0 and f and g are given functions of the class H^*.

N. Muskhelishvili [66], [68] proved the following theorems.

a) A necessary and sufficient condition of solvability in the given class $h = h(c_1,\dots,c_q)$ of equation $k\varphi = f$ is that

$$\int\limits_\Gamma f(t)\psi_j(t)dt = 0, \quad j = 1,\dots,k',$$

where ψ_j $(j = 1,2,\dots,k')$ is a complete system of linearly independent solutions of the associate class $h' = h(c_{q+1},\dots,c_m)$ of the associate homogeneous equation $k'\psi = 0$.

b) The difference $k - k' = \varkappa$, where $k(k')$ is the number of linearly independent solutions of the class $h(h')$ of the homogeneous equation $k\varphi = 0$, $(k'\psi = 0)$ and $\varkappa$ is the index of $G = (A-B)(A+B)^{-1}$ in the class h.

For construction of the theory of singular integral equations the Poincaré-Bertrand permutation formulas are of exceptional importance:

$$\int\limits_\Gamma \frac{d\tau}{\tau - t_0}\int\limits_\Gamma \frac{\varphi(\tau,\tau_1)}{\tau_1 - \tau}d\tau_1 =$$

$$= -\pi^2\varphi(t_0,t_0) + \int\limits_\Gamma d\tau_1\int\limits_\Gamma \frac{\varphi(\tau,\tau_1)}{(\tau - t_0)(\tau_1 - \tau)}d\tau, \quad t_0 \in \Gamma.$$

N. Muskhelishvili justified this formula in the general case when Γ is a general piecewise smooth curve with the knots $c_1,\dots,c_n$, and φ is a function such that after multiplying by the product of n multipliers of the kind $|t - c_k|^{\alpha_k}|\tau - c_k|^{\beta_k}$, $\alpha_k + \beta_k < 1$ it belongs to the class $H_0(\Gamma \times \Gamma)$, then the Poincaré–Bertrand formula is valid at all points t_0 different from the knots.

The connection of the problem (I) with singular integral equations has been noticed by T. Carleman [10] in 1922. He proposed the idea of constructing solutions of the problem (I) in terms of Cauchy type integrals. I. Vekua [92] justified a method of solving a complete singular integral equation in classical assumptions.

N. Muskhelishvili [66], [67] proposed a new method of investigation of the Riemann-Hilbert problem (II) in both continuous and piecewise continuous cases, when the boundary curve is a circle or a straight line. By reducing it to the problem of linear conjugation he obtained an effective solution of the above-given problem. This method attracted attention of many researches, of his pupils and successors to the investigation of the Riemann-Hilbert problems in different statements. The above-mentioned one-sided boundary value problem is more general than the Dirichlet problem for harmonic functions. As is shown by N. Muskhelishvili [66], the Neumann problem can be reduced to solution of the problem (II).

It should be noted that a number of important results on one-sided boundary value problems of the theory of functions of a complex variable were obtained in the works of N. Muskhelishvili's disciples and followers. Among these results are various boundary value problems of elliptic linear differential equations, systems of such equations or mixed differential equations. Here first and foremost we should mention the works of his most talented disciples I. Vekua and A. Bitsadze.

1. Piecewise Continuous Problems

In this section we present a survey of results on boundary value problems for holomorphic functions in the piecewise continuous statement which were obtained in recent years. These investigations are penetrated with the ideas of N. Muskhelishvili.

In [5] the following problem is investigated: find a function φ, holomorphic in the annulus $D = \{z : 1 < |z| < R\}$, according to the following conditions: a) φ is continuously extendable to the boundary, except possibly a finite number of points $c_1, \dots, c_k$ at the vicinity of which the conditions

$$|\varphi(z)| < c|z - c_i|^{-\mu}, \quad 0 < \mu < 1, \quad i = 1, 2, \dots, n \tag{1.1}$$

are fulfilled; b) on the circle $\gamma = \{t : |t| = 1\}$ φ satisfies the boundary condition

$$\varphi(at) = G(t)\varphi(t) + f(t), \tag{1.2}$$

where $a = \mathrm{Re}^{i\varphi}$.

It is assumed that the functions G and f have discontinuities of the first kind at the points of the boundary and on each closed arc, whose ends are the points $c_1, \dots, c_n$ of discontinuity, they satisfy the Hölder condition; moreover, $G(t) \neq 0$ everywhere on γ.

This problem is, in fact, the problem of conjugation with the Carleman shift for the annulus. It should be noted that the investigation of the above-formulated problem was preceded by an effective solution of the infinite system of algebraic equations

$$a^n \varphi_n - \sum_{m=-\infty}^{\infty} k_{n-m} \varphi_m = f_n \quad (n = 0, \pm 1, \pm 2, \dots)$$

with $|a| \neq 1$, $\{f_n\} \in l_1$, $\{k_n\} \in l_1$. As a class of unknown functions the class of sequences was considered for which $\{\varphi_n\} \in l_1$, $\{a^n \varphi_n\} \in l_1$, where l_1 is the space of infinite sequences $\{\alpha_k\}$ satisfying

$$\sum_{k=-\infty}^{\infty} |\alpha_k| < \infty.$$

Without loss of generality, it is assumed that $|a| > 1$. Solving the above-mentioned system was reduced to finding a holomorphic function φ in the domain $D = \{z : 1 < |z| < a\}$, whose boundary values satisfy the condition (1.2) with $G(t) = \sum_{n=-\infty}^{\infty} k_n t^n$, $f(t) = \sum_{n=-\infty}^{\infty} f_n t^n$, $\varphi(z) = \sum_{n=-\infty}^{\infty} \varphi_n z^n$, $z \in D$, $k(t) \neq 0$, and belong to the class of functions whose Fourier series converge absolutely.

Get now back to more detailed treatment of the problem which is formulated at the beginning of this section.

Following N. I. Muskhelishvili, the points of discontinuity at which the condition

$$\arg G(c-) = \arg G(c+)$$

is satisfied, are called singular points; all other points are called nonsingular.

Let t_0 be a point on γ at which G is continuous. We choose any value $\ln G(t_0-)$. Starting from the point t_0 and moving in the positive direction, we can continuously change the function $\ln G(t)$ until t reaches the first singular point c. By this, we get a well-defined value $\arg G(c-)$.

When passing through the point c, we choose the value $\arg G(c+)$ so that one of the conditions

$$0 < \frac{1}{2\pi}(\arg G(c-) - \arg G(c+)) < 1 \tag{1.3}$$

or

$$-1 < \frac{1}{2\pi}(\arg G(c-) - \arg G(c+)) < 1 \tag{1.4}$$

is fulfilled. Continuing the movement of the point t in the positive direction on γ, choosing the value $\arg G(t)$ so that one of the conditions (1.3) or (1.4) to be fulfilled at every nonsingular point and then getting back to the initial point t_0, we obtain quite definite value for the function $\ln G(t)$ on each of the arcs, into which the discontinuity points and the point t_0 divide the contour γ.

Suppose

$$\varkappa = \frac{1}{2\pi i}[\ln G(t_0 - 0) - \ln G(t_0+)] = \frac{1}{2\pi}[\arg G(t)]_\gamma .$$

It is evident that $\varkappa$ is an integer, which does not depend on t_0.

Introduce a function

$$G_0(t) = \Big(\frac{at - z_0}{t - z_0}\Big)^{\varkappa} G(t),$$

where z_0 is a fixed point of the annulus D.

Obviously,

$$[\ln G_0(t)]_\gamma = 0$$

under the above-mentioned choice of $G(t)$.

It is proved that the function $G(t)$ is representable in the form

$$G(t) = \mu \frac{X(at)}{X(t)}, \quad t \in \gamma, \tag{1.5}$$

where

$$\begin{aligned} X(z) &= (z - z_0)^{\varkappa} \exp\left(\frac{1}{2\pi i} \int\limits_\gamma K_1^*\Big(\frac{z}{t}\Big) \ln \frac{G_0(t)}{\mu} \frac{dt}{t}\right), \\ \mu &= \exp\left(\frac{1}{2\pi i} \int\limits_\gamma \ln G_0(t) \frac{dt}{t}\right). \end{aligned} \tag{1.6}$$

Here $K_1^*\Big(\frac{z}{t}\Big) = \frac{at}{at-z} + \frac{t}{t-z} + K_0\Big(\frac{z}{t}\Big)$ and

$$K_0\Big(\frac{z}{t}\Big) = \sum_{n=1}^{\infty} \Big(\frac{z}{at}\Big)^n + \sum_{n=-\infty}^{-1} \frac{a^n}{a^n - 1} \Big(\frac{z}{t}\Big)^n.$$

The last function is holomorphic in the annulus $\frac{1}{|a|} < \left|\frac{z}{t}\right| < |a|^2$.

The function $X(z)$ is continuously extendable at the points of the contour of the annulus D, except possibly the points $c_k \in \gamma$ of discontinuity of the function $G(t)$ and the corresponding to them points ac_k. In the neighborhood of those points ac_k the function X can be represented as

$$X(z) = \big((z - c)(z - ac)\big)^{\alpha + i\beta} H(z), \tag{1.7}$$

where

$$\alpha = \frac{1}{2\pi} \arg \frac{G(c-)}{G(c+)}, \quad \beta = \frac{1}{2\pi} \ln \left|\frac{G(c-)}{G(c+)}\right|.$$

H is holomorphic in the vicinity of the points c and ac and tends to a definite, different from zero, limit as $z \to c$ or $z \to ac$.

As is seen from the formula (1.7), the function X is bounded in the vicinity of all singular points and of those nonsingular points for which the condition (1.3) is fulfilled. It is also bounded in the neighborhoodof corresponding to them points.

In applications it is very important to find solutions of the problem (1.2) bounded near some prescribed nonsingular points $c_1, c_2, \dots, c_p$. Then it will simultaneously be bounded in the vicinity of the corresponding points $ac_1, ac_2, \dots, ac_p$. Following N. I. Muskhelishvili [66], [68], solutions of the problem (1.2) satisfying the latter condition are called solutions of the class $h(c_1, \dots, c_p)$.

The class corresponding to $p = 0$ is denoted by h_0. If m is the number of all nonsingular points and $c_1, c_2, \dots, c_m$ are all these points, then the class $h(c_1, \dots, c_m)$ is denoted by h_m. The class h_0 contains all the other classes and the class h_m is contained in all the other classes.

If the condition (1.3) is fulfilled at the nonsingular points $c_1, c_2, \dots, c_m$ the condition (1.4) at the remaining nonsingular points, then the function defined by the formula (1.6) will be bounded at the points $c_{p+1}, \dots, c_m$ (and hence at the points $c_1, c_2, \dots, c_p$). This function is called the canonical function of the problem (1.2) in the class $h(c_1, \dots, c_p)$, and the corresponding to it number $\varkappa$ is called the index of the problem of the class $h(c_1, \dots, c_p)$. As it is easily seen, the function $X(z)$ is holomorphic in the annulus D and differs from zero everywhere except the point z_0, where it has a zero of order $\varkappa$ for $\varkappa > 0$ and a pole of order $-\varkappa$ for $\varkappa < 0$.

Let $c_{m+1}, \dots, c_n$ be singular points. If all points of discontinuity of the function $G(t)$ are nonsingular, then $n = m$; on the other hand, if all points are singular, then $m = 0$.

The canonical function $X(z)$ of the class $h(c_1, \dots, c_p)$ is continuously extendable to the boundary of the annulus D, except the points $c_1, \dots, c_n$ and the corresponding points, bounded near the points $c_{m+1}, \dots, c_n, ac_{m+1}, \dots,$ ac_n and admits in the vicinity of the nonsingular points c_k, ac_k, $k = p+1, \dots, m$, the estimate

$$|X(z)| < \frac{\text{const}}{|z - c_k|^\alpha}, \quad |X(z)| < \frac{\text{const}}{|z - ac_k|^\alpha}, \quad 0 < \alpha < 1.$$

Inserting in (1.2) the value of the function $G(t)$ defined by the formula (1.5), we obtain the equality

$$\frac{\varphi(at)}{X(at)} - \mu \frac{\varphi(t)}{X(t)} = \frac{f(t)}{X(at)}, \quad t \in \gamma. \tag{1.8}$$

For $\varkappa \neq 0$, we can choose a point z_0 such that $a^n - \mu \neq 0$, $n = 0, \pm 1, \pm 2, \dots$. For $\varkappa > 0$, using the formula (1.6), we arrive at the representation

$$\varphi(z) = \frac{X(z)}{2\pi i} \int\limits_\gamma K_\mu\Big(\frac{z}{t}\Big) \frac{f(t)}{tX(at)} dt + X(z)\varphi_\varkappa(z), \tag{1.9}$$

where

$$K_\mu(z) = \frac{a}{a - z} + \frac{1}{\mu(1 - z)} + \mu \sum_{n=0}^{\infty} \frac{1}{a^n - \mu}\Big(\frac{z}{a}\Big)^n + \frac{1}{\mu} \sum_{n=-\infty}^{-1} \frac{a^n z^n}{a^n - \mu},$$

$$\varphi_\varkappa(z) = \sum_{j=0}^{\varkappa-1} c_j \frac{d^j\varphi(z,\lambda)}{d\lambda^j}\Big|_{\lambda=z_0}, \quad z = D,$$

and

$$\varphi(z,\lambda) = \frac{z}{z-\lambda} + \mu\frac{z}{z-a\lambda} + \sum_{n=0}^{\infty} \frac{\lambda^n}{(a^n\mu-1)z^n} + \mu^2 \sum_{n=-\infty}^{-1} \frac{a^{2n}\lambda^n}{(a_n\mu-1)z^n},$$

where c_j $(j = 0,1,\dots,\mu-1)$ are some constants.

One can show that the solution belongs to the class $h(c_1,\dots,c_p)$, and in the neighborhood of the singular points it is almost bounded.

For $\varkappa < 0$, a solution of the given class $h(c_1,\dots,c_p)$ exists if and only if

$$\int_\gamma \frac{d^i K_\mu(\frac{z}{t})}{dz^i}\frac{f(t)}{tX(at)}dt = 0, \quad z = z_0, \quad i = 0,1,\dots,-\varkappa-1. \tag{1.10}$$

In case the conditions (1.10) are fulfilled, the problem has a unique solution given by the formula (1.8) in which one has to put $\varphi_\varkappa = 0$.

If $\varkappa = 0$ and $a^m = \mu$ for some integer m, then the problem is solvable under the fulfilment of the condition

$$\int_\gamma \frac{f(t)}{t^{m+1}X(at)}dt = 0, \tag{1.11}$$

and the solution is given by the formula

$$\varphi(z) = \frac{X(z)}{2\pi i}\int_\gamma K^*_\mu\Big(\frac{z}{t}\Big)\frac{f(t)}{X(at)t}dt + bz^m, \tag{1.12}$$

where b is a complex constant. Here K^*_μ is obtained from the expression of K_μ where we eliminate the term with the vanishing denumerator.

If now we replace the condition (1.3) by the condition (1.4), then we get a quite general solution of the problem (1.2) on which no restrictions at the nonsingular points are imposed. Such a solution is referred to the class h_0. It is obvious that the index $\varkappa_0$ of that class is greater than those of all the rest classes. The index $\varkappa$ of the class $h(c_1,\dots,c_p)$ is connected with $\varkappa_0$ by the relation

$$\varkappa = \varkappa_0 - p.$$

The problem

$$\psi(\overline{a}t) = \overline{G}(t)\psi(t), \quad t \in \gamma, \tag{1.13}$$

is called associated to the problem (1.2). It is obvious that the singular (nonsingular) points of the problem (1.2) are simultaneously singular (nonsingular) points of the problem (1.13). Correspondingly, the class $h = h(c_1,\dots,c_p)$ of solutions of the problem (1.2) and the class $h' =$

$h(c_{p+1},\dots,c_m)$ of the solutions of the problem (1.13) are called associated classes.

Canonical functions of the associated problems (1.2) and (1.13) from associated classes are connected by the relation

$$X'(z)=\frac{1}{X(\frac{a}{z})},\quad z\in D, \tag{1.14}$$

and the corresponding indices by the relation

$$\varkappa'=-\varkappa.$$

For $\varkappa>0$, the homogeneous problem (1.2) in the class $h(c_1,\dots,c_p)$ has linearly independent solutions of the type

$$\varphi_i(z)=X(z)\frac{d^i\varphi(z,\lambda)}{d\lambda^i},\quad i=0,1,\dots,\varkappa-1. \tag{1.15}$$

For $\varkappa<0$, the associated homogeneous problem of the class $h(c_{m+1},\dots,c_n)$ has $-\varkappa$ linearly independent solutions of the type

$$\varphi(z)=X'(z)\frac{d^i\varphi(\lambda,z)}{d^i\lambda}\Big|_{\lambda=\frac{a}{z}},\quad i=0,1,\dots,-\varkappa-1. \tag{1.16}$$

For $\varkappa=0$ and $a^m\neq\mu$, the associated problems, homogeneous in the corresponding classes, have no solutions.

For $\varkappa=0$ and $a^m=\mu$, each associate homogeneous problem in the associated classes has a solution of the type

$$\varphi(z)=AX(z)z^m,\quad \psi(z)=BX'(z)z^m. \tag{1.17}$$

Taking into account (1.14), the second formula (1.17) results in

$$\overline{\psi}(t)=1/X(t)t^m.$$

It is also easy to prove that

$$\overline{\psi}_i(t)=\frac{d^iK_\mu(\frac{z}{t})}{dz^i}\frac{1}{X(at)},\quad z=z_0,\quad i=0,1,\dots,-\varkappa-1. \tag{1.18}$$

Consequently, for the problem (1.2) the Noether theorem is valid:

1. If $\varkappa>0$ or $\varkappa=0$ and $a^n\neq\mu$, then for arbitrary natural n the homogeneous problem (1.13) in the class $h(c_{p+1},\dots,c_m)$ has no nonzero solution, while the problem (1.2) is always solvable in the class $h(c_1,c_2,\dots,c_p)$ and the solution is given by the formula (1.15).

2. If $\varkappa<0$, then the homogeneous problem (1.13) has $-\varkappa$ linearly independent solutions of type (1.16), and for the solvability of the problem (1.2), it is necessary and sufficient that the condition

$$\int\limits_\gamma f(t)\overline{\psi}_i(t)\frac{dt}{t}=0,\quad i=0,1,\dots,-\varkappa-1,$$

be fulfilled.

In case these conditions are fulfilled, the solution of the problem (1.2) is given by the formula (1.9) in which $\varphi_{\varkappa}(z) \equiv 0$.

3. If $\varkappa = 0$ and $\mu = a^m$ for some integer m, then each associate homogeneous problem in associate classes has only one solution given by the formula (1.17), while for the solvability of the inhomogeneous problem (1.2) it is necessary and sufficient that the condition

$$\int_{\gamma} f(t)\overline{\psi}_0(t)\frac{dt}{t} = 0$$

be fulfilled.

If one of these conditions is fulfilled, then the solution of the problem (1.2) has the form (1.12).

A more general problem than the one stated above has been investigated in [14].

Let now $\gamma_1 = \{t : |t| = 1\}$ and $\gamma_2 = \{t : |t| = R\}$. Moreover, let f_k and G_k be functions given on γ_k $(k = 1, 2)$, with G_k nonzero everywere on γ_k and having discontinuities of the first kind at a finite number of points, while on each of the closed arcs whose ends are the points of discontinuities satisfying the Hölder condition.

The following problem is investigated in [14]: find $\varphi(z)$ and $\psi(z)$, holomorphic functions in the annulus $D = \{z : 1 < |z| < R\}$, continuously extendable on $\overline{D}$ except possibly the points of discontinuity of the functions $f_k(t)$ and $G_k(t)$, $(k = 1, 2)$ and near these points satisfying the estimate

$$|\varphi(t)| < \frac{\text{const}}{|z - c|^{\alpha}}, \quad 0 \le \alpha < 1, \tag{1.19}$$

by the boundary conditions

$$\varphi(t) + G_1(t)\overline{\varphi}(t) = f_1(t), \quad t \in \gamma_1$$

and

$$\psi(t) + G_2(t)\overline{\psi}(t) = f_2(t), \quad t \in \gamma_2.$$

For the above-stated problem Noetherian theorems are proved, and in case of the existence of solutions the formulas are written out explicitly. The case where solutions have on the boundary first order singularities is considered in the same work.

The mixed problem for holomorphic functions appearing in the problems of elasticity has been investigated in [6].

The problem is to find an analytic function in the annulus $S = \{z : 1 < |z| < R^2\}$ cut along the arcs of the circle $|z| = R$. The boundary value of the real part is prescribed on one set of the annulus boundary, the imaginary part on the other set, and the difference of boundary values of the unknown analytic functions at the points $R^2 t$ and t on the exterior and interior boundaries, respectively, is given.

Let S be the plane cut along a finite number of arcs of the circle $|z| = R$. The set of these cuts is denoted by l. The set of cuts is divided into two subsets l_1 and l_2. Then suppose that the circle $|\gamma| = \{t : |t| = 1\}$ is divided into a finite number of arcs and the set of these arcs is divided into three subsets γ_1, γ_2 and γ_3. The arcs in each subset have no common points. The positive direction of the circuit on the circle is assumed to be counter-clockwise.

Let us consider the following problem: find a function $\phi(z)$, analytic in S and continuously extendable to the boundary, except possibly at a finite number of points near which (1.19) hold, and satisfying the boundary conditions

$$\begin{gathered}
\operatorname{Re}\phi^{\pm}(t) = f_1^{\pm}(t), \quad t \in l_1,\\
\operatorname{Im}\phi^{\pm}(t) = f_2^{\pm}(t), \quad t \in l_2,\\
\operatorname{Re}\phi(t) = g_1(t), \quad \operatorname{Re}\phi(R^2t) = g_1^*(t), \quad t \in \gamma_1,\\
\operatorname{Im}\phi(t) = g_2(t), \quad \operatorname{Im}\phi(R^2t) = g_2^*(t), \quad t \in \gamma_2,\\
\phi(R^2t) - \phi(t) = g_3(t), \quad t \in \gamma_3.
\end{gathered}$$

In [6] the effective solution of the above-stated problem is obtained.

Let us also recall the effective solution of the Riemann-Hilbert problem for doubly-connected domains. As we have mentioned above, N. I. Muskhelishvili suggested a new effective method of solution of the Riemann-Hilbert problem by reducing it to a problem of linear conjugation. Using this method, in [4] we can find the solution of the problem formulated as follows.

Let D be a doubly-connected domain, bounded by simple closed smooth contours L_0 and L_1. Find a function $\varphi(z)$, holomorphic in D and continuous in $\overline{D}$, by the boundary condition

$$\operatorname{Re}[\overline{a_j(t)}\varphi(t)] = c_j(t), \quad t \in L_j, \quad j = 0, 1,$$

where a_j and c_j are prescribed on L_j functions of the class H, with $a_j(t) \neq 0$.

The recent work [7], also should be noted in which integral representations are constructed for functions holomorphic in a strip. Using these representations, an effective solution of Carleman type problem is given for the strip. For instance, the following problem is investigated: find a piece-wise holomorphic function, bounded throughout the plane, by the boundary condition

$$\phi^+(x) = G(x)\phi^-[\alpha(x)] + f(x), \quad -\infty < x < \infty,$$

where G and f are given functions satisfying the Hölder condition, $G(x) \neq 0$, $G(\infty) = G(-\infty) = 1$, $f(+\infty) = f(-\infty) = 0$ and

$$\alpha(x) = \begin{cases} x, & x < 0,\\ bx, & x \geq 0, \end{cases}$$

b is a real constant.

In [55] a boundary value problem is solved for an infinite plane with cuts (with cracks) along the segments of two mutually perpendicular straight lines. Here we formulate the problem.

Let L and Λ be a family of cuts along the segments of the coordinate axes, and let S be the plane with the cuts $L \cup \Lambda$.

The problem is to find in S a piecewise holomorphic function $F(z)$, vanishing at infinity and satisfying the boundary conditions

$$[F(t) + \overline{F}(t)]^{\pm} = f^{\pm}(t), \quad t \in L$$
$$[F(t) + \overline{F}(t)]^{\pm} = g^{\pm}(t), \quad t \in \Lambda.$$

Here $f^{\pm}(t)$ and $g^{\pm}(t)$ are real functions of the class H given on L and Λ, respectively. The signs "+" and "−" indicate the boundary values on the contours of the cut $L(\Lambda)$ from above (from the left) and from below (from the right), respectively. The plane $z = x + iy$ is assumed to be cut along $2p$ segments $a_k \le x \le b_k$, $-b_k \le x \le -a_k$, $k = 1, 2, \dots, p$ of the real axis and along $2m$ segments $\alpha_j \le y \le \beta_j$, $-\beta_j \le y \le \alpha_j$, of the imaginary axis, i.e., the segments are symmetric with respect to the coordinate axes.

Owing to the fact that the segments are located symmetrically, the problem is reduced to four boundary value problems of the theory of holomorphic functions. These problems are reduced to the problem of linear conjugation studied by N. Muskhelishvili.

The solution of the above-formulated problem is of the form

$$F(z) = \sum_{n=0}^{1} \sum_{\nu=1}^{2} \Bigg\{ [\chi_q(z)]^{2-\nu} [Q_s(z)]^n p_{2\nu-n}(z) +$$
$$+ \frac{[Q_s]^n}{[\chi_q(z)]^{\nu-2}} \Bigg[\frac{1}{2\pi i} \int_L \frac{f^+(\tau) + (3-2\nu) f^-(\tau)}{[\chi_q^+(\tau)]^{2-\nu} [Q_S(\tau)]^n} \Big(\frac{1}{\tau - z} - \frac{(-1)^{2-\nu}(2n-1)}{\tau + z} \Big) d\tau +$$
$$+ \frac{1}{2\pi i} \int_\Lambda \frac{g^+(\tau) + (2n-1) g^-(\tau)}{[\chi_q(\tau)]^{2-\nu} [Q_s^+(\tau)]^n} \Big(\frac{1}{\tau - z} - \frac{(-1)^n (3-2\nu)}{\tau + z} \Big) d\tau \Bigg] \Bigg\}, \quad (1.20)$$

where

$$\chi_q(z) = \prod_{k=1}^{q} (z^2 - c_k^2)^{\frac{1}{2}} \prod_{k=q+1}^{2p} (z^2 - c_k^2)^{-\frac{1}{2}} \quad (1.21)$$

and

$$Q_s(z) = \prod_{j=1}^{s} (z^2 + d_j^2)^{\frac{1}{2}} \prod_{j=s+1}^{2m} (z^2 + d_j^2)^{-\frac{1}{2}} \quad (1.22)$$

are canonical functions of the class h_q and h_s, respectively; c_k and d_j are the ends of the cuts L and Λ enumerated in an arbitrary order. Note that in the vicinity of symmetrical ends canonical functions are chosen from the same class. Obviously, under the radical signs it is meant a definite branch, holomorphic in the domain S. Polynomials $p(z)$ are chosen in a well-defined manner.

The picture of solvability of the problem under consideration looks as follows: If $q+s<p+1$ for $q \leq p$, $s<m$ or $q<p$, $s \leq m$, then the problem is always solvable; if $q>p$, $s<m$ or $q<p$, $s>m$, then for the solvability of the problem it is necessary and sufficient that the conditions

$$\int\limits_L \frac{f^+(\tau)+f^-(\tau)}{\chi_q^+(\tau)}\tau^{2\mu}d\tau + \int\limits_\Lambda \frac{g^+(\tau)-g^-(\tau)}{\chi_q(\tau)}\tau^{2\mu}d\tau = 0, \qquad (1.23)$$
$$\mu = 0, 1, \ldots, q-p-1,$$

or

$$\int\limits_L \frac{f^+(\tau)-f^-(\tau)}{Q_s(\tau)}\tau^{2\mu}d\tau + \int\limits_\Lambda \frac{g^+(\tau)-g^-(\tau)}{Q_s^+(\tau)}\tau^{2\mu}d\tau = 0, \qquad (1.24)$$
$$\mu = 0, 1, \ldots, s-m-1,$$

be satisfied, respectively.

Let now $q+s=p+m$. If $q=p$ and $s=m$, then the problem is uniquely solvable. For $q>p$, $s<m$ or $q<p$, $s>m$ the problem is solvable under the conditions (1.23) and (1.24), respectively.

Let $q+s>p+m$. When $q>p$, $s>m$, then the problem is solvable if (1.23), (1.24) and the condition

$$\int\limits_L \frac{f^+(\tau)+f^-(\tau)}{\chi_q^+(\tau)Q_s(\tau)}\tau^{2\mu+1}d\tau + \int\limits_\Lambda \frac{g^+(\tau)+g^-(\tau)}{\chi_q(\tau)Q_s^+(\tau)}\tau^{2\mu N+1}d\tau = 0, \quad (1.25)$$
$$\mu = 0, 1, \ldots, q+s-p-m-1,$$

are fulfilled, while when $q>p$, $s \leq m$ or $q \leq p$, $s>m$, then the solution exists if the conditions (1.23), (1.24) or (1.24), (1.25) are satisfied, respectively.

In a similar way the problem is solved for the domain S, when the real part of the unknown function is given on the cuts to within constant summands, i.e, we have the modified Dirichlet problem. As it was expected, the problem has a unique solution, and the constants in this solution are defined uniquely.

In the same work a mixed problem of the theory of holomorphic functions is solved, when the real and the imaginary parts of the unknown function are given on the cuts.

In all the cases the solutions are constructed effectively in terms of the Cauchy type integrals.

The boundary value problem and the mixed boundary value problem for the plane with linear and arcwise cuts (cracks) are solved in [57]. It is assumed that arcwise cuts are located along the circumference $|z|=r$

symmetrically with respect to the real axis, while linear cuts are located on the real axis, symmetrically with respect to the circumference $|z| = r$. For the mixed problem, the imaginary part of the unknown function is given on the linear cuts, while the real part of that function is given on the arcwise cuts.

The above-described problem is solved in classes of functions which are bounded in the neighborhood of all cut ends.

Next, the problem for the half-plane with cuts along the arcs of the circle $|z| = r$ is solved by means of the method of analytic continuation. In this case the real part of the unknown function is given on the cuts and on the linear boundary of the half-plane, i.e., on the symmetrical with respect to the circumference $|z| = r$ segments, while the imaginary part of that function is given on the rest of the segments.

Solutions in these cases as well are constructed effectively, in the classes of functions representable by the Cauchy type integrals.

Based on the above results, as applications in [57] and [58] there were studied two cases of torsion of a prismatic bar weakened by longitudinal cuts. In the first case the normal cross-section of the bar is a circle with four linear cuts located symmetrically with respect to the coordinate axes. In the second case the cross-section of the bar is a semi-circle with the above-mentioned cuts.

Let now D be the half-plane $y > 0$ with cuts along the segments L of the y-axis or along the arcs L of the semi-circle $|z| = 1$, $y > 0$. Consider the following problem:

Define in D a holomorphic function $\phi(z)$, vanishing at infinity, by the condition

$$\mathrm{Re}[\phi(t)] = f(t), \quad t \in R$$

and by one of the conditions

$$\mathrm{Re}[\phi^{\pm}(t)] = \varphi^{\pm}(t), \quad t \in L$$
$$\mathrm{Re}[\phi^{+}(t)] = \varphi^{+}(t), \quad \mathrm{Im}[\phi^{-}(t)] = \varphi^{-}(t), \quad t \in L$$
$$\mathrm{Re}[\phi^{\pm}(t)] = \varphi^{\pm}(t), \quad t \in L_1, \quad \mathrm{Im}[\phi^{\pm}(t)] = \psi^{\pm}(t) \in L_2, \quad L_1 \cup L_2 = L.$$

Using the Riemann-Schwartz principle of reflection, in [72] this problem is reduced to the corresponding problem for the plane with cuts along the segments of the straight line $x = 0$ or along the arcs of the circle $|z| = 1$. Using N. Muskhelishvili's results, this solution is represented in quadratures.

In [72], the following mixed boundary value problem has been studied. Let D be the half-plane or the circular domain $\{z : |z| < 1\}$ inside of which smooth lines $L_1, L_2, \ldots, L_n$ are located. Define a piecewise holomorphic in D function with jump lines $L = L_1 \cup \ldots L_n$ by the conditions

$$\phi^{+}(t) = G(t)\phi^{-}(t) + g(t), \quad t \in L,$$

and $\mathrm{Re}[\phi^{+}(t)] = f(t)$, $t \in R$ or $|t| = 1$.

The solution is reduced to the Riemann problem for holomorphic function in the half-plane or in the circular domain and is solved effectively.

In [30] on a finite Riemann surface the Poincaré problem is investigated under the assumptions of meromorphy of the coefficient. Conditions of nontrivial solvability of the homogeneous Poincaré problem are established.

The question on nontriviality of a family of minimal surfaces is investigated in [1] by means of the Riemann problem in the piecewise continuous statement.

In [91], the problems of the theory of filtration are solved on the basis of boundary value problems of the theory of holomorphic functions.

As is mentioned above, the boundary value problems in the piecewise constinuous statement are tightly connected with the solutions of singular integral equations in classes H^*. In the case where the integration curve Γ is a family of a countable set of smooth open arcs located doubly periodically, the problem of solvability of the equation

$$\frac{1}{\pi i}\int\limits_{\Gamma}\frac{u(t)}{t-z}dt = f(z),$$

in the classes H^*, provided $f \in H$, has been investigated in [31].

2. On the Factorization of Matrix-Functions

The problem of linear conjugation for several unknown functions in the continuous statement has been investigated by N. Muskhelishvili in his joint with N. Vekua paper [69]. Developing significantly Plemelj's ideas, the homogeneous problem has been studied. N. Muskhelishvili introduced the notions of particular indices and of an index of the problem. For the index of the problem of conjugation there takes place N. Muskhelishvili's well-known formula

$$\varkappa = \frac{1}{2\pi i}[\ln \det G(t)]_L = \frac{1}{2\pi}[\arg \det G(t)]_L\,, \tag{2.1}$$

where the symbol $[\ ]_L$ denotes the increment of the function in brackets under a clock-wise revolution about L.

In the above-mentioned paper, for the first time it has been constructed the theory of solvability of the continuous, inhomogeneous problem for several unknown functions and it has been shown that all its piecewise holomorphic solutions, having a finite order at infinity, are represented by the formula

$$\phi(z) = \frac{X(z)}{2\pi i}\int\limits_{\Gamma}\frac{[X^+(t)]^{-1}g(t)}{t-z}dt + X(z)P(z), \tag{2.2}$$

where $X(z)$ is the canonical matrix. N. Muskhelishvili proved that construction of the canonical matrix is quite possible not only by reducing to a system of Fredholm integral equations, but more simply by means of a system of singular integral equations whose necessary properties are established independent of the problem of conjugation.

Further development of these results, when the matrix has discontinuities (in the sense of N. Muskhelishvili) at a finite number of points of a boundary, can be found in the works of N. Muskhelishvili's pupils.

Natural statement of the problem on the factorization of a matrix consists in the following:

Let R be a normed ring of functions defined on the unit circle of the complex plane, decomposed into direct sum of its subrings $R = R^+ + R_0^-$ with the continuous Riesz projector. Let $R^+ \ni 1$, and let R^- be the extension of R_0^- by unit. Let $M_n(R)$ be the ring of $n \times n$ matrices with elements from R. It is well-known that the invertible matrix $G \subset M_n(R)$ is factorizable as $G(t) = G^+(t)D(t)G^-(t)$, where $(G^{\pm})(t) \in M_n(R^+)$, $(G^{\pm})^{-1} \in M_n(R^{\pm})$ and $D(t) = \|d_{ij}(t)\|$ is the diagonal matrix with $d_{jj} = t^{\varkappa_j}$. Following N. Muskhelishvili, the integers $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ are called particular indices of $G(t)$. The particular indices determine the number of independent solutions of the corresponding homogeneous singular integral equation [66].

As is shown in [8], particular indices are unstable, therefore the choice of classes of unitary matrix functions with particular indices, equal to zero, is the matter of special interest. Since in this case we have a stability.

Since particular indices of positive matrices are equal to zero, the problem of calculation of these indices is actually reduced to the problem of such indices for unitary matrix-functions.

In [19], the problem on factorization of matrix-functions and on finding particular indices has been investigated for above-mentioned matrix-functions $U(t) = \|u_{ij}(t)\|$ of the type

$$\begin{gathered} \det U(t) = 1, \quad |t| = 1, \\ u_{ij}(t) = u_{ij}^+(t), \quad 1 \le i \le n-1, \quad 1 \le j \le n, \\ u_{nj}(t) = \overline{u_{nj}^+(t)}, \quad 1 \le j \le n, \end{gathered} \tag{2.3}$$

where $u_{ij}^+(t)$ are polynomials. Further in [19], using the solution of the well-known corona-problem [11], the above problem has been solved in a more general case, where $u_{ij}^+(t) \in L_\infty^+$, i.e., when $u_{ij}^+(t)$ are boundary values of the function of the class H_∞. The corresponding result runs as follows:

Theorem 1 [19]. *Let $u_{ij}^+(t) \in L_\infty^+$. Particular indices of the unitary matrix-function satisfying condition* (2.3) *are equal to zero if and only if*

$$\sum_{j=1}^{n} |u_{nj}^+(z)| > \delta > 0, \quad |z| < 1. \tag{2.4}$$

With the help of this theorem, for the unitary matrix of the type

$$U(t) = \begin{pmatrix} \alpha^+(t) & \beta^+(t) \\ -\beta^+(t) & \alpha^+(t) \end{pmatrix}, \tag{2.5}$$

where α^+, $\beta^+ \in L^+_\infty$, $|\alpha^+(t)|^2 + |\beta^+(t)|^2 = 1$ the validity of the following statement is proved.

Theorem 2 [19]. *For the unitary matrix-function of the type* (2.5) *the particular indices are equal to k and $-k$ if the functions $\alpha^+(z)$ and $\beta^+(z)$ have k common zeros in the space of maximal ideals of the space H_∞, lying inside the unit disk.*

Using Theorem 1, it can be managed to write the factorization explicitly in case u_{ij} are rational.

Corollary. *If the unitary matrix* (2.5) *is rational, then the factorization*

$$U(t) = F^+(t)F^-(t)$$

holds, where the elements of the matrix $F^+(t)$ are defined by the equalities $f^+_{in}(t) = \delta_{ij}$, $1 \le i \le n$, $1 \le j < n$, $f_{nn}(t) = 1$ and $f^+_{in}(t) = \varphi^+_i(t)$, $1 \le i < n$. Here φ^+_i, $1 \le i < n$ are rational functions with poles outside the unit disk (*not excluding infinity*). *Moreover, the functions φ^+_i can be constructed explicitly.*

The problem of factorization of unitary matrix-functions has arisen when investigating the factorization of positive definite matrix-functions.

For the matrix-function S given on the unit circle of the complex plane, the problem of factorization is, as far as possible, to represent it as a product

$$S(t) = S^+(t)S^+(t)', \tag{2.6}$$

where S^+ are boundary values of holomorpic in the unit disk matrix function $S^+(z)$, provided $\ln\det S^+(t) \in H^1$, i.e., $S^+(t)$ is an outer matrix-function. Here $'$ denotes Hermitian conjugacy.

The case $n = 1$, when S is a trigonometric polynomial, has been considered by Fejér and Riesz. Szegö [90] has proved that if $S \in L^1$, $S(t) > 0$ then the condition $\ln S(t) \in L^1$ is necessary and sufficient for the representation (2.6) to take place. Moreover, $S^+(t) = \exp\left(\frac{1}{2}\ln S(t) + ig(t)\right)$, where g is the conjugate to $\frac{1}{2}\ln S(t)$ function.

In the 40s of the last century, in the theory of stationary processes founded by A. Kolmogorov [54] and N. Wiener [95] a process is characterized by a spectral function $S(t) > 0$ and characteristics of the process (prediction, for example) are expressed through the Fourier coefficient of the function $S^+(t)$ obtained by means of factorization of the spectral function $S(t)$. For regular processes, $S(t) \in L^1$ and $\ln S(t) \in L^1$. For multidimensional processes, $S(t)$ itself is a positive definite matrix.

In 1956–1958, N. Wiener and P. Mazani and Helson proved independently from each other the following fact: in order for a positive summable matrix-function to admit the factorization (2.6), it is necessary and sufficient that $\ln\det S(t) \in L^1$. Under certain restrictions the algorithm for finding the above-mentioned factorization was given in the works of N. Wiener–P. Mazani and P. Mazani.

In [28], a method of factorization of a positive definite matrix has been proposed which is convenient in case the necessary and sufficient (and no additional) conditions for the factorization are fulfilled. This method is based on the traditions of investigation and applications of boundary value problems for holomorphic functions developed and justified by N. Muskhelishvili.

The essence of the method is to represent $S(t)$ in the form $S(t) = A(t)A'(t)$, where $A(t)$ is a lower triangular matrix and further, if one manages to find a unitary matrix $U(t)$ such that $A \cdot U = S^+$, then $S = AU \cdot U'A'$ will be the factorization, if S^+ is an outer holomorphic function. For finding the matrix $U(t)$, we obtain a boundary value problem for holomorphic functions whose effective solution for $A_n(t)$-special approximations of $A(t)$ enables one to construct a sequence of outer matrices $S_n^+(t)$, converging to $S^+(t)$ in L^2 [29].

Below we will endeavour to describe a method of approximate factorization of the positive definite two-dimensional matrix-function, which appeared in [29]. The main point of this achievement consists in that only under the necessary and sufficient condition for the factorization of the matrix function

$$S(t) = \chi^+(t) \cdot (\chi^+(t))',$$

where χ^+ is an outer matrix-function with elements from the Hardy H_2 space (no additional conditions were supposed), the authors managed to find an effective method of factorization, namely, they succeeded in constructing a sequence of positive definite matrix-functions $S_n(t)$, converging to $S(t)$ in the norm L^1 and possessing factorization in the explicit form:

$$S_n(t) = \chi_n^+(t)(\chi_n^+(t))'. \tag{2.7}$$

Moreover, the convergence of $\chi_n^+(t)$ to $\chi^+(t)$ was proved in the sense of the norm L^2.

Denote by $L^p_+(L^p_-)$, $p \geq 1$, the class of those complex-valued functions given on the unit circle, whose all negative (positive) Fourier coefficients are equal to zero.

A matrix-function is called of the class L^p or L^p_+, if the elements of the matrix belong to those classes, respectively. A sequence of matrix-functions converges in the norm L^p, if its elements converge in the same norm. Indices "$+$" and "$-$" denote the belonging of the function to the spaces L^p_+ and L^p_-, respectively.

If $f \in L^2$, then by $[f]^+$ ($[f]^-$) we denote the function from $L^2_+(L^2_-)$ whose positive (negative) Fourier coefficients are the same as for the function f.

Let E_r be a matrix of dimension r.

Denote by D the unite circle and let

$$D' = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Let

$$S(t) = \begin{pmatrix} a(t) & b(t) \\ \overline{b(t)} & c(t) \end{pmatrix}, \tag{2.8}$$

where a, b, $c \in L^1$ and $a(t)$, $c(t)$, $a(t)c(t) - |b(t)|^2 \geq 0$ for almost all t. The above-mentioned condition $\log\det(S(t)) \in L^1$ means that $\log(a(t)c(t) - |b(t)|^2) \in L^1$, whence we have

$$\log a(t),\ \log\Big(\frac{a(t)c(t) - |b(t)|^2}{a(t)}\Big) \in L^1. \tag{2.9}$$

Under this condition the matrix-function $S(t)$ admits the representation

$$S(t) = \begin{pmatrix} f_1^+(t) & 0 \\ \varphi(t) & f^+(t) \end{pmatrix} \begin{pmatrix} \overline{f_1^+}(t) & \overline{\varphi(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix}, \tag{2.10}$$

where f_1^+ and f^+ are outer holomorphic functions of the class H_2 for whom squares of modules of the boundary values on the unit circle coincide almost everywhere with $a(t)$ and $c(t) - |b(t)|^2/a(t)$, respectively. Note that $\varphi(t) = \overline{b(t)}/\overline{f_1^+(t)}$. It is clear that $\varphi \in L^2$, since $|\varphi(t)|^2 = |b(t)|^2/a(t) \leq c(t) \in L^1$.

Let $\varphi = \varphi^+ + \varphi^-$, where $\varphi^+ \in L^2_+$ and $\varphi^- \in L^2_-$. Rewrite 2.10 in the form

$$S(t) = \begin{pmatrix} f_1^+(t) & 0 \\ \varphi^+(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi^-(t) & f^+(t) \end{pmatrix} \begin{pmatrix} 1 & \overline{\varphi^-(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix} \begin{pmatrix} \overline{f_1(t)} & \overline{\varphi^+(t)} \\ 0 & 1 \end{pmatrix}.$$

Let

$$\varphi_n^-(t) = \sum_{k=0}^{n} \gamma_k t^{-k}, \quad n = 1, 2, \dots, \tag{2.11}$$

where $\varphi^- \sim \sum_{k=0}^{\infty} \gamma_k t^{-k}$, and let $S_n(t)$, $n = 1, 2, \dots$, be the following sequence of positive definite matrix functions:

$$S_n(t) = \begin{pmatrix} f_1^+(t) & 0 \\ \varphi^+(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi_n^-(t) & f^+(t) \end{pmatrix} \times$$
$$\times \begin{pmatrix} 1 & \overline{\varphi_n^-(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix} \begin{pmatrix} \overline{f_1(t)} & \overline{\varphi^+(t)} \\ 0 & 1 \end{pmatrix}. \tag{2.12}$$

As is easily seen, $\|S_n - S\|_{L^1} \to 0$.

Our main task now is to construct a factorization for the matrix $S_n(t)$.

We seek for a unitary matrix $U_n(t)$,

$$U_n(t) \cdot (U_n(t))' = E_2,$$

with the determinant equal almost everywhere to unity, satisfying the condition

$$\begin{pmatrix} 1 & 0 \\ \varphi_n^-(t) & f^+(t) \end{pmatrix} \cdot U_n(t) \in L_+^2. \tag{2.13}$$

A unitary matrix with the determinant equal to unity has the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Consequently, (2.13) takes the form

$$\begin{pmatrix} \alpha_n^+(t) & \beta_n^+(t) \\ \varphi_n^-(t)\alpha_n^+(t) - f^+(t)\overline{\beta_n^+(t)} & \varphi_n^-(t)\beta_n^+(t) + f^+(t)\overline{\alpha_n^+(t)} \end{pmatrix} \in L_+^2. \tag{2.14}$$

Since $\varphi_n^-(t)$ has only nonzero negative coefficients, the unknown functions α_n^+ and β_n^+ must be polynomials of the same order n.

Hence

$$U_n(t) = \begin{pmatrix} \alpha_n^+(t) & \beta_n^+(t) \\ -\overline{\beta_n^+(t)} & \overline{\alpha_n^+(t)} \end{pmatrix}, \tag{2.15}$$

where

$$\alpha_n^+(t) = \sum_{k=0}^{n} a_k t^k, \quad \beta_n^+(t) = \sum_{k=0}^{n} b_k t^k \tag{2.16}$$

and

$$|\alpha_n^+(t)|^2 + |\beta_n^+(t)|^2 = 1, \quad |t| = 1.$$

From (2.14) we have

$$\begin{cases} \varphi_n^-(t)\alpha_n^+(t) - f^+(t)\overline{\beta_n^+(t)} = \psi_{1n}^+(t), \\ \varphi_n^-(t)\beta_n^+(t) + f^+(t)\overline{\alpha_n^+(t)} = \psi_{2n}^+(t), \end{cases} \tag{2.17}$$

where ψ_{1n}^+ and ψ_{2n}^+ are functions of the class L_+^2. Equating the negative Fourier coefficients of the functions from (2.15) to zero, we construct a system of linear equations and prove that this system has a nontrivial solution.

For the sake of simplicity, the use will be made of the following notation for the matrices:

$$\Gamma_n = \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{n-1} & \gamma_n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_n & 0 & \dots & 0 & 0 \end{pmatrix}, \quad F_n = \begin{pmatrix} l_0 & l_1 & \dots & l_{n-1} & l_n \\ 0 & l_2 & \dots & l_{n-2} & l_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$A_n = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad B_n = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where γ_k, $k=1,2,\dots$ are the Fourier coefficients of φ^- and

$$f^+(z)=\sum_{k=0}^{\infty} l_k z^k.$$

The corresponding system has the form

$$\begin{cases}\Gamma_n A_n - F_n\overline{B}_n = 0,\\ \Gamma_n B_n + F_n\overline{A}_n = 1\end{cases} \tag{2.18}$$

(it is assumed that $\psi_{2n}^+(0)=1$).

Since f^+ is the outer holomorphic function, $1/f^+$ is holomorphic in D. From this we have

$$\frac{1}{f^+(z)}=\sum_{k=0}^{n} d_k z^k, \quad |z|<1,$$

where $d_0=(f^+(0))^{-1}\neq 0$ and

$$F_n^{-1}=\begin{pmatrix} d_0 & d_1 & d_2 & \dots & d_{n-1} & d_n\\ 0 & d_0 & d_1 & \dots & d_{n-2} & d_{n-1}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & d_0\end{pmatrix}$$

Defining B_n from the first equation of (2.17) and substituting it in the second one, we obtain

$$B_n=\overline{F}_n{}^{-1}\cdot\overline{\Gamma}_n\cdot\overline{A}_n, \quad \Gamma_n\cdot\overline{F}_n{}^{-1}\cdot\overline{\Gamma}_n\cdot\overline{A}_n+F_n\overline{A}_n=1.$$

Consequently,

$$(F_n^{-1}\Gamma_n\cdot\overline{F}_n{}^{-1}\overline{\Gamma}_n+E_n)\overline{A}_n=F_n^{-1}\cdot 1. \tag{2.19}$$

But the matrix $Q=F_n^{-1}\Gamma_n$ is symmetric, therefore $Q\cdot\overline{Q}$ is positive definite and the determinant of the matrix in the left-hand side of (2.19) differs from zero (moreover, its eigenvalues are greater than 1). Hence, having found A_n from (2.19), we can define the coefficients a_k, b_k, $k=0,1,\dots,n$.

Further it is proved that the equality

$$|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2=\text{const}, \quad |t|=1, \tag{2.20}$$

holds for polynomials of type (2.15) with the condition (2.16).

From (2.16) it follows

$$f^+(t)(|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2)=\psi_{2n}^+(t)\alpha_n^+(t)-\psi_{1n}^+(t)\beta_n^+(t).$$

Consequently,

$$|\alpha_n^+(t)|^2+|\beta_n^+(t)|^2=\frac{1}{f^+(t)}(\psi_{2n}^+(t)\alpha_n^+(t)-\psi_{1n}^+(t)\beta_n^+(t)). \tag{2.21}$$

The right-hand side of the above equality can be considered as boundary value of the holomorphic function

$$\phi^+(z) = \frac{1}{f^+(z)}(\psi^+_{2n}(z)\alpha^+_n(z) - \psi^+_{1n}(z)\beta^+_n(z)), \quad z \in D.$$

As far as $f^+(z)$ is the outer holomorphic function, we can conclude that $\phi^+ \in H_\infty$. But due to the fact that the left-hand side of (2.18) is positive, this implies that ϕ^+ is constant and (2.20) is valid.

Having in hand the solutions (2.18), the numbers a_k, b_k, $k = 0, 1, \dots, n$, after substitution of $t = 1$ in (2.20) we arrive at

$$\text{const} = \left|\sum_{k=0}^{n} a_k\right|^2 + \left|\sum_{k=0}^{n} b_k\right|^2.$$

The coefficients further are normalized so as the equality (2.20) be fulfilled and the matrix (2.15) to be unitary.

Next it is proved that

$$\chi^+_n(t) = \begin{pmatrix} f^+_1(t) & 0 \\ \varphi^+(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi^-_n(t) & f^+(t) \end{pmatrix} U_n(t). \tag{2.22}$$

The equality (2.8) holds owing to (2.12), (2,22) and to the equality $U_n(t)(U_n(t))' = E_2$. To show that $\chi^+_n(t)$ is an outer function, we use the equality

$$\det(\chi^+_n(z)) = f^+_1(z)f^+(z), \quad |z| < 1, \tag{2.23}$$

which in its turn is obtained as follows: the equality (2.22) can be continued to D' by using definitions of the functions φ^-_n, α^+_n and β^+_n and putting

$$\overline{\alpha}^+_n(t) = \sum_{k=0}^{n} \overline{a}_k t^{-k}, \quad \overline{\beta}^+_n(t) = \sum_{k=0}^{n} \overline{b}_k t^{-k} \quad 0 < |t| < 1.$$

Then, according to the equality $|\alpha^+_n(t)|^2 + |\beta^+_n(t)|^2 = 1$, we get

$$\det(U_n(t)) = 1, \quad t \in D'.$$

Consequently, from (2.22) it follows that (2.23) holds for $z \in D'$. On the other hand, since both sides of (2.23) are holomorphic, they will coincide on the unit disk.

As is mentioned above, in [29] it has been proved that $\chi^+_n(t)$ converge in the norm of L^2 to $\chi^+(t)$.

The basic problem which appears when finding an algorithm consists in that $S^+_n \to S^+$ does not follow from $S_n \to S$. In this connection it seems very important to us to cite the following result.

On the basis of the above algorithm for the factorization of the positive definite matrix-function, in [20] a necessary and sufficient condition has been found for the convergence of the factor functions. Here we formulate this result.

Theorem 3 [20]. *Let $S_n(t)$, $n = 0, 1, \ldots$ be a sequence of positive definite matrix functions with integrable logarithms of determinats, and let*

$$\|S_n(t) - S_n^+(t)\|_{L^1} \to 0.$$

Then

$$\|\chi_n^+(t) - \chi^+(t)\|_{H_2} \to 0$$

if and only if $\log\det(S_n(t))$ *converge weakly to* $\log\det(S(t))$.

At the end of the present section we recall one result on the approximate factorization of spectral measures in stationary processes.

It has been proved in [18] that the factorization of a measure approximation, by means of the Poisson integral kernel, converges weakly to that of the regular part of the measure (as a function). The strong convergence takes place when the measure is regular and the logarithm of its derivative is integrable. Moreover, the Poisson kernel can be replaced by many other approximate units.

3. Discontinuous Boundary Value Problems for Holomorphic Functions and Related topics of the Theory of Integral Operators

The class of piecewise holomorphic functions is a very narrow subclass of the class of holomorphic functions representable by Cauchy type integrals with density from the space L^p. Early in the 1940s, N. Muskhelishvili, having completed his investigations of continuous and piecewise continuous problems of conjugation and closely connected with them theory of singular integral equations in classical assumptions, has formulated a number of problems. In particular, the question arose: can the problem of linear conjugation, without changing the assumption on the given functions in the boundary condition (I), have a solution representable by the Cauchy type integral, different from the solutions in the class of piecewise holomorphic functions? A similar question has been put in connection with the singular integral equations: can the equation, without changing the assumption on the given functions in equation (III), have an integrable solution not belonging to the class $H^*(\Gamma)$? These problems have been investigated by B. Khvedelidze [45], [46]. He considered the problem in the following statement: find a function, holomorphic in the plane cut along a smooth curve Γ, having a finite order at infinity and continuous up to the boundary Γ except possibly a finite number of points $c_k \in \Gamma$, provided the boundary values $\phi^\pm \in L^p(\Gamma)$, $(1 < p < \infty)$, and at every point Γ (except possibly the points c_k) they are connected by the relation (I), where G and g satisfy the conditions adopted in the piecewise continuous problem. It has been proved that problem (I) has the same set of solutions in the class of piecewise holomorphic functions and in $\widehat{K}_p(\Gamma)$, i.e., in the class of holomorphic functions representable in the form

$$\phi(z) = (K_\Gamma f)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\tau + p(z), \quad f \in L^p(\Gamma), \quad p > 1,$$

where $p(z)$ is a polynomial. In the sequel we denote by $K^p(\Gamma)$ the subset of $\widehat{K}^p(\Gamma)$ consisting of functions vanishing at infinity.

The answer to the second N. Muskhelishvili's question is given in [45]. It states: any solution of equation (III) from the class $L^p(\Gamma)$ $(p > 1)$ belongs automatically to the class $H^*(\Gamma)$.

Further, in the case of Lyapunov boundary curve, in [45] it has been studied a discontinuous boundary value problem of linear conjugation, when a piecewise continuous function G is different from zero at every point of the boundary curve, $g \in L^p(\Gamma, w)$, $\phi \in \widehat{K}^p(\Gamma, w)$, $p > 1$ where w is a power function and (I) is understood in terms of the equality almost everywhere. The picture of the solvability turned out to be similar to the one in the continuous statement when one is restricted, however, to Lyapunov boundary curves. The above result stimulated further investigations of discontinuous boundary value problems in different directions.

In this section we will discuss the aspects of extension of the classes of coefficients G and of boundary curves Γ. In discontinuous problems, it is more natural to assume for the coefficient of conjugacy G to be bounded and measurable. Significant progress in this direction has been achieved by I.B.Simonenko [88], [89], who gave an effective solution of the discontinuous boundary value problem in the case of Lyapunov boundary curve and for a general class of coefficients G, provided $g \in L^p(\Gamma)$. Here we present the definition of the class of coefficients.

Definition 1. A measurable function $G(t)$ defined on the curve Γ belongs to the class $A(p)$, $p > 1$, if:

(1) there are constants c_1 and c_2 such that

$$0 < c_1 < |G(t)| < c_2 < \infty \tag{3.1}$$

for any $t \in \Gamma$;

(2) there is $\delta > 0$ such that for any point $t \in \Gamma$ there exists an arc neighborhood on Γ in which the function $G(t)$ takes values contained in a sector of angle $\frac{2\pi - \delta}{\max(p, p')}$ $\left(p' = \frac{p}{p-1}\right)$ with vertex at the origin.

In connection with the above-mentioned investigations, in discontinuous problems there arose the necessity in simultaneous extension of classes of the given and unknown functions as well as of boundary curves. In [49], (see also [39]) the boundary curve is taken from a wide class containing, in particular, piecewise smooth curves and those with bounded rotation without cusps. Assumptions regarding the coefficients are such that along with the above-considered coefficients with finite index, they cover some new ones.

Detailed account of these results can be found in [49], [39]. We will dwell on the description of conditions on the coefficient G, because the solution of the discontinuous problem of linear conjugation under these assumptions has, as it will be shown below, an essential impact on the solution of the Riemann-Hilbert problem in domains with piecewise smooth boundaries involving cusps, as well as on problems of conformal mappings.

Along with the qualitative investigation, the authors have managed to construct the solutions in quadratures under these general assumptions as well.

Definition 2. A measurable on Γ function is said to be of the class $\widehat{A}(p)$, $1 < p < \infty$, if the following conditions are fulfilled:

(i) $0 < \operatorname{ess\,inf} |G|$; $\operatorname{ess\,sup} |G| < \infty$;

(ii) for all $t \in \Gamma$ with the possible exception of a finite number of points $c_k = t(s_k)$, $s_k < s_{k+1}$, $k = 1, \dots, n$, there exists on Γ a neighborhood in which the values of G lie in some sector with the vertex at the origin and the angle less than $a(p) = \frac{2\pi}{\max(p,p')}$, $p' = \frac{p}{p-1}$;

(iii) at the points c_k there exist limits $G(c_k-)$ and $G(c_k+)$; let the angles δ_k between the vectors corresponding to $G(c_k-)$ and $G(c_k+)$ be such that

$$\frac{2\pi}{p} < \delta_k \leq \frac{2\pi}{p'}, \quad \text{for} \quad p > 2 \quad \text{and}$$

$$\frac{2\pi}{p'} \leq \delta_k < \frac{2\pi}{p}, \quad \text{when} \quad 1 < p < 2, \quad k = 1, 2, \dots, n.$$

The points c_k will be called p-points of discontinuity of the function G.

The existence of p-points of discontinuity makes it possible to cover by the class $\widehat{A}(p)$ most of those functions satisfying condition (i) which were considered in terms of the coefficients of the problem of conjugation with a finite index. The class $\widehat{A}(p)$ contains any admissible piecewise continuous coefficients and the functions whose argument φ is representable in the form $\varphi = \varphi_0 + \varphi_1$, where φ_0 is continuous and φ_1 is of a bounded variation (see [12]).

Combining the definitions of the argument for piecewise continuous functions and for a function from $A(p)$, it is determined for given p the argument of $G(t)$ at every point $t \in \Gamma$ so that the increment of the argument resulting from going around Γ appears to be exactly the same characteristic for the problem with the coefficient G as the increment of the argument is for the continuous coefficient. For the details of the above-mentioned definitions we refer to [49] (see also [39], p.92).

The integer

$$\varkappa = \varkappa_p = \varkappa_p(G) = \frac{1}{2\pi}[\arg_p G(c-) - \arg_p G(c^+)] \tag{3.2}$$

is called the index of the function G in the class $K_p(\Gamma)$ and is denoted by $\operatorname{ind}_p G$.

Suppose that the boundary curve is piecewise smooth or with bounded rotation, or it represents a union of such arcs without cusps on it.

The following statement is true.

Theorem 1 [49]. *Let* $G \in \widehat{A}(p)$, $1 < p < \infty$ *and* $\varkappa = \varkappa_p(G)$ *be its index. Then*

I. *For problem* (I) *in the class* $K_p(\Gamma)$ *the following assertions are valid:*

(i) *if* $\varkappa \geq 0$, *then the problem is solvable for any* $g \in L^p(\Gamma)$, *and its general solution is given by the equality*

$$\phi(z) = X(z) K_\Gamma\Big(\frac{g}{X^+}\Big)(z) + X(z) p_{\varkappa-1}(z), \tag{3.3}$$

where the function $X(z)$ *is constructed in quadratures in terms of* G *and* $p_{\varkappa-1}$ *is an arbitrary polynomial of degree not higher than* $\varkappa - 1$, $p_{-1}(z) \equiv 0$;

(ii) *if* $\varkappa < 0$, *then the homogeneous problem has only the zero solution, while the inhomogeneous problem is solvable only for the functions* g *satisfying the condition*

$$\int\limits_\Gamma \frac{t^k}{X^+(t)} g(t)dt = 0, \quad k = 0, 1, \dots, |\varkappa| - 1.$$

If this condition is fulfilled, then the solution is given by (3.3), *where* $p_{\varkappa-1} \equiv 0$.

II. *For the singular integral equation* (III) *in the space* $L^p(\Gamma)$, $1 < p < \infty$, *the Noether theorem is valid under the assumption that* $G = (a-b)(a+b)^{-1} \in \widehat{A}(p)$.

Moreover,

$$l - l' = \varkappa_p(G),$$

where l *and* l' *are the numbers of linearly independent solutions of the homogeneous equation and its adjoints.*

Solutions of the singular integral equation with $V = 0$ are given by the formula

$$\varphi = \phi^+ - \phi^-,$$

where ϕ is the solution of the problem of conjugation

$$\phi^+ = (1-b)(a+b)^{-1}\phi^- + f(a+b)^{-1}.$$

Of the main tools applied in solution of discontinuous boundary value problems for holomorphic functions, the basic are the methods of factorization of coefficients given on the boundary, as well as the theory of Cauchy type integrals with densities from Lebesgue spaces. All these investigations need an improvement of the techniques of Cauchy type integrals and Cauchy singular integral operators, the derivation of new weighted inequalities for these operators and the development of some methods of functional analysis.

Here we present some results stated in above-mentioned works and having considerable impact in the subsequent development of investigations in this direction.

When a function G is factored in the class $K^p(\Gamma)$, there naturally arises the problem of finding the conditions on Γ and φ under which the function

$$X(z) = \exp\Big(\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\varphi(t)dt}{t-z}\Big) = \exp(K_\Gamma(\varphi))$$

belongs to the Smirnov class in domains bounded by the curve Γ.

Let us recall the definition of Smirnov class of holomorphic functions.

Let D be a simply connected domain bounded by a rectifiable curve Γ.

Then $E^p(D)$, $p>0$, is the Smirnov class of holomorphic in D functions ϕ for which there exists a sequence of closed curves $\Gamma_n \subset D$ converging to Γ such that

$$\sup_n \int\limits_{\Gamma_n} |\phi(z)|^p|dz| < \infty.$$

If D is an infinite domain bounded by Γ, then $\widehat{E}^p(D)$ is a set of analytic in D functions ϕ for which $h(w) = \phi\Big(\frac{1}{w}+z_0\Big) \in E^p(\widehat{D})$, where $\widehat{D}$ is the finite domain into which the function $w = \frac{1}{z-z_0}$, $z_0 \overline{\in} D$ maps D. (It is clear that this class does not depend on the choice of z_0). Assume $E^p(D) = \{\phi : \phi \in \widehat{E}^p(D), \quad \phi(\infty) = 0\}$.

By R we denote the set of all Jordan curves for which the singular operator S_Γ is continuous in $L^p(\Gamma)$, $1<p<\infty$. G. David's famous result says that for the curve Γ to belong to the class R, it is necessary and sufficient that the condition

$$\sup_{\substack{z\in\Gamma\\ r>0}} \nu(B(z,r)\cap\Gamma)r^{-1} < \infty$$

be fulfilled, where ν is the arclength measure defined on Γ, $B(z,r)$ is the ball with center $z\in\Gamma$ and radius r. Such curves are called also regular curves or curves with Carleson conditions.

Theorem 2 [49], [39]. *Let Γ be a closed Jordan curve of the class R bounding a finite domain D^+ and an infinite domain D^-. Then:*

(i) *for any bounded, measurable on Γ function φ there exist numbers $\delta>0$ and an integer $n_0 \ge 0$ such that*

$$\exp(K_\Gamma\varphi) = X(z) \in E^\delta(D^+), \quad (z-z_0)^{-n_0}[X(z)-1] \in E^\delta(D^-);$$

(ii) *for an arbitrary continuous on Γ function φ we have*

$$X(z) \in \cap_{p>1}E^p(D) \quad \textit{and} \quad [X(z)-1] \in \cap_{p>1}E^p(D^-).$$

This theorem can be considered as an extension of the classical Smirnov and Zygmund's theorems.

The other important tool of investigation of discontinuous problems is the theorem on boundedness of the operator S_Γ in weighted $L^p(\Gamma, w)$ spaces $1 < p < \infty$. We present the general statement in this direction.

Theorem 3 [39]. *Let $1 < p < \infty$ and $\Gamma \in R$. For the validity of the inequality*

$$\int_\Gamma |S_\Gamma f(t)|^p w(t) d\nu \leq c \int_\Gamma |f(t)|^p w(t) d\nu,$$

where c does not depend on f, it is necessary and sufficient that $w \in A_p(\Gamma)$, i.e.,

$$\sup_{z \in \Gamma} \frac{1}{\nu\Gamma(z,r)} \int_{\Gamma(z,r)} w(t) d\nu \left(\frac{1}{\nu\Gamma(z,r)} \int_{\Gamma(z,r)} w^{1-p'}(t) d\nu \right)^{p-1} < \infty,$$

where $\Gamma(z,\rho) = \Gamma \cap B(z,\rho)$.

For one of the proofs of this theorem we refer to [39], Theorem 4.2, p. 52.

The class $A_p(\Gamma)$ is an anologue of the well-known B. Muckenhoupt A_p class.

Get back to the classes $\widehat{A}(p)$ of coefficients G. Their introduction should not at first sight extend the class of oscillating coefficients in discontinuous boundary value problems, but the above-mentioned solution of the problem with the boundary coefficient $G \in \widehat{A}(p)$ has in fact exerted great influence on the investigation of a number of complicated problems. We will mention only some of them.

First of all it should be noted that on the basis of the above results, in [50-53], see also [39], we succeeded in studying the Riemann-Hilbert problem

$$\mathrm{Re}[(a+ib)\phi^+] = f, \quad f \in L^p, \quad p > 1,$$

in classes $E^p(D)$ under the assumption that $(a-ib)(a+ib)^{-1} \in \widehat{A}_p$ and Γ is a piecewise smooth curve, involving cusps.

The above-mentioned extensions of the class of coefficients in the problem of linear conjugation have been widely used in finding most important from theoretical point of view representations of derivatives which map conformally the unit circle onto a simply connected domain. One of appropriate results is given in the form of

Theorem 4. *Let the boundary of a simply connected domain D be a simple, closed, oriented curve such that its every point, except a finite number of points $t_1, \dots, t_k$, has a neighborhood such that the existing directed tangential vectors lie in an angle of size $\frac{\pi}{\lambda}$, $\lambda > 2$. Suppose, moreover, that at the points t_i $(i = 1, \dots, k)$ the curve has angles of size $\nu_i \pi$, $0 \leq \nu_i \leq 2$*

and is smooth near them. Then for the function z, mapping conformally the unit circle onto the simply connected domain D, we have

$$z'(w) = \prod_{i=1}^{k} (w - \tau_i)^{\nu_i - 1} z_0(w), \quad z(\tau_i) = t_i,$$

where $[z_0(w)]^{\pm 1} \in H^{\lambda}$, $[z_0(\tau)]^{\frac{1}{\lambda}} \in A_{\lambda}$.

In the subsequent sections we will get back to the two series of the above-discussed questions and present some other results.

When studying the boundary value problems, there naturally arise Smirnov's weighted classes and also classes of those Cauchy type integrals whose densities belong to the Lebesgue weighted classes as those of unknown functions. It turned out that solution of linear boundary value problems in the above-mentioned classes can be easily reduced to the problems in unweighted classes. Indeed, if $\rho = e^{iS\varphi}$, $\varphi \in \operatorname{Re} L^{\infty}$, then the boundary value problem with coefficient G from the class with weight ρ is equivalent to the problem with coefficient $G_{\rho} = Ge^{i\varphi}$ in the unweighted class. Note that weighted functions of the type mentioned above represent a highly wide class of all admissible weighted functions, while in the case of Lyapunov contours they exhaust the class of such functions.

Reduction of the problem of linear conjugation in the class $K^p(\Gamma, \rho)$ to an analogous unweighted problem for $\Gamma \in R$ has been discussed in [49]. The idea of the above-mentioned reduction arose in 60s (see, for example, [75] and cited there the earlier papers by the same author).

The application of the idea to the reduction of Riemann-Hilbert problem in the class $K^p(D, \rho)$ to the linear conjugation problem with circular boundary contour γ, whose coefficient absorbs all singularities of the boundary, weight and initial coefficient, has been considered in [78]. Here $K^p(D, \rho)$ denotes the restriction on D of the functions from the class $K^p(\Gamma, \rho)$.

In the last investigations, the well-known N. Muskhelishvili's method of reduction of the Riemann-Hilbert problem to the linear conjugation problem is employed heavily.

As is mentioned above, in investigating problem (I) in $K^p(\Gamma)$, the boundary curves might be piecewise smooth or of bounded rotation or they might be a union of arcs from the above-mentioned classes, but with the only condition that the curves have no cusps.

An attempt to become free from that restriction is contained in [23-25]. Imposing at the cusp some additional restriction on the coefficient G from the class $A(p)$, one can retain the well-known picture of solvability of the linear conjugation problem.

By Γ_{ab} is denoted an arc with the ends a and b directed from a to b. An arc Γ_{ab} is said to be of class R_A, if it can be supplemented to the closed

Jordan curve Γ_0 so that for any function $G(t) = \exp i\varphi(t)$, where

$$\operatorname{vrai\,sup} |\varphi(t)| < \frac{\pi}{\max(p, p')}, \quad t \in \Gamma_{b_k a_k},$$

and $\varphi(t) = 0$ for $t \in \Gamma_0 \backslash \Gamma_{ab}$, we have the well-known statement of solvability of (I).

The basic result of the above-mentioned work looks as follows: Assume that a simple closed Jordan curve $\Gamma \in R$, $\Gamma = \cup_{k=1}^{n-1} \Gamma_{a_k a_{k+1}}$, $a_n = a_1$, $\Gamma_{a_k, a_{k+1}} \in R_{_A}$ $(k = 1, \dots, n-1)$ and $G(t)$ is a measurable function on Γ satisfying Simonenko's conditions and, moreover, for the points d_k $(k = 1, \dots, n)$ there exist on Γ arc neighborhoods $\Gamma_{b_k a_k}$ and $\Gamma_{a_k c_k}$ in which for some $\alpha \in [0, 1]$ the conditions

$$\operatorname{vrai\,sup} |\varphi(t)| < \alpha \frac{\pi}{\max(p, p')}, \quad t \in \Gamma_{b_k a_k}, \tag{3.4}$$

and

$$\operatorname{vrai\,sup} |\varphi(t)| < (1 - \alpha) \frac{\pi}{\max(p, p')}, \quad t \in \Gamma_{a_k, c_k}, \tag{3.5}$$

are fulfilled. Then the known picture of solvability is retained and formulas of the solution are written out explicitly.

Obviously, under the above-mentioned conditions on the boundaries Γ the cusps are quite possible at the points a_k.

Thus, if the coefficient G satisfies the condition $A(p)$ and, in addition, conditions (3.4)–(3.5) are valid for piecewise smooth boundary curves Γ, as well as for lines of bounded variation with cusps the picture of solvability in $K^p(\Gamma)$ is usual.

In connection with this result we note the pioneering work [24], where the boundedness of a singular integral operator in case the integration curve is a piecewise Lyapunov curve with an individual cusp, was established.

Let us now pass to consideration of the case where the right-hand side g in the discontinuous boundary value problem (I) belongs to $L^1(\Gamma)$. Then it is well-known that the boundary value problem in the class of functions representable by Cauchy-Lebesgue integral has no solution even in the simplest case. This is connected with the fact that boundary function of the Cauchy type integral is not always summable, and a Cauchy type integral with a summable density fails to be representable by the Cauchy integral.

In [37], [39] it has been shown that most of the results obtained for the conjugate function and for Cauchy-Lebesgue type integrals do not depend on specific properties of these integrals. They hold also true for any generalization of the Lebesgue integral in whose sense the conjugate function is integrable and its integral equals zero. Thus the notion of the $\widehat{L}$-integral has arisen. It turned out that if density of the Cauchy type integral is summable, then its angular boundary values are $\widehat{L}$-integrable and the Cauchy type integral is representable in the domain by the Cauchy $\widehat{L}$-integral.

Let Γ be a simple, closed, rectifiable curve dividing the complex plane into domains $D^+ = \operatorname{Int}\Gamma$ and $D^- = \operatorname{Ext}\Gamma$. Γ is said to be of class (C), if the following statement is valid: If the angular boundary value F^+ of Cauchy-Lebesgue type integral $F(z)$ is summable, then

$$\int\limits_\Gamma F^+(t)dt = 0$$

(the Cauchy theorem).

Condition (C) is satisfied, for example, by regular curves.

One of the main tools of investigation of the linear conjugation problem with summable right side is

Theorem 5 [37]. *Let the functions ϕ and F be representable in $D^+(D^-)$ by the Cauchy-Lebesgue integral with densities φ and f, respectively, satisfying the conditions*

$$\int\limits_\Gamma \Big|\frac{\varphi(t)-\varphi(t)}{\tau - t}\Big| d\nu < M, \quad t \in \Gamma$$

and $f \in L^1(\Gamma)$.

Then the product ϕF is also representable in $D^+(D^-)$ by the Cauchy type integral.

From this theorem it follows that if the functions ϕ and F are representable by the Cauchy type integral respectively with densities φ and f from $\widehat{L}(\Gamma)$, satisfying the condition of the previous theorem, then the product ϕF is representable in $D^+ \cup D^-$ by the Cauchy type $\widehat{L}$-integral.

On the basis of the above results in [37] the inhomogeneous linear conjugation problem (I) is solved when the boundary curve is regular, i.e., satisfies the Carleson condition, the left side in the boundary condition is summable and the coefficient G belongs to Hölder class on Γ and differs on Γ from zero.

A solution of the above-formulated problem is found in the class of functions representable in the form

$$\phi(z) = \begin{cases} K_\Gamma(\varphi_1)(z) & \text{for} \quad z \in D^+, \\ K_\Gamma(\varphi_2)(z) + p(z) & \text{for} \quad z \in D^-, \end{cases}$$

where φ_1 and φ_2 are summable and p is a polynomial. This class of functions coincides with that, of functions representable by the Cauchy type $\widehat{L}$-integral with the polynomial principal part at infinity.

In particular, the investigation of the above-mentioned problem shows that if the functions g and $S_\Gamma g$ are summable on Γ, then any solution of the problem is representable by the Cauchy-Lebesgue type integral. For the details we refer to [37], [39, Ch.II].

As is mentioned above, investigation of discontinuous boundary value problems in classes of holomorphic functions representable by a Cauchy type integral with density from the Lebesgue class has given an impetus to quite a number of studies of integral operators.

We present here one kind of such results.

Theorem 6 [40], [81]. *Let Γ be the union of a countable set of concentric circles. For the boundedness of S_Γ from $L^p(\Gamma)$ to $L^q(\Gamma)$, $p > q \geq 1$, it is necessary and sufficient that the condition*

$$\int_\Gamma [h(t)]^\sigma d\nu < \infty,$$

where $h(t) = \sup_{r>0} \frac{\nu(B(t,r)\cap\Gamma)}{r}$, be fulfilled.

We complete this section description of some results concerning the boundedness of singular integral operators with a weak singularity.

Let Γ be a simple, rectifiable curve of the complex plane. Assume

$$T_\Gamma^\alpha f(t) = \int_\Gamma \frac{f(\tau)}{|\tau - t|^{1-\alpha}} d\nu, \quad 0 < \alpha < 1.$$

The following statement is true.

Theorem 7 [47]. *Let $1 < p < q < \infty$. The operator T_Γ^α acts boundedly from $L^p(\Gamma)$ into $L^q(\Gamma)$ if and only if there exists a constant $c > 0$ such that*

$$\nu(B(z,r) \cap \Gamma) \leq cr^\beta,$$

for $\beta = \frac{pq(1-q)}{pq+p-q}$ and arbitrary $z \in \Gamma$ and $r > 0$.

From this theorem it follows that if $\frac{1}{q} = \frac{1}{p} - \alpha$, then T_Γ^α is bounded from $L^p(\Gamma)$ to $L^q(\Gamma)$, iff Γ is regular.

Theorem 8. *Let $1 < p < \infty$ and Γ be regular, finite, closed curves. If $w \in A_p(\Gamma)$, then T_α acts boundedly from $L^p(\Gamma, w)$ into $L^p(\Gamma, w)$.*

4. On Conformal Mappings of a Circle onto a Domain with a Piecewise Smooth Boundaries via the Boundary Value Problem of Linear Conjugation

In recent years, in a number of works [38], [76], [77], [80], [39] by using the solution of discontinuous boundary value problems for holomorphic functions, the properties of the derivative and those of the argument of the derivative of a function which maps conformally the unit circle onto a domain with a non-smooth boundary have been investigated. The authors have given important representations of the derivative of conformal maps in terms of Cauchy type integrals. This leads to provides new insights into the behaviour of the derivative in the neighborhood of angular points and cusps of the boundary. The authors' approach provides new and lucid proofs of classical theorems due to Lindelöf, Kellog and Warschawski as well as generalizations of these theorems.

Let us mention some well-known facts on conformal mappings. If a function $z = z(w)$ maps conformally the unit circle U onto a finite domain D bounded by a closed, rectifiable curve Γ, then $z' \in H^1$, z is continuous on $\overline{U}$ and absolutely continuous on its boundary γ, for almost all $\theta \in [0, 2\pi]$ there exists an angular boundary limit of the functions $z'(w)$ and

$$\lim_{w \widehat{\to} e^{i\theta}} z'(w) = -ie^{i\theta} \frac{dz(e^{i\theta})}{d\theta}. \tag{4.1}$$

Further we have that $\frac{dz(e^{i\theta})}{d\theta} = \left|\frac{dz(e^{i\theta})}{d\theta}\right| e^{i\alpha(\theta)}$, where $\alpha(\theta)$ is some angle between the oriented tangent to Γ at the point $z(e^{i\theta})$ and the real axis.

Let $z'(0) > 0$, and consider a holomorphic in D function

$$\ln z'(w) = \ln|z'(w)| + i \arg z'(w), \quad \arg z'(0) = 0.$$

From the equality (4.1) we can conclude that there exists an angular limit

$$\lim_{q \widehat{\to} e^{i\theta}} \arg w'(z) = \alpha(\theta) - \theta - \frac{\pi}{2} + 2k(\theta)\pi = \beta(\theta) + 2k(\theta)\pi, \tag{4.2}$$

where $k(\theta)$ is a function taking integer values.

In the presence of additional information about the boundary one can elicit more knowledge than that available here. For example, if Γ is a smooth curve, then $z' \in \cap_{p>0} H^p$, and in the case of Lyapunov boundary curves z' belongs to Hölder class in a closed disk.

Using the solutions of discontinuous boundary value problems it becomes possible to investigate properties of the function z' for a wide class of rectifiable curves. In particular, the following statements are proved in [76]: If every point of the boundary Γ has a neighborhood in which tangential oscillation is less than π, then for $\arg z'(w)$ the analogue of Lindelöf's theorem, known for smooth curves, is valid. Indeed, for almost all $\theta_0 \in [0, 2\pi]$ the equality

$$\lim \arg_{re^{i\theta} \widehat{\to} e^{i\theta_0}} z'(re^{i\theta}) = \alpha(\theta_0) - \theta_0 - \frac{\pi}{2},$$

is valid, where $\alpha(\theta_0)$ is an appropriately chosen value of a multi-valued angular function made up of the tangent and positively directed Ox-axis.

On the basis of Theorem 4 from previous section the well-known theorems of Kellog, Lindelöf and Warschowski and some their generalizations are derived. In particular, for a piecewise Lyapunov curve with only one angular point c, when the size of angle equals $\nu\pi$, $0 < \nu \leq 2$, according to the Warschawski theorem we have

$$z'(w) = (w - c)^{\nu - 1} z_0(w), \tag{4.3}$$

where $z(c) = C$, and z_0 is a function continuous everywhere and different from zero in $\overline{U}$. It turns out that the last representation remains valid for $\nu = 0$ with the only difference that $(z_0)^{\pm 1} \in \cap_{p>0} H^p$. Moreover, under the same assumptions regarding z_0, the equality (4.3) remains valid for arbitrary piecewise smooth curves.

Let us cite as an example the assertion generalizing Warschawski's theorem.

Theorem 1 [41], [39]. *Let Γ be a piecewise smooth, closed, oriented Jordan curve with angular points t_k, bounding the finite domain D, and let $\nu_k\pi$, $0 \le \nu_k \le 2$, $k = 1, 2, \dots, n$ be a size of interior (with respect to D) angles at these points.*

Then if the function $z = z(w)$ maps conformally the unit disk U onto D, and

$$z'(0) > 0, \quad z(c_k) = t_k, \quad c_k \in \gamma,$$

then

$$z'(w) = \prod_{k=1}^{n} (w - c_k)^{\nu_k - 1} z_0(w), \tag{4.4}$$

where

$$z'(w) = \exp\left(\frac{1}{\pi}\int\limits_{\gamma} \frac{\delta(\tau)d\tau}{\tau - w}\right)$$

and $\delta = \delta(t)$ is a continuous on γ function. Moreover, if Γ is a piecewise Lyapunov contour and $0 < \nu_k \le 2$, then $\delta(t)$ belongs to the Hölder class.

Let us consider briefly the method of obtaining the above-mentioned results. By means of construction of a certain branch $\alpha(\theta)$ of the angular function of Γ, it is proved that $\sqrt[p]{z'}$, $p > \frac{1+\lambda}{\lambda}$, is a restriction to the unit circle of the unique in $K^p(\Gamma)$ solution for the conjugacy problem

$$\phi^+ = \exp\frac{2\beta}{p}\phi^-, \quad \beta(\theta) = \alpha(\theta) - \theta - \frac{\pi}{2}.$$

This problem, as it has been shown in Section 3, is well investigated.

Following this way, there is a good perspective to achieve much more in the theory of conformal mappings in the case of further progress in the problems of linear conjugation.

We will especially dwell on a representation of the derivative of a conformal mapping obtained incidentally in the above-cited investigations. The question is about the following representation:

$$z'(w) = z'(0)\exp\left(\frac{1}{\pi}\int\limits_0^{2\pi} \frac{\alpha(\theta) - \theta - \frac{\pi}{2}}{e^{i\theta} - w} de^{i\theta}\right) \tag{4.5}$$

established in [76] for the curves from Theorem 4. In (4.5) the angle $\alpha(\theta)$ is well-defined by the geometry of the curve (for details see [76], [77] and [39], pp. 147-149, see also [83], p. 65). The formula (4.5) is an analogue of Cisotti's formula which has been obtained for a narrow class of curves.

As for Cisotti formula, the function $\alpha(\theta)$ (inclination angle of the tangent to the circle at $w(e^{i\theta})$) is unknown and Cizotti's formula fails to give an efficient solution of the problem of conformal mapping. (See M. A. Lavrent'jev and B. N. Shabat, The Methods of the Theory of Functions of Complex Variables (Russian). Gosizdat, Moscow 1958, p. 213). However, this formula turns out to be useful whenever $\beta(t)$ is found by some reasons. In [83, p. 65] it is for certain said that "the difficulty in applying this formula is that $\beta(t)$ is the forward tangent direction at $w(e^{it})$ and therefore refers to the conformal parametrization which is not explicitly known if the boundary is given geometrically".

If the domain is such that $\ln z'(w) \in H^1$, then it can be easily proved that

$$z'(w) = z'(0)\exp\left(\frac{1}{\pi}\int\limits_0^{2\pi}\frac{\arg z'(e^{i\theta})de^{i\theta}}{e^{i\theta}-w}\right). \tag{4.6}$$

On the other hand, we have (4.2), where $\alpha(\theta)$ is, we emphasize, the angle at $z(e^{i\theta})$ formed by the tangent and the Ox-axis, and $k(\theta)$ is a function admitting integer values.

On the basis of the formulas (4.2) and (4.6) it is impossible to state that formula (4.5) is valid for an arbitrary domain with the condition $\ln z'(w) \in H^1$, where $\alpha(\theta)$ is the angle formed by the tangent at the point $z(e^{i\theta})$ and the Ox-axis, since it is not clear what value of that angle we speak about.

The more so it is erroneous to speak about its validity for any rectifiable curve (see [17], p.94). In this case $\arg z'(w)$ is not always summable. The real situation clears up by the result below [43].

Recall the definition of Smirnov's domain. So are called the domains for which the representation

$$\lg z'(w) = \frac{1}{2\pi}\int\limits_0^{2\pi}\lg|z'(e^{i\delta})|\frac{e^{i\sigma}+w}{e^{i\sigma}-w}d\sigma$$

holds.

Theorem 2 [43]. *For the formula*

$$\begin{aligned}\lg z'(w) &= \lg z'(0) + \frac{i}{2\pi}(\widehat{L})\int\limits_0^{2\pi}\arg z'(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}d\sigma = \\ &= \lg z'(0) + \frac{1}{\pi}(\widehat{L})\int\limits_{|\tau|=1}\frac{\arg z'(\tau)}{\tau-w}d\tau, \quad |w|<1\end{aligned} \tag{4.7}$$

to be valid, it is necessary and sufficient that D be Smirnov's domain.

From the above theorem follows

Corollary. *For the equality* (4.7) *with the summable function* $\arg z'(e^{i\theta})$ *to take place, it is necessary and sufficient that* $\lg z'$ *to belong to the Hardy class* H^1.

It is interesting to note that further extension of the notion of the integral with a view to extend the formula (4.7) to non-Smirnov domains becomes impossible [43].

As it has been shown, the results quoted in this section are widely used in Riemann-Hilbert boundary value problems, in particular, in the investigation of Dirichlet and Neumann problems for harmonic functions of Smirnov class in domains with piecewise smooth boundaries involving cusps.

5. Application of Muskhelishvili's Method to the Solution of Boundary Value Problem in Domains with Piecewise Smooth Boundaries

In this section we will present the results on Riemann-Hilbert boundary value problems, particularly, on the Dirichlet and Neumann problems for harmonic functions in domains with "bad" boundaries, in particular, for piecewise smooth boundaries involving cusps of various types.

For the Dirichlet and Neumann problems, in [50], [51], [53], [39] a picture of solvability is described completely for harmonic functions which are real parts of holomorphic functions from the Smirnov class $E^p(D)$ and boundary data in $L^p(\partial D)$, $p > 1$. The influence of geometric properties of ∂D on the solvability of the problems is revealed completely; the non-Fredholm case is exposed; in all cases of solvability the explicit formulas for the solutions in terms of Cauchy type integrals and conformal mapping functions are given. The results are extended to Riemann-Hilbert problems with coefficients in $\widehat{A}(p)$ and domains D whose boundary ∂D has a single angular point involving cusps.

Note that the following original sources were the basis of these investigations: N. Muskhelishvili's method of reduction of Riemann-Hilbert problem to the problem of linear conjugation; behaviour of a function the derivative of mapping conformally the unit disk onto a simply connected domain in the neighborhood of angular points (see previous section);solution of the linear conjugation problem with the coefficient from $\widehat{A}(p)$; and finally, two-weight estimates for singular integral operators.

Let D be a simply connected domain bounded by a simple, piecewise smooth curve Γ. Let $e^p(D)$ be the set of harmonic functions representing a real parts of functions from the class $E^p(D)$. The functions of this class possess almost everywhere on Γ angular boundary values belonging to $L^p(\Gamma)$.

We pose the following Dirichlet problem:

$$\left.\begin{array}{ccc} \Delta u = 0, & u \in e^p(D), & p > 1 \\ u(t) = f(t), & t \in \Gamma, & f \in L^p(\Gamma) \end{array}\right\}. \tag{5.1}$$

Let $u = \operatorname{Re}\phi$, and assume

$$\psi(w) = \sqrt[p]{z'(w)}\,\phi(z(w)), \quad |w| < 1, \tag{5.2}$$

where $z = z(w)$ is a function mapping conformally the unit disk U onto D. Following N. Muskhelishvili, suppose

$$\Omega(w) = \begin{cases} \psi(w), & |w| < 1, \\ \overline{\psi}\left(\frac{1}{w}\right), & |w| > 1, \end{cases} \tag{5.3}$$

$$\Omega_*(w) = \overline{\Omega}\left(\frac{1}{w}\right), \quad |w| \neq 1, \tag{5.4}$$

The problem (5.1) is equivalent to the problem in the following statement:

$$\Omega^+(\tau) = -\frac{\overline{\sqrt[p]{z'(\tau)}}}{\sqrt[p]{z'(\tau)}}\Omega^-(\tau) + g(\tau), \tag{5.5}$$

$$g(\tau) = 2f(z(\tau))\sqrt[p]{z'(\tau)}, \quad \tau \in \gamma,$$

$$\Omega \in \widehat{K}^p(\gamma), \quad \Omega_*(w) = \Omega(w), \quad \gamma = \{\tau;\ |\tau| = 1\} \tag{5.6}$$

in the sense that any solution of $u = \operatorname{Re}\phi$ of (5.1) generates by equalities (5.2) and (5.3) the function Ω which satisfies the condition (5.5) and vice versa, if Ω satisfies (5.5)–(5.6) and ψ is its restriction on U, then

$$u = \operatorname{Re}\Big[\frac{\psi(w)}{\sqrt[p]{z'(w)}}\Big]$$

is a solution of (5.1).

Theorem 1 [53], [39]. *Let Γ be a simple, closed piecewise smooth curve containing one angular point C with interior angle of size $\nu\pi$, $0 \le \nu \le 2$ and $X_1(w) = (w-c)^{-1}X(w)$, where*

$$X(w) = \begin{cases} -\sqrt[p]{z'(w)}, & |w| < 1, \\ \overline{\sqrt[p]{z'\left(\frac{1}{w}\right)}}, & |w| > 1, \end{cases} \tag{5.7}$$

and $c = w(C)$.

The problem (5.1)*:*

– is uniquely solvable for $0 < \nu < p$;

– has for $p < \nu$ a set of solutions depending on one parameter;

– is, in general, unsolvable for $p = \nu$ and becomes solvable, iff the function

$$Tg(\zeta_0) = (\zeta_0 - c)^{\frac{1}{p'}} z_1(\zeta_0) \int\limits_{\gamma} \frac{g(\zeta)d\zeta}{(\zeta - c)^{\frac{1}{p'}} z_1(\zeta)(\zeta - \zeta_0)}, \tag{5.8}$$

belongs to $L^p(\gamma)$*. Here* $z_1 = z_0^{\frac{1}{p}}$*,* z_0 *is from the representation of the derivative of the conformal mapping (see the previous section);*

– moreover, it has a unique solution, if $X_1 \notin H^p$*, and a set of solutions depending only on one parameter for* $X_1 \in H^p$*;*

– is, in general, unsolvable for $\nu = 0$*. If* $T_g \in L^p(\gamma)$ *is fulfilled, it has a unique solution.*

As is mentioned above, in all cases of solvability the explicit formulas of solutions are given.

The inclusion $T_g \in L^p(\gamma)$ *is true, for example, when* $f(t)\ln|w(t) - C| \in L^p(\Gamma)$*.*

Other optimal conditions can possibly be easily derived from [48].

For the sake of simplicity of our exposition we have presented the result in the case of one angular point only. For a finite number of angular points and for some other details we refer the reader to [39, Ch.IV].

As is mentioned above, the same work gives a comprehensive investigation both of the Riemann problem for harmonic functions of an appropriate class and of the Riemann-Hilbert problem with oscillating coefficients in the domains bounded by piecewise smooth curves involving cusps.

In Smirnov classes a more general problem has been studied. Let A be the class of holomorphic functions whose m-th derivative possess boundary values on Γ. The generalized Riemann-Hilbert-Poincaré problem can be formulated as follows: find a function $\phi(z)$ from the class A, satisfying the boundary condition

$$\operatorname{Re}\left[\sum_{i=0}^{m}\left[a_i(t)\phi^{(i)}(t) + \int\limits_{\Gamma} h_i(t_0,t)\phi^{(i)}(t_0)dt_0\right]\right] = f(t),$$

where f, a_i, h_i, $i = 1, \ldots, m$ are the given functions. Under $\phi^{(i)}(t)$ it is meant the boundary value of the function $\phi^{(i)}(z)$ at the point t.

This problem has been posed by I. Vekua [94] in the case where $\phi^{(m)}(z)$ is continuous in D^+, and $\phi^{(m)}(t)$ belongs to the Hölder class. The case where the m-th derivative of the unknown function belongs to $E^p(D^+)$ has been considered in [45]. In the mentioned papers the boundary curve is assumed to be of Lyapunov class. Investigation of the problem depends essentially on a certain representation of the unknown function. In the case of piecewise smooth curve the appropriate representation depends on the geometry of the curve [3].

Based on these representations, the above boundary value problem has been investigated in the class of holomorphic functions, the m-th derivatives of which belong to $E^p(D^+)$. In particular, the index of that singular operator to which the boundary value problem is equivalently reduced, has been established. This index, besides the order of the derivative m, depends essentially both on the geometry of the boundary curve and on the number p. Moreover, the so-called kernel (the function of two variables) is constructed, whose closure for a positive index ensures the solvability of the problem under consideration (see [2]).

In [62] the Dirichlet and Neumann problems for harmonic functions are investigated in the classes $e_p(D,\rho) = \operatorname{Re} E^p(D,\rho)$, where the weight function is of the form $\rho(z) = (z-A)^\beta$, where A is an angular point of the boundary, the angle size equals $\nu\pi$, $0 \le \nu \le 2$, and β is an arbitrary real number. In this case, too, a complete picture of solvability is given.

Let $k^* = \nu\beta + \frac{\nu}{p}$ for the Dirichlet problem, and let $k^* = \nu\beta + \frac{\nu}{p} - \nu - 1$ for the Neumann problem. Assume that $k = [k^*]$ is the integer part of k^*. Then we have the following.

If $z^* \overline{\in} \mathbb{Z}$ and $k > 0$, then both problems are, undoubtedly, solvable, and the homogeneous problems have k linearly independent solutions. For $k < 0$, there arise additional conditions for the solvability (we mean $|k|-1$ orthogonality conditions).

For $k^* \in \mathbb{Z}$, the problem is, generally speaking, unsolvable. For the Neumann problem there occur cases, when (a) it is undoubtedly solvable and (b) for its solvability it is required of the boundary conditions that more than one condition for solvability be fulfilled (as is known, in the classical case one condition of orthogonality is quite sufficient). In cases, when the problems are unsolvable for an arbitrary right side from $L^p(\Gamma,\rho)$, one manages to point out wide enough classes for them, ensuring their solvability.

In all the cases when the problem is solvable, they are constructed in quadratures by means of conformal mappings and Cauchy type integrals.

In [17], in weighted classes $e_p(D,\rho)$ the boundary value problems have been considered under the condition $\rho(z) = |z-A|^\alpha$, $-\frac{1}{p} < \alpha < \frac{1}{p'}$ only. Note that the question on the construction of solutions was not even mentioned.

There naturally arises the question on the investigation of boundary value problems for harmonic functions from a more wide class $\widehat{e}_p(D)$, the set of those harmonic functions in domains D for which

$$\sup_{r>0} \int_{\Gamma_r} |u(z)|^p |dz| < \infty,$$

where Γ_r is an image of circumferences of radius r for the conformal mapping of the unit circle onto a simply connected domain. Describe in a few words the results from [63]. As it turned out, the class $\widehat{e}^p(D)$ is considerably wider than $e^p(D)$. In particular, there exist a domain D_0 (bounded by a curve involving a cusp) and a harmonic function u_0 such that $u_0 \in \cap_{p>1} \widehat{e}^p(D)$ and $u_0 \overline{\in} \cup_{p>1} e^p(D)$.

The following statement is true.

Theorem 2 [64]. *If the domain D is such that for boundary values of the derivative of the function mapping conformally the unit circle onto that* *domain we have $z' \in A_p$, then*

$$\widehat{e}_p(D) = e_p(D).$$

In the same paper it is established that for the unique solvability of the Dirichlet problem in $e^p(D)$ for any f it is necessary and sufficient that $z'(\tau) \in A_p$. As for the boundary value problems for harmonic functions in classes $\widehat{e}_p(D)$, the picture of solvability of the Dirichlet problem substantially differs from the case $e_p(D)$. In particular, if the boundary curve Γ is piecewise smooth with one angular point of angle size $\nu\pi$, $1 < \nu \leq 2$, then the homogeneous problem has in $\widehat{e}_\nu(D)$ a nonzero solution while in $e_\nu(D)$ has only the zero solution.

Note finally that recently the work [17] has been published which deals with the Dirichlet and Neumann problems in classes, analogous to $e_p(D)$, in case where the boundary curve is piecewise Lyapunov's only. The above-mentioned problems are investigated in classes of harmonic functions representing the real part of Cauchy type integral with densities from certain functional class X (for example, Hölder and Zygmund classes, weighted classes L^p with weight $\rho(t) = \prod_{k=1}^{n} |t - c_k|^{\alpha_k}$, $-\frac{1}{p} < \alpha_k < \frac{1}{p}$ etc.).

The Fredholm property of singular integral operators related to these problems is proved, the index is calculated and the way of construction of solutions is indicated, if and only if the solution of the corresponding singular integral operator is known.

As is mentioned above, for $X = L^p(\Gamma)$ the more general Riemann-Hilbert problem has been fully studied in [39] in case the boundary curve is piecewise smooth. Along with that, when the problem is solvable, its solutions are constructed in quadratures.

In the series of papers [32-36], the Fredholm aspects of abstract analogues of classical singular operators and some generalized Riemann boundary value problems are investigated.

6. On Algebras of Singular Integral Operators

Let $P_\Gamma = (1 + S_\Gamma)/2$ and $Q_\Gamma = (1 - S_\Gamma)/2$. As is well-known, Fredholm properties of the singular integral operator $aP_\Gamma + Q_\Gamma$ depend substantially on the coefficient a which is, generally speaking, a measurable, essentially bounded function. For the present moment, for different classes (continuous, piecewise continuous, sectorial with almost periodic discontinuities) of functions a the necessary and sufficient conditions for the Fredholmity of the above-mentioned operator in Lebesgue spaces L^p ($p > 1$) are well-known [22]. In this direction the paper [87] is worth mentioning. There, in terms of locally essential values of functions a the author established the necessary conditions of Fredholmity. In particular, he has proved that if the set of one-sided essential values at some point of the boundary omits at least two arcs whose lengths are more than $2\pi/\max(p, p')$, while the set of two-sided values omits only one arc, then the singular operator fails to possess the

Fredholm property in $L^p(\Gamma)$. From this theorem follow, in particular, the earlier known results of K. F. Clancey, I. Spitkovskiĭ, A. Bötcher and Yu. I. Karlovich for the case of two or three essential values.

In the 90s of the past century, the Fredholm theory of singular integral operators with piecewise continuous coefficients for regular boundary curves and weighted Lebesgue spaces with general weights from the Muckenhoupt class has attained its completion in [9]. Along with that, the investigation of highly important from applicational viewpoint singular integral operators with complex conjugations has faced with the known difficulties. In this direction it should be noted the work [15], in which the algebra generated by singular integrals and operators of complex conjugation in Lebesgue spaces with power weight is studied, when the boundary curve is piecewise smooth and involves cusps of the first order only. The symbol is constructed and formula for the index is written out. Based on the above mentioned results, the generalized Riemann boundary value problem is investigated.

In [85] it has been investigated the algebra generated by a singular integral operator with discontinuous coefficients, as well as by an operator of complex conjugation and with a shift. A set of integration, i.e., a finite union of piecewise smooth curves and the shifts are admitted to be outside of the line of integration.

In [16] we can find the criteria of Fredholm properties for two-dimensional singular integral operators with a conformal Carleman shift.

Finally, of special interest are the results obtained in [86], and we will dwell on them in more details.

In that paper, the boundary conditions and the norm of integral operators with fixed singularities in the kernel are established in weighted Lebesgue spaces. The second kind integral equations containing the above-mentioned singularities are studied. These equations have important applications in the theory of automatized design of complex systems.

7. On Some Other Boundary Value Problems

Many of N. Muskhelishvili's pupils and followers were engaged in the so-called boundary value problems with shifts. In Section 1 we have already discussed the problem of conjugation with Carleman shift for a ring, as well as integral and discrete equations closely related to that problem. In [60], the boundary value problem of conjugation with a shift for holomorphic matrix-functions is studied, when boundary values of an unknown function join at the displaced points. Special attention in this paper is granted to the construction of canonical matrices which allow one to construct general solutions of the problems under consideration. Differential boundary value problems, i.e., the problems whose conditions contain boundary values of derivatives of unknown functions, are considered.

The earliest results concrening boundary value problems and singular operators with shifts are reflected in well-known monograph by G. S. Litvinchuk, Boundary Value Problems and Singular Integral Equations with Shifts (Russian). Nauka, Moscow, 1977.

A few words should be said about some results stated in [73], [74] boundary value problems for the multi-dimensional case.

For the pluriholomorphic function $\phi(z)$,

$$\frac{\partial^m \phi}{\partial z^m} = 0, \quad m > 1 \tag{7.1}$$

one has the representation

$$\phi(z) = \sum_{k=0}^{m-1} \overline{z}^k \phi_k(z) \quad \text{or} \quad \phi(z) = \sum_{k=0}^{m-1} (|z|^2 - 1)^k \phi_k(z),$$

where $\phi_k(z)$ are harmonic functions. It is obvious that $\phi(z)$ is at the same time a solution of the polyharmonic equation in R^2,

$$\Delta^m \phi = 0. \tag{7.2}$$

For ϕ, the boundary value problems are posed and their solutions are given explicitly. Moreover, for the above equations non-local problems are solved using the Wiener-Hopf generalized integral and dual integral equations with two kernels depending both on the difference and on the sum of the arguments. They are reduced to the Riemann-Hilbert boundary value problems for two holomorphic functions, and the cases are found when these systems are solved explicitly.

As a multi-dimensional analogue of holomorphic and pluriharmonic functions, the regular and pluriregular equations in Clifford's analysis are considered:

$$\overline{\partial}^m u(x) = 0, \quad m \geq 1, \quad x = (x_0, x_1, \ldots, x_n), \tag{7.3}$$

where $\overline{\partial}$ is the Dirac operator in the elliptic case, $u(x)$ is at the same time a solution of polyharmonic equations in R^{n+1}.

The well-known Liouville's theorem for the harmonic function of a complex variable for the pluriregular function $u(x)$ is formulated as follows:

Let $u(x)$ be a solution of (7.3) in the whole R^{n+1} and at infinity satisfy the conditions

$$\lim_{|x| \to \infty} \overline{\partial}^k u(x) = 0, \quad k = 0, 1, \ldots, m-1.$$

Then $u(x) \equiv 0$ everywhere.

Moreover, the extension theorem and the Riemann-Schwartz principle of reflection are proved. For the iterated multi-dimensional Cauchy type singular integral the Poincare-Bertrand formula is proved by using a slightly modified method which is given in the well-known monograph of N. Muskhelishvili [66].

Let D^+ be the half-space, $x_2 > 0$, S^+ be the circular domain, $x_1^2 + x_2^2 \leq a^2$, and S^- be the domain, $\{(x_1, x_2) : x_1^2 + x_2^2 > a^2\}$. Boundary value problems are considered in three-dimensional spaces with cracks for polyharmonic functions. Let, for instance, $m = 2$.

Consider the following problems:

1. Define a biharmonic function in D^+, vanishing at infinity, by the conditions

$$u(x_0, x_1, 0) = f(x_0, x_1), \quad (x_0, x_1) \in S^2$$
$$\Delta u|_{x_2=0} = \varphi(x_0, x_1), \quad (x_0, x_1) \in S^-$$
$$\frac{\partial u}{\partial x_2}|_{x_2=0} = \psi(x_0, x_1), \quad (x_0, x_1) \in S^+,$$

2. Define in the space R^3 with a crack along the S^+ a polyharmonic function, vanishing at infinity, by the conditions

$$\frac{\partial^k u}{\partial x_2^k}|^{\pm}_{x_2=0} = \varphi^{\pm}(x_0, x_1), \quad (x_0, x_1) \in S^+, \quad k = 0, 1, \dots, m-1.$$

Using the representation u_k, these problems are reduced to the problems for harmonic $u = \sum_{k=0}^{m-1} x_2^k u_k$ functions, and then the solution is given explicitly with the help of Hobson's formula. For the details we refer the reader to [73], [74].

Finally, it should be noted that N. Muskhelishvili's ideas exert considerable influence upon the investigation of a number of problems in statistics. We mean first results obtained at the A. Razmadze Mathematical Institute on random Riemann boundary value problems and random singular integral equations which are closely connected with many physical and engineering problems, such as problems of crack mechanics, air-dynamics, etc.

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